Consistently modeling unisex mortality rates

Dr. Peter Hieber, Longevity 14, 20.09.2018
University of Ulm, Germany

Motivation

- **European Court of Justice**, 2011

  Different insurance premiums according to gender are prohibited (Gender directive 2004/113EC).

- But: Life insurance risk differs by gender (statistically significant).

Motivation, research question, introduction

Consistent mortality models

Numerical examples
- Consistency: Lee-Carter mortality model
- Reserves: (un)observed heterogeneity

Research question

Given: *Two* groups with *differing mortality risk*
and mortality model for each group.

-  male/female
-  smoker/non-smoker

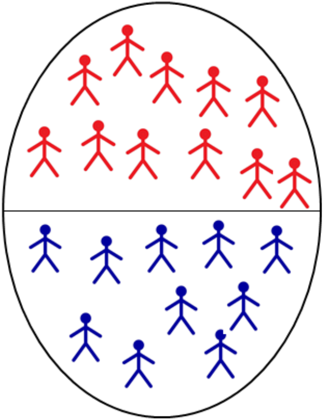
How to create **unisex mortality models** / unisex mortality tables that are **consistent** with a given male/female mortality model?

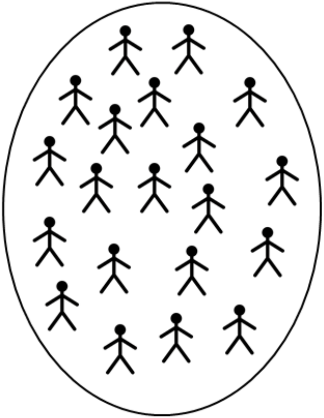
- male/female model for risk management.
- unisex model for premium calculation.

Portfolio at time $t = 0$

female/male portfolio ($n = 20$)
age y, survival probability ${}_Tp_y$

unisex portfolio ($n = 20$)
age z, survival probability ${}_Tp_z = ?$



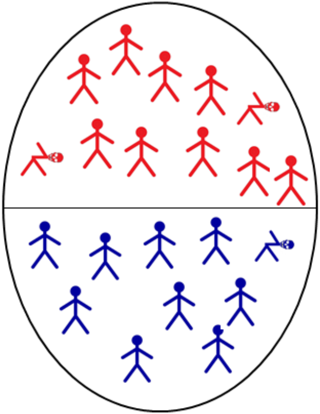


$N_0^y = N_0^x = 10$
age x, survival probability ${}_Tp_x$

$N_0^z = n = 20$

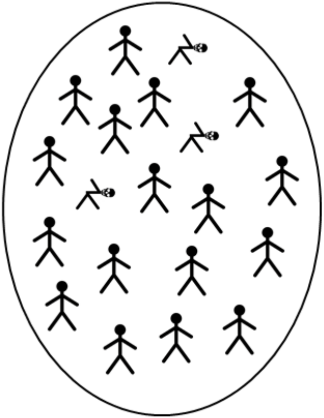
Portfolio at time $t = T$

female/male portfolio ($n = 20$)
age $y + T$



$N_T^y = 8$, $N_T^x = 9$
age $x + T$

unisex portfolio ($n = 20$)
age $z + T$



$N_T^z = 17$

Consistency: Example

Consider an annuity portfolio of N_0^y females and N_0^x males. Mortality risk is specified by two **Lee-Carter models** with parameters $(A_t^y, B_t^y, \theta_y, c_y)$ and $(A_t^x, B_t^x, \theta_x, c_x)$. This implies a time-$T$-survival probability

$$_Tp_y := \mathbb{P}(\text{“female survives } T\text{”}) \text{ and } {_Tp_x} := \mathbb{P}(\text{“male survives } T\text{”})$$

For a unisex portfolio, this leads to the survival probability:

$$_Tp_z := \frac{N_0^y}{N_0^x + N_0^y} \cdot \mathbb{P}(\text{“female survives } T\text{”}) + \frac{N_0^x}{N_0^x + N_0^y} \cdot \mathbb{P}(\text{“male survives } T\text{”})$$

What is the **consistency error** if we use a unisex **Lee-Carter model** with parameters $(A_t^z, B_t^z, \theta_z, c_z)$?

What happens if the group composition (N_0^y, N_0^x) is **not** observable?

Consistency: Deterministic mortality tables

In this talk, 2 consistent unisex mortality models are introduced.

female/male portfolio — unisex portfolio

Consistency criterion 1 (unobservable)

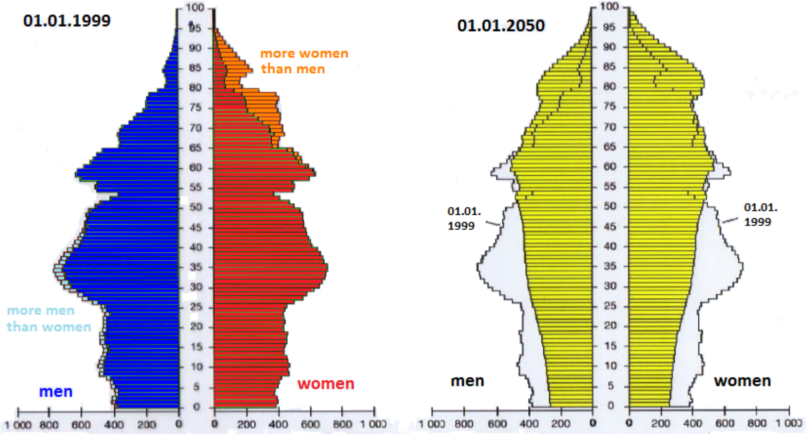
(C1) survival probability $\hat{\xi}_0 \cdot {}_tp_x + (1 - \hat{\xi}_0) \cdot {}_tp_y \quad = \quad {}_tp_z$, for all $t \in [0, T]$.

($\hat{\xi}_0$: initial guess of share of group x).

Consistency criterion 2 (observable)

(C1*) portfolio members $N_t^x + N_t^y \quad = \quad N_t^z$, for all $t \in [0, T]$.

Demography in Germany



M1

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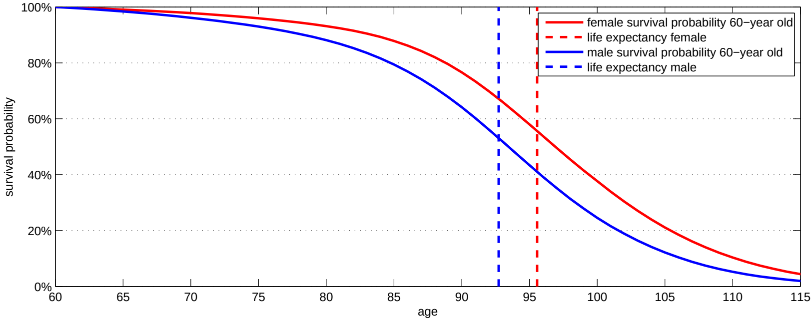
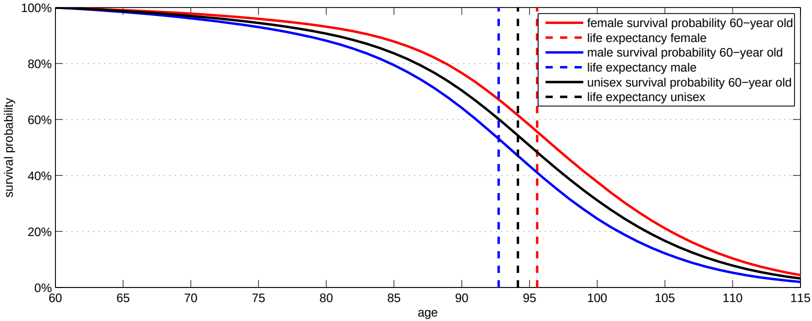
Survival curve $\{_t p_y\}_{t\in[0,T]}$, $\{_t p_x\}_{t\in[0,T]}$



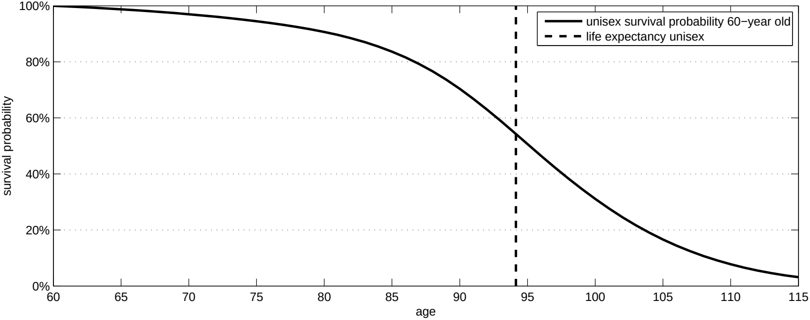
DAV 2004R, annuity table (includes risk margins).

Survival curve



DAV 2004R, annuity table (includes risk margins).

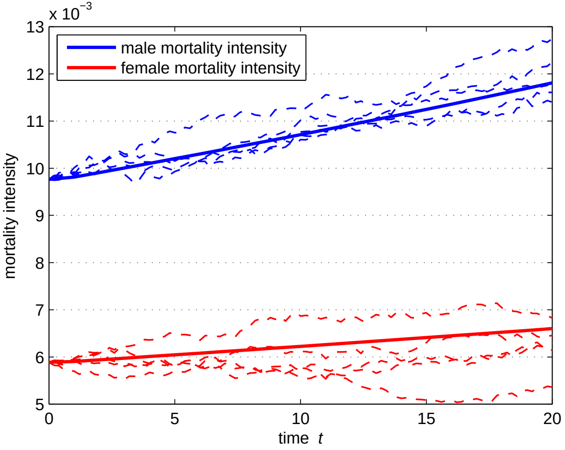
Unisex survival curve $\{{}_tp_z\}_{t\in[0,T]}$



For an initial share of males ξ_0, choose:

$$ {}_tp_z = \xi_0 \cdot {}_tp_x + (1-\xi_0) \cdot {}_tp_y \, . $$

Stochastic mortality rates



Plots $\{\lambda_t^x\}_{t\in[0,20]}$ (male) and $\{\lambda_t^y\}_{t\in[0,20]}$ (female).

Survival curves ${}_tp_x := e^{-\int_0^t \lambda_s^x \mathrm{d}s}$, ${}_tp_y := e^{-\int_0^t \lambda_s^y \mathrm{d}s}$.

Assumption (Mortality model)

For $i \in \{x, y, z\}$, we assume that

- **Given** *the survival curve* $\{{}_tp_i\}_{t\in[0,T]}$, **individual deaths are independent**. *Choose* ${}_tp_i := e^{-\int_0^t \lambda_s^i \mathrm{d}s}$. *Number of survivors at ti e $t > 0$ is binomially distributed:*

$$N_t^i \sim \text{Bin}\left(N_0^i, {}_tp_i\right).$$

- *Randomness in the survival curve* $\{{}_tp_i\}_{t\in[0,T]}$ *(***systematic mortality risk***) is conditionally independent of the binomial distribution (***unsystematic mortality risk***).*
- *The intensity* $\{\lambda_t^i\}_{t\geq 0}$ *is continuous.*

Definition (Unisex mortality model (unobservable))

For initial share of males $\hat{\xi}_0$, define

$$\lambda_t^z := \underbrace{\frac{\hat{\xi}_0 \cdot e^{-\int_0^t \lambda_s^x \mathrm{d}s}}{\hat{\xi}_0 \cdot e^{-\int_0^t \lambda_s^x \mathrm{d}s} + (1-\hat{\xi}_0) \cdot e^{-\int_0^t \lambda_s^y \mathrm{d}s}}}_{\text{time-}t\text{ share of males}} \cdot \lambda_t^x$$

$$+ \underbrace{\frac{(1-\hat{\xi}_0) \cdot e^{-\int_0^t \lambda_s^y \mathrm{d}s}}{\hat{\xi}_0 \cdot e^{-\int_0^t \lambda_s^x \mathrm{d}s} + (1-\hat{\xi}_0) \cdot e^{-\int_0^t \lambda_s^y \mathrm{d}s}}}_{\text{time-}t\text{ share of females}} \cdot \lambda_t^y .$$

We obtain: $N_T^z \sim \text{Bin}\left(n, \hat{\xi}_0 \cdot {}_t p_x + (1-\hat{\xi}_0) \cdot {}_t p_y\right)$.

How to obtain λ^z: Solve ${}_t p_z = \hat{\xi}_0 \cdot {}_t p_x + (1-\hat{\xi}_0) \cdot {}_t p_y$ for λ_t^z.

Definition (Unisex mortality model (observable))

For initial share of males $\hat{\xi}_0$, define

$$\mu_t^z := \underbrace{\frac{N_t^x}{N_t^x + N_t^y}}_{\text{time-}t\text{ share of males}} \cdot \lambda_t^x + \underbrace{\frac{N_t^y}{N_t^x + N_t^y}}_{\text{time-}t\text{ share of females}} \cdot \lambda_t^y \,. \qquad (1)$$

We obtain: $N_T^{z*} = N_t^x + N_t^y$, where $N_t^x \sim \text{Bin}\left(\xi_0 n,\ {}_t p_x\right)$ and $N_t^z \sim \text{Bin}\left((1-\xi_0)n,\ {}_t p_y\right)$.

(μ_t^z is still the "instantaneous" death probability, but does not define a mortality model).

- For the observable case, it is necessary, to observe deaths immediately (no reporting delays etc.) and to observe the group membership.

- For the unobservable case, we do not observe the group membership or deaths immediately.

Implications for risk management

M2

Unisex portfolio: $N_T^z \sim \text{Bin}\left(N_0^z,\ {}_Tp_z\right)$, where

$$ {}_tp_z = \xi_0 \cdot {}_tp_x + (1 - \xi_0) \cdot {}_tp_y . $$

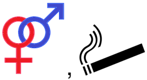
Female/male portfolio: $N_T^y \sim \text{Bin}\left(N_0^y,\ {}_Tp_y\right)$, $N_T^x \sim \text{Bin}\left(N_0^x,\ {}_Tp_x\right)$.

Lemma (Prudence of the unisex mortality model (C1))

$$ \mathbb{E}[N_T^z] = \mathbb{E}[N_T^x + N_T^y]\,, \tag{2} $$

$$ \text{Var}(N_T^z) \geq \text{Var}(N_T^x + N_T^y)\,. \tag{3} $$

Proof: special cases: e.g. Feller [1950].



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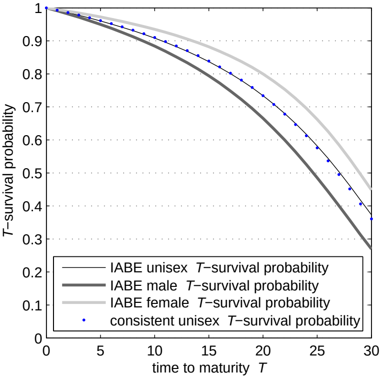
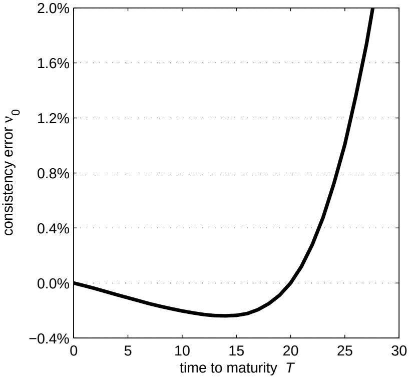
Consistency: Lee-Carter mortality model





Parameters: Belgian Actuarial Society, IA|BE (available online).

Reserves: (un)observed heterogeneity

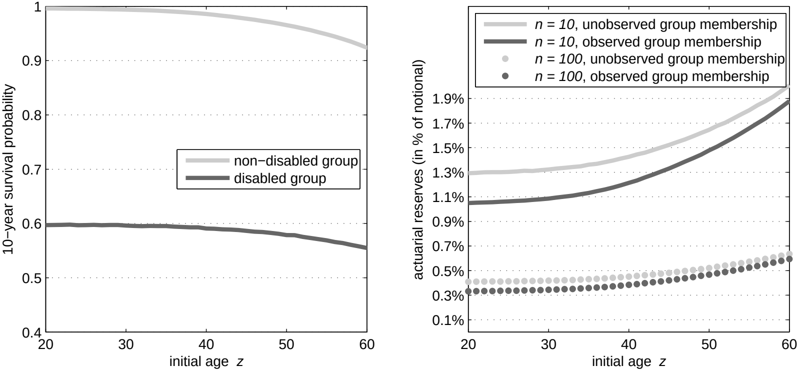
Consider a portfolio of n pure endowment insurance contracts with **survival benefit** $S =€\,1$ at maturity $T = 10$. Risk-free rate $r = 0\%$. 10% of the portfolio is disabled with life expectancy:

$$_{T}p_{z}^{\text{disabled}} = 60\% \cdot {}_{T}p_{z}\,.$$

We choose the standard deviation principle and define the **per-contract actuarial reserve** (in % of the contract's nominal €1) as

$$R^{j} := \frac{1}{n} \cdot \frac{\alpha}{2} \sqrt{\mathrm{Var}\left(N_{T-}^{z}\right)}\,. \qquad (4)$$

Reserves: (un)observed heterogeneity



Reserves annuity portfolio with 10% disabled persons.

M2

Conclusion

How to create **unisex mortality models** / unisex mortality tables that are **consistent** with a given male/female mortality model?

M1 Change/stochasticity in male/female mortality rates affects also male/female share in the annuity portfolio (also stochastic!).

M2 Observed heterogeneity reduces mortality risks (e.g. the portfolio's variance), compare two consistency criteria.

Further interesting aspects: **adverse selection**, **effect of portfolio size** *n*.

Literature

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