Robust Estimation of Real Exchange Rate Process Half-life

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Background

- The real exchange rate (in logs) is defined as

\[ q_t \equiv s_t - \bar{p}_t^h + \bar{p}_t^f \]

with \( s_t \) denoting the spot exchange rate, \( \bar{p}_t^h \) the domestic price index and \( \bar{p}_t^f \) the foreign price index.
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- If PPP holds exactly, \( q_t \) should equate 0 for all \( t \) though deviations due to sticky prices are theoretically postulated (Rogoff, 1996).
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- If PPP holds exactly \( q_t \) should equate 0 for all \( t \) though deviations due to sticky prices are theoretically postulated (Rogoff, 1996).

- The measure adopted in the literature to quantify the persistence of these deviations is the half-life of \( \{q_t\} \) (Mark, 2001; Rossi, 2005) defined as the smallest \( h \) such that

\[ \psi(h) = \frac{1}{2} \mid \psi(0) = 1 \]

with \( \psi(t), t \geq 0 \) denoting the IRF of \( \{q_t\} \).
Motivation

According to theory, half-lives should be in the range of 1-2 years, yet empirical estimates imply much larger persistence ⇒ PPP puzzle.
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  2. **Correct inference** of the model parameters.
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- In the present paper, we reconsider the extent of the PPP puzzle using outliers robust inference for ARMA processes.

- Unaccounted outliers distort the half-life estimates since they alter the autocorrelation structure of the observed time-series (Tsay, 1986) and hence the IRF.
Motivation (cont’d)

Figure: USD/GBP Real Exchange Rate.
This paper

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- We introduce a modelling framework for the real exchange rate that allows for **outlying observations**.

- We consider a **Dummy Saturation** type procedure to detect and model the outliers observations; we check the procedure to deliver the correct retention rates of the dummies.

- We test the **PPP** for a group of countries by estimating the half-life of the real exchange rates with and without outliers detection.
Real Exchange Rate Model

Let the process followed by \( \{q_t\} \) be described by

\[
q_t = q_0 + \sum_{i=1}^{k} \delta_i V_i(L)1(t = T_i) + v_t
\]

(1)

\[
\phi(L)v_t = \theta(L)\varepsilon_t \quad t = 1, \ldots, T
\]

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\]  \hspace{1cm} (1)

\[
\phi(L)v_t = \theta(L)\xi_t \quad t = 1, \ldots, T
\]  \hspace{1cm} (2)

where

- \( k \) denotes the number of outlying events;
- \( \delta_i \) is the outlier or level shift size;
- \( V_i(L) \) (with \( L \) denoting the lag operator) defines the outlier type;
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- \( \delta_i \) is the outlier or level shift size;
- \( V_i(L) \) (with \( L \) denoting the lag operator) defines the outlier type;
- \( 1(t = T_i) \) is an impulse indicator assuming value 1 for \( t = T_i \) and 0 otherwise;
- \( \phi(L) = 1 - \phi_1L - \cdots - \phi_pL^p \) and \( \theta(L) = 1 - \theta_1L - \cdots - \theta_qL^q \) are lag polynomials with roots outside the unit circle, and
- \( \varepsilon_t \sim iid \mathcal{N}(0, \sigma^2_\varepsilon) \).
Real Exchange Rate Model (cont’d)

- Three specifications of $V_i(L)$ are particularly relevant to our analysis:

  - $V_i(L) = 1$ \hspace{1cm} \text{Additive Outlier (AO)}
  - $V_i(L) = \phi^{-1}(L)\theta(L)$ \hspace{1cm} \text{Innovative Outlier (IO)}
  - $V_i(L) = (1 - L)^{-1}$ \hspace{1cm} \text{Level Shift (LS)} \quad ((1 - L)^{-1}1(t = T_i) = 1(t \geq T_i)).
Estimation

- It is convenient to rewrite (1)-(2) in matrix notation, to give

\[ q = W \delta + v \quad \quad v = \phi^{-1}(L)\theta(L)\varepsilon. \]

where \( W \) is a matrix of size \( (1 + k^A + k^L + k^I) \times T \) mostly made up of 0-1 entries.
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- We are thus in the framework of a **regression with ARMA errors** and under the normality assumption we have

\[ v \sim \mathcal{N}_T(0, \sigma^2\varepsilon\Omega). \]
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- Estimation of the unknowns (regression coefficients and time series parameters) can be obtained maximising the following likelihood

\[
\ell(\delta, \phi, \theta, \sigma^2_\varepsilon) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log\sigma^2_\varepsilon - \frac{1}{2} \log|\Omega| - \frac{1}{2\sigma^2_\varepsilon} (q - W\delta)^\top \Omega^{-1} (q - W\delta).
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\]

- Problem: the matrix of regressors \( W \) is not known. This amounts to the problem of selecting the outlying observations.
Outliers Detection Approach

We consider a ML procedure built around the Dummy Saturation principle (Hendry, 1999; Hendry et al., 2008; Johansen and Nielsen, 2009).
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- The procedure searches for outliers in \( \{ q_t \} \) by saturating in turn with AOs, IOs and LSs.
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Steps

1. **Initial ARMA order**: select using AIC the ARMA order under the null of no outliers, i.e. find the model

   \[
   \phi(L)(q_t - q_0) = \theta(L)\varepsilon_t,
   \]

   and denote the corresponding order with \((\tilde{p}, \tilde{q})\).
Outliers Detection Approach (cont’d)

2. **Search for outliers**: Look sequentially for AOs, IOs and LSs using a significance level \( \alpha \) and keep track of the selected outliers.
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   2.1 Saturate the model determined at Stage 1 with AOs.
      
      (a) Add the first half of AOs, say $x_{j,t}$, $j = 1, \ldots, \lfloor T/2 \rfloor$, and estimate by ML the following regression

      \[ q_t = q_0 + \sum_{j=1}^{\lfloor T/2 \rfloor} \delta^A_j x_{j,t} + \phi^{-1}(L)\theta(L)\varepsilon_t. \quad (3) \]
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(b) Store all $x_j$ such that $|t_{\hat{\delta}^A_j}| > c_{\alpha/2}$. Denote the matrix of retained AOs with $\tilde{X}(1)$. 

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(b) Store all $x_{j,t}$ such that $|t_{\hat{\delta}_j^A}| > c_{\alpha}/2$. Denote the matrix of retained AOs with $\bar{X}_{(1)}$.

(c) Repeat by saturating with the second half of AOs, i.e. estimating (3) with $x_{j,t}$, $j = \lfloor T/2 \rfloor + 1, \ldots, T$, and again define $\bar{X}_{(2)}$ the matrix of the outliers for which $|t_{\hat{\delta}_j^A}| > c_{\alpha}/2$. 
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   (d) Estimate (3) including only the AOs selected at the two previous stages and denote $\tilde{X}$ the matrix with the statistically significant outliers.
Outliers Detection Approach (cont’d)

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(b) Store all $x_j$ such that $|\hat{\delta}^A_j| > c_{\alpha/2}$. Denote the matrix of retained AOs with $\hat{X}_{(1)}$.

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(d) Estimate (3) including only the AOs selected at the two previous stages and denote $\hat{X}$ the matrix with the statistically significant outliers.

2.2 Repeat steps (a)-(d) for LSs and then IOs in order to get $\hat{Y}$ and $\hat{Z}$, the matrix containing the retained LSs and IOs respectively.
3. **Final model selection**: estimate by ML the following regression with ARMA errors

$$q_t = q_0 + \hat{x}_t^\top \delta^A + \hat{y}_t^\top \delta^L + \phi^{-1}(L)\theta(L)(\varepsilon_t + \hat{z}_t^\top \delta^I)$$  (4)

and drop the not significant outliers. Estimation of (4) is iterated until the included outliers are all statistically significant and the ARMA order is modified accordingly.
We check by simulation that the procedure delivers the correct retention rates of the dummies:

<table>
<thead>
<tr>
<th></th>
<th>T = 100</th>
<th></th>
<th>T = 200</th>
<th></th>
<th>T = 300</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AO</td>
<td>IO</td>
<td>LS</td>
<td>AO</td>
<td>IO</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>α = 0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = 2</td>
<td>1.242</td>
<td>6.472</td>
<td>0.901</td>
<td>1.215</td>
<td>6.488</td>
</tr>
<tr>
<td>n = 5</td>
<td>1.189</td>
<td>2.125</td>
<td>0.989</td>
<td>1.079</td>
<td>2.088</td>
</tr>
<tr>
<td>n = 10</td>
<td>1.116</td>
<td>1.530</td>
<td>0.993</td>
<td>1.096</td>
<td>1.493</td>
</tr>
<tr>
<td>n = 20</td>
<td>1.116</td>
<td>1.462</td>
<td>1.041</td>
<td>1.054</td>
<td>1.273</td>
</tr>
</tbody>
</table>

α = 0.05

| n = 5   | 5.245  | 7.756 | 4.567   | 5.096 | 7.822| 4.951 | 5.092 | 7.875| 4.981 |
| n = 10  | 5.206  | 6.346 | 4.894   | 5.188 | 6.235| 4.965 | 5.038 | 6.271| 5.010 |
| n = 20  | 5.332  | 6.140 | 4.995   | 5.096 | 5.680| 4.994 | 5.132 | 5.669| 5.022 |
Robust Half-life Computation

- Robust half-life estimates are obtained from the following “cleaned” series

\[ \tilde{q}_t \equiv q_t - q_0 - x_t^\top \delta^A - y_t^\top \delta^L - \phi(L)^{-1} \theta(L) z_t^\top \delta^I = \phi^{-1}(L) \theta(L) \varepsilon_t. \]
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Let \( \psi(L) = \phi^{-1}(L) \theta(L) \) to give \( \tilde{q}_t = \sum_{j=0}^{+\infty} \psi_j \varepsilon_{t-j}, \) with \( \sum_{j=0}^{+\infty} \psi_j^2 < \infty \) (under the assumption that the roots of \( \phi(L) \) all lie outside the unit circle), such that \( \lim_{j \to \infty} \psi_j = 0. \)
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Let \( \psi(j) = \psi_j, j = 1, 2, \ldots, T \) and \( \psi(0) = 1 \) be the IRF, we are interested in finding the first instant \( h \) such that \( \psi(h) = 0.5 \).
Robust Half-life Computation (cont’d)

Assuming \( \{\tilde{q}_t\}_{t=1}^T \sim ARMA(p, q) \),

\[
\begin{bmatrix}
\tilde{q}_t \\
\tilde{q}_{t-1} \\
\vdots \\
\tilde{q}_{t-p+1} \\
\varepsilon_t \\
\varepsilon_{t-1} \\
\vdots \\
\varepsilon_{t-q+1}
\end{bmatrix}
= \begin{bmatrix}
\phi_1 & \phi_2 & \ldots & \phi_p & \theta_1 & \ldots & \theta_q \\
1 & 0 & \ldots & & & & \\
0 & \ddots & 0 & \ldots & & & \\
0 & \ldots & 1 & 0 & \ldots & & \\
0 & \ldots & & & & & \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ddots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{q}_{t-1} \\
\tilde{q}_{t-2} \\
\vdots \\
\tilde{q}_{t-p} \\
\varepsilon_{t-1} \\
\varepsilon_{t-2} \\
\vdots \\
\varepsilon_{t-q}
\end{bmatrix}
+ \begin{bmatrix} 1 \\
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0 \end{bmatrix} \varepsilon_t
\]

The IRF can be obtained as

\[
\psi(j) = e(F^j G)
\]

where \( e = [1 \ 0 \ \ldots \ 0]^T \) is a selection vector.
Robust Half-life Computation (cont’d)

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\varepsilon_{t-1} \\
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\end{bmatrix} = 
\begin{bmatrix}
\phi_1 & \phi_2 & \cdots & \phi_p & \theta_1 & \cdots & \theta_q \\
1 & 0 & \cdots \\
0 & \ddots & 0 & \cdots \\
0 & \cdots & 1 & 0 & \cdots \\
0 & \cdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{q}_{t-1} \\
\tilde{q}_{t-2} \\
\vdots \\
\tilde{q}_{t-p} \\
\varepsilon_{t-1} \\
\varepsilon_{t-2} \\
\varepsilon_{t-q}
\end{bmatrix} + 
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
1 \\
0 \\
0
\end{bmatrix} \varepsilon_t
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The IRF can be obtained as

\[
\psi(j) = \mathbf{e}(\mathbf{F}^j \mathbf{G})
\]

where \( \mathbf{e} = [1 \ 0 \ \cdots \ 0]^\top \) is a selection vector.

To find the smallest value \( h \) such that \( \mathbf{e}(\mathbf{F}^h \mathbf{G}) = 0.5 \) we use interpolating splines.
Empirical Application: Data

▶ We analyse **US dollar bilateral exchange rates** for a group of developed countries: United Kingdom, Germany, France, Italy, Switzerland, Japan, South Africa, Mexico and the Euro Area (EMU).
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- Real exchange rate for the $i^{th}$ country ($q_{i,t}$) is computed from the nominal exchange rate ($s_{i,t}$, currency units for $1$) and the **CPIs** ($p_{i,t}$ and $p_{US,t}$) as

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- Data are obtained from the FRED database at a **quarterly frequency** over the period 1971:1-2013:3 (max 171 obs, min 59 obs).
Empirical Application: Half-life Computation
Without Outliers Detection

<table>
<thead>
<tr>
<th></th>
<th>( \hat{h} )</th>
<th>( \hat{c}_{\text{low}} )</th>
<th>( \hat{c}_{\text{upp}} )</th>
<th>((p, q))</th>
<th>AIC</th>
<th>J-B</th>
</tr>
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<td>UK</td>
<td>1.78</td>
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<td>2.26</td>
<td>4,3</td>
<td>-584.61</td>
<td>[0.0050]**</td>
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<td>4.82</td>
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<td>11.02</td>
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<td>3,3</td>
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<tr>
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<td>0.67</td>
<td>4.38</td>
<td>1,1</td>
<td>-208.00</td>
<td>[0.2370]</td>
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</tbody>
</table>

**Notes:** \( \hat{h} \) denotes the annualised half-life estimate, \( \hat{c}_{\text{low}} \) and \( \hat{c}_{\text{upp}} \) are the lower and upper endpoint of the bootstrapped confidence interval, \((p, q)\) denotes the ARMA order, \( \text{AIC} \) the Akaike Information Criterion and J-B the \( p \)-value of the Jarque-Bera test with ‘***’ and ‘*’ denoting rejection of the null of Normality at 1% and 5% significance level respectively.
### Empirical Application: Half-life Computation With Outliers Detection

<table>
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<tr>
<th></th>
<th>( \hat{h} )</th>
<th>( \hat{c}_{\text{low}} )</th>
<th>( \hat{c}_{\text{upp}} )</th>
<th>(p, q)</th>
<th>AOs</th>
<th>IOs</th>
<th>LSs</th>
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<td>1984(3)</td>
<td>1992(3)</td>
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<td>[0.0117]*</td>
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<td>4.82</td>
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<td>1985(1)</td>
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<td>5.07</td>
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<td>1979(4)</td>
<td>1971(1)</td>
<td>1998(4)</td>
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<td>1995(1)</td>
<td>1995(2)</td>
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<td>2000(4)</td>
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<td>2004(1)</td>
<td>-218.31</td>
</tr>
</tbody>
</table>
Empirical Application: Evidence

- For “problematic” countries, **modelling outliers** seems to drastically **reduce** the extent of the **PPP puzzle**.
Empirical Application: Evidence

▶ For “problematic” countries, modelling outliers seems to drastically reduce the extent of the PPP puzzle.

▶ For countries where the puzzle is less evident or absent, half-life estimates are not affected.
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Empirical Application: Evidence

- For “problematic” countries, **modelling outliers** seems to drastically **reduce** the extent of the PPP puzzle.

- For countries where the puzzle is less evident or absent, half-life estimates are not affected.

- Benefits of accounting for outliers are in any case evident in **tighter confidence intervals** and **restored normality**.

- **Four** outliers retained on average. The most recurring is the IO in the fourth quarter of 2008.
Testing for Non-linear Effects

- Part of the literature models \( \{ q_t \} \) using **non-linear models** (TAR, SETAR, ...) following the “bands of inaction” argument raised by Taylor (2001).
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- We test **with and without outliers removal** to see whether non-linearities (if any) can be due to unaccounted outliers or, conversely, outliers inclusion is masking non-linearities.

- We employ the **BDS statistic** (Brock et al., 1996) which is based on the concept of **correlation integral** and aims at measuring the frequency with which temporal patterns repeat over time.
### Testing for Non-linear Effects (cont’d)

<table>
<thead>
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<th>4</th>
<th>5</th>
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<td>0.005**</td>
<td>0.002**</td>
<td>0.000**</td>
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<tr>
<td></td>
<td>0.484</td>
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<td>0.313</td>
<td>0.004**</td>
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<td>0.960</td>
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<td>0.002**</td>
<td>0.285</td>
<td>0.689</td>
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<td>0.037*</td>
<td>0.006**</td>
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<td>0.001**</td>
<td>0.000**</td>
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<td>0.001**</td>
<td>0.000**</td>
<td>0.000**</td>
<td>0.000**</td>
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<td>0.757</td>
<td>0.407</td>
<td>0.711</td>
<td>0.332</td>
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<tr>
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<td>0.555</td>
<td>0.522</td>
<td>0.119</td>
<td>0.555</td>
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</tr>
</tbody>
</table>

**Notes:** ** and * denote presence of non-linear effects at 1% and 5% significance level.
Concluding Remarks

- We conjectured that existing half-life estimates are influenced by outlying observations.
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- We allowed the real exchange rate process to follow an **ARMA** dynamics contaminated by **AOs, IOs** and **LSs**.
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- We proposed a sequential **Dummy Saturation** approach combined with ML estimation to detect the outlying observations.

- When the PPP puzzle is rather pronounced, including outliers helps to reduce the half-life by a factor of 2 or 3.

- In any case, robust estimation allows to obtain **tighter CIs** and to restore normality.

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