On Discrete-Time Greek Hedging of Longevity Risk

Kenneth Q. Zhou
Joint-work with Johnny S.-H. Li

University of Waterloo

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Agenda

1. Introduction
2. Longevity Greek Hedging
3. Empirical Analysis
4. Conclusion
Standardization of Longevity Risk

- Standardization could resolve the misalignment of incentives between longevity hedgers and capital market investors.

- Capital markets could provide sufficient supply for acceptance of longevity risk.

- Capital market investors could enjoy diversification benefits and risk premiums.

- High liquidity and symmetric information could be achieved through standardized longevity-linked securities.
Greek Hedging for Longevity Risk

- Greeks measure the sensitivity of the value of a security to changes in certain underlying parameters on which the value depends.

- Longevity risk is embedded in mortality liabilities that depend on multiple ages over multiple years.

- Greek hedging for longevity requires a mortality model that combines the multiple underlying forces into a handful of period effects.

- Longevity Greeks are calculated based on the underlying period effects.
Greek Hedging in Longevity Literature

- Cairns (2011) developed a dynamic hedging strategy considering delta-only hedges.

- Luciano et al. (2012) developed a delta-gamma hedging strategy in the context of continuous-time modelling.

- Cairns (2013) developed the concept of nuga-hedging to mitigate the recalibration risk of period effects.
Our Objectives

1. The discrete-time Lee-Carter model with conditional heteroscedasticity (Chen et al., 2015; Wang et al., 2015).

2. Multiple Greeks hedging with the consideration of ‘vega’.


4. Suggestions on the selection of hedging instruments.

5. Empirical analysis with the England and Wale mortality data.
The Lee-Carter Model

\[ \ln(m_{x,t}) = a_x + b_x k_t \]

where \( k_t \) is the period effect following a random-walk process

\[ k_t = k_{t-1} + \theta + \epsilon_t. \]
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The GARCH Model

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\[ \epsilon_t = \sqrt{h_t} \eta_t \]

\[ h_t = \omega + \alpha \epsilon_{t-1}^2 + \beta h_{t-1} \]
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GARCH(1,1) Conditional Variance Model:

Conditional Probability Distribution: Gaussian

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Error</th>
<th>t Statistic</th>
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<tr>
<td>Constant</td>
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<td>0.707719</td>
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<tr>
<td>ARCH{1}</td>
<td>0.135123</td>
<td>0.0632534</td>
<td>2.13621</td>
</tr>
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The Conditional Volatility of the Period Effect
Survival Rate

Survival rate:

\[ S_{x,t}(T) = e^{-\sum_{s=1}^{T} m_{x+s-1,t+s}} = e^{-\sum_{s=1}^{T} e^{a_{x+s-1}+b_{x+s-1}k_{t+s}}} \]

\[ = e^{-\sum_{s=1}^{T} e^{Y_{x,t}(s)}} = e^{-W_{x,t}(T)} \]

where

\[ k_{t+s} = k_t + s\theta + \sum_{i=1}^{s} \epsilon_{t+i} \]

\[ \epsilon_{t+i} = \eta_{t+i}\sqrt{h_{t+i}} \]

\[ h_{t+i} = \omega + \omega \sum_{v=1}^{i-1} \prod_{n=1}^{v} (\alpha \eta_{t+i-n}^2 + \beta) + (\alpha \epsilon_{t}^2 + \beta h_{t}) \prod_{n=1}^{i-1} (\alpha \eta_{t+i-n}^2 + \beta). \]
Survival Probability

Survival probability:

\[ p_{x,t}(T, k_t, h_t) : = \mathbb{E}[S_{x,t}(T) \mid k_t, h_t] \]
\[ = \mathbb{E}\left[e^{-W_{x,t}(T)} \mid k_t, h_t\right] \]
\[ = \mathbb{E}\left[e^{-\sum_{s=1}^{T} e^{Y_{x,t}(s)}} \mid k_t, h_t\right]. \]

Annuity Liability:

\[ L = \sum_{s=1}^{\infty} (1 + r)^{-s} p_{x,t}(s, k_t, h_t). \]

q-forwards:

\[ Q = (1 + r)^{-T^*} (p_{x_f,t+T^*-1}(1, k_t, h_t) - (1 - q^f)). \]

where \( r \) is the constant interest rate and \( q^f \) is the predetermined forward mortality rate.
Calculation of Delta

Annuity liability:

\[ \Delta_{x,t}(T) = \frac{\partial}{\partial k_t} p_{x,t}(T, k_t, h_t) = -\sum_{s=1}^{T} b_{x+s-1} \mathbb{E}[e^{Y_{x,t}(s) - W_{x,t}(T)} | k_t, h_t] \]

\[ \Delta_L = \frac{\partial}{\partial k_t} \sum_{s=1}^{\infty} (1 + r)^{-s} p_{x,t}(s, k_t, h_t) = \sum_{s=1}^{\infty} (1 + r)^{-s} \Delta_{x,t}(s). \]

q-forwards:

\[ \Delta_{xf,t+T^*-1}(1) = \frac{\partial}{\partial k_t} p_{xf,t+T^*-1}(1, k_t, h_t) \]

\[ \Delta_Q = \frac{\partial}{\partial k_t} (1 + r)^{-T^*} p_{xf,t+T^*-1}(1, k_t, h_t) = (1 + r)^{-T^*} \Delta_{xf,t+T^*-1}(1). \]
Delta of q-forwards

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Calculation of Gamma

Annuity liability:

\[
\Gamma_{x,t}(T) = \frac{\partial^2 p_{x,t}(T, k_t, h_t)}{\partial k_t^2}
\]

\[
= E \left[ e^{-W_{x,t}(T)} \left( \left( \sum_{s=1}^{T} b_{x+s-1} e^{Y_{x,t}(s)} \right)^2 - \sum_{s=1}^{T} b_{x+s-1}^2 e^{Y_{x,t}(s)} \right) \right]_{k_t, h_t}
\]

q-forwards:

\[
\Gamma_{x_f,t+T^* - 1(1)} = b_{x_f}^2 E \left[ e^{Y_{x_f,t+T^* - 1(1)} - e^{Y_{x_f,t+T^* - 1(1)}} \left( e^{Y_{x_f,t+T^* - 1(1)}} - 1 \right) \right]_{k_t, h_t}
\]
Gamma of q-forwards

\[ \Gamma_q \]

- Value vs. reference age
- Value vs. time-to-maturity

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Calculation of Vega

Annuity liability:

\[ V_{x,t}(T) = \frac{\partial p_{x,t}(T, k_t, h_t)}{\partial h_t} \]

\[ = - \sum_{s=1}^{T} b_{x+s-1} \mathbb{E} \left[ e^{Y_{x,t}(s)-W_{x,t}(T)} \left( \sum_{i=1}^{s} \frac{\beta \epsilon_{t+i}}{2h_{t+i}} \prod_{n=1}^{i-1} (\alpha \eta_{t+i-n}^2 + \beta) \right) \bigg| k_t, h_t \right] \]

q-forwards:

\[ V_{x_f,t+T^*-1}(1) = b_{x_f} \mathbb{E} \left[ e^{Y_{x_f,t+T^*-1} - Y_{x_f,t+T^*}} \left( \sum_{i=1}^{T^*} \frac{\beta \epsilon_{t+i}}{2h_{t+i}} \prod_{n=1}^{i-1} (\alpha \eta_{t+i-n}^2 + \beta) \right) \bigg| k_t, h_t \right] \]
Vega of $q$-forwards

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Hedge Ratios

Hedge ratios \((u)\) are determined by matching the Greeks of the liabilities and q-forwards.

- Delta-only:
  \[ u_1 \Delta_{Q1} = \Delta_L \]

- Delta-Gamma:
  \[
  \begin{bmatrix}
  \Delta_{Q1} & \Delta_{Q2} \\
  \Gamma_{Q1} & \Gamma_{Q2}
  \end{bmatrix}
  \begin{bmatrix}
  u_1 \\
  u_2
  \end{bmatrix}
  =
  \begin{bmatrix}
  \Delta_L \\
  \Gamma_L
  \end{bmatrix}
  \]

- Delta-Gamma-Vega:
  \[
  \begin{bmatrix}
  \Delta_{Q1} & \Delta_{Q2} & \Delta_{Q3} \\
  \Gamma_{Q1} & \Gamma_{Q2} & \Gamma_{Q3} \\
  V_{Q1} & V_{Q2} & V_{Q3}
  \end{bmatrix}
  \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
  \end{bmatrix}
  =
  \begin{bmatrix}
  \Delta_L \\
  \Gamma_L \\
  V_L
  \end{bmatrix}
  \]
Example

- A 30-year temporary annuity sold to a male individual aged 60 at time 0.

<table>
<thead>
<tr>
<th>$\Delta_L$</th>
<th>$\Gamma_L$</th>
<th>$V_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.067485</td>
<td>-0.0016493</td>
<td>-0.0078816</td>
</tr>
</tbody>
</table>

- q-forwards with reference age from age 60 to 89 and time-to-maturity from 1 to 30 years.

- Delta-only, vega-only, delta-gamma and delta-vega hedges are considered.

- Hedges are evaluated at time 30, where the hedge effectiveness is

$$HE = 1 - \frac{\text{var}(PH - PL)}{\text{var}(PL)}$$
Single Greek Hedging

Heat maps for the hedge effectiveness of delta-only (left) and vega-only (right) hedges.

Kenneth Q. Zhou
University of Waterloo

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Heat maps for the hedge effectiveness of delta-gamma (left) and delta-vega (right) hedges.
Multiple Greeks Hedge Ratios
Multiple Greeks Hedge Ratios

For multiple Greek hedging with q-forwards, hedge ratios need to be all positive. Recall that the hedge ratios are calculated as

$$
\begin{bmatrix}
\Delta Q_1 & \Delta Q_2 \\
\Gamma Q_1 & \Gamma Q_2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
= \begin{bmatrix}
\Delta L \\
\Gamma L
\end{bmatrix}
\Rightarrow
u_1 = \frac{\Delta L \Gamma Q_2 - \Gamma L \Delta Q_2}{\Delta Q_1 \Gamma Q_2 - \Delta Q_2 \Gamma Q_1}
\quad
u_2 = \frac{\Gamma L \Delta Q_1 - \Delta L \Gamma Q_1}{\Delta Q_1 \Gamma Q_2 - \Delta Q_2 \Gamma Q_1}.
$$

To have positive $u_1$ and $u_2$, we want the following inequality to hold:

$$
\frac{\Delta Q_1}{\Gamma Q_1} > \frac{\Delta L}{\Gamma L} > \frac{\Delta Q_2}{\Gamma Q_2}.
$$
Ratios of Delta, Gamma and Vega

Ratios of delta over gamma (left) and delta over vega (right) for q-forwards with different reference ages. The horizontal Bold line is the ratio for the liability.
Conclusion

- Greek hedging can be an useful hedging tool in mitigating longevity risk.
- The choice of reference ages and time-to-maturities of the hedging instrument used is important.
- Single Greek hedging is robust to the choice of reference ages, but not to the choice of time-to-maturities.
- Multiple Greeks hedging can further improve the hedge effectiveness, but careful calibrations are required.
- With limited choices of reference ages, delta-vega hedging is more robust than delta-gamma hedging.
Further Remarks

- The empirical results heavily depend on the assumptions underlying the Lee-Carter model.

- Other mortality models and their model estimates, especially non-parametric models, should be considered to verify the results.

- The robustness of Greek hedging relative to other risks such as population basis risk deserves further studies.

- Other benefits of using multiple Greeks such as reduction of left-tail risk should be further investigated.