Forecasting portfolio credit default rates

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1The opinions expressed in this presentation are those of the author and do not necessarily reflect views of the Bank of England.
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Expected loss forecast

Conclusions

References
Moody’s corporate issuer and default counts in 2008 and 2009.$^2$

<table>
<thead>
<tr>
<th>Grade</th>
<th>2008 Issuers</th>
<th>2008 Defaults</th>
<th>2009 Issuers</th>
<th>2009 Defaults</th>
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</thead>
<tbody>
<tr>
<td>Caa-C</td>
<td>421</td>
<td>63</td>
<td>528</td>
<td>182</td>
</tr>
<tr>
<td>B</td>
<td>1158</td>
<td>25</td>
<td>962</td>
<td>72</td>
</tr>
<tr>
<td>Ba</td>
<td>527</td>
<td>6</td>
<td>511</td>
<td>12</td>
</tr>
<tr>
<td>Baa</td>
<td>1025</td>
<td>5</td>
<td>1011</td>
<td>9</td>
</tr>
<tr>
<td>A</td>
<td>981</td>
<td>5</td>
<td>964</td>
<td>2</td>
</tr>
<tr>
<td>Aa</td>
<td>595</td>
<td>4</td>
<td>527</td>
<td>0</td>
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<tr>
<td>Aaa</td>
<td>145</td>
<td>0</td>
<td>136</td>
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<td>All</td>
<td>4852</td>
<td>108</td>
<td>4639</td>
<td>277</td>
</tr>
</tbody>
</table>

$^2$Source: Moody’s (2013)
### Forecast problem

Moody’s corporate issuer proportions and default rates in 2008 and issuer proportions in 2009. All numbers in %.

<table>
<thead>
<tr>
<th>Grade</th>
<th>2008</th>
<th></th>
<th>2009</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Issuers</td>
<td>Default rate</td>
<td>Issuers</td>
<td>Default rate</td>
</tr>
<tr>
<td>Caa-C</td>
<td>8.7</td>
<td>15.0</td>
<td>11.4</td>
<td>?</td>
</tr>
<tr>
<td>B</td>
<td>23.9</td>
<td>2.2</td>
<td>20.7</td>
<td>?</td>
</tr>
<tr>
<td>Ba</td>
<td>10.9</td>
<td>1.1</td>
<td>11.0</td>
<td>?</td>
</tr>
<tr>
<td>Baa</td>
<td>21.1</td>
<td>0.5</td>
<td>21.8</td>
<td>?</td>
</tr>
<tr>
<td>A</td>
<td>20.2</td>
<td>0.5</td>
<td>20.8</td>
<td>?</td>
</tr>
<tr>
<td>Aa</td>
<td>12.3</td>
<td>0.7</td>
<td>11.4</td>
<td>?</td>
</tr>
<tr>
<td>Aaa</td>
<td>3.0</td>
<td>0.0</td>
<td>2.9</td>
<td>?</td>
</tr>
<tr>
<td>All</td>
<td>100.0</td>
<td>2.2</td>
<td>100.0</td>
<td>?</td>
</tr>
</tbody>
</table>

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3Source: Moody’s (2013)

Dirk Tasche (PRA)
Introduction

Comparison of two approaches

Observed default rates (DR) for 2008 and 2009 and Total Probability (TP) and Kullback-Leibler (KL) forecasts for 2009. All numbers in %.

<table>
<thead>
<tr>
<th>Grade</th>
<th>DR 2008</th>
<th>TP 2009</th>
<th>KL 2009</th>
<th>DR 2009</th>
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</thead>
<tbody>
<tr>
<td>Caa-C</td>
<td>14.96</td>
<td>12.09</td>
<td>30.22</td>
<td>34.47</td>
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<td>B</td>
<td>2.16</td>
<td>3.24</td>
<td>9.53</td>
<td>7.48</td>
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<td>Ba</td>
<td>1.14</td>
<td>1.46</td>
<td>4.47</td>
<td>2.35</td>
</tr>
<tr>
<td>Baa</td>
<td>0.49</td>
<td>0.78</td>
<td>2.42</td>
<td>0.89</td>
</tr>
<tr>
<td>A</td>
<td>0.51</td>
<td>0.33</td>
<td>1.02</td>
<td>0.21</td>
</tr>
<tr>
<td>Aa</td>
<td>0.67</td>
<td>0.12</td>
<td>0.36</td>
<td>0.00</td>
</tr>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td>0.03</td>
<td>0.10</td>
<td>0.00</td>
</tr>
<tr>
<td>All</td>
<td>2.23</td>
<td>2.46</td>
<td>6.69</td>
<td>5.97</td>
</tr>
</tbody>
</table>

4 Default rates for 2008 were smoothed by quasi-moment matching (Tasche, 2013) before being used for the TP and KL forecasts.
Default rate forecasts are often based on
- regression on macroeconomic variables or
- assumptions on shared portfolio characteristics (e.g. with credit bureau data collections).

Drawbacks:
- Long time series of observations are required.
- Firm specific underwriting policies are not taken into account.

We investigate methods that
- allow for period-to-period forecasts and
- only rely on internal data.
Setting

- Formalise setting of slide 4. We only consider **binary classification problem**.

- **Known:**
  - Probability space $(\Omega, \mathcal{A}, P_0)$ (training set).
  - $\sigma$-field $\mathcal{C} \subset \mathcal{A}$ (covariates).
  - Event $A \in \mathcal{A}$, $A \notin \mathcal{C}$ (class of example).
  - Probability measure $P_1$ on $(\Omega, \mathcal{C})$ (test set without class labels).

- In rating example (slide 4):
  - $\mathcal{C}$ is information provided by rating grade.
  - $A$ means issuer’s default. Issuer’s default status is not known at the beginning of the year.
  - $P_0$ is known joint distribution of rating grades at beginning of 2008 and default status at end of 2008.
  - $P_1$ is known distribution of rating grades at beginning of 2009.
Problem

- Find **extension** $P_1^*$ of $P_1$ to $\sigma(\{A\} \cup C)$ such that we can compute $P_1^*[A]$ and $P_1^*[A|C]$.
- The extension should meaningfully incorporate features of $P_0$.
- In the rating example $P_1^*[A]$ is a forecast of the portfolio-wide 2009 default rate and $P_1^*[A|C]$ is a forecast of the grade-level default rates.

**Assumptions:**

- $P_0|_C$ has a density $f$ with respect to some measure $\mu$ on $(\Omega, C)$.
- Suppose $p_0 = P_0[A] \in (0, 1)$.

In the example:

- $\mu$ is the Laplace distribution on $\{Caa-C, \ldots, Aaa\}$ and $f$ is given by the rating frequencies.
- $p_0 = 2.2\%$. 
The Law of Total Probability approach

- **Classical case:** For $\mathcal{C}$-measurable partition $C_1, C_2, \ldots$ of $\Omega$

  $$P[A] = \sum_{k=1}^{\infty} P[A \mid C_k] P[C_k].$$

- **For classification problem:**
  - Replace $P[C_k]$ by $P_1[C_k]$ and $P[A \mid C_k]$ by $P_0[A \mid C_k]$.
  - In general, $P^*_1[B] = \mathbb{E}_1[P_0[B \mid C]]$, $B \in \mathcal{A}$ defines a probability measure on $(\Omega, \mathcal{A})$ if $P_1 \ll P_0 \mid C$.

- This gives column “TP 2009” on slide 5 (assuming that $P_0[A \mid C] = P^*_1[A \mid C]$).

- In the machine learning literature, this solution is called **covariate shift** approach ([Moreno-Torres et al., 2012](#)).
Class Density Ratios

- Since \( P_0 \mid_C \ll \mu \) we have \( \mu \)-densities \( f_A \) and \( f_{A^c} \) of \( P_0[\cdot \mid A] \mid_C \) and \( P_0[\cdot \mid A^c] \mid_C \).

- **Assumption:** \( f_{A^c} > 0 \).

- Define the **density ratio** \( \lambda_0 = \frac{f_A}{f_{A^c}} \).

- On \((\Omega, C, P_0)\) then we have

\[
f = p_0 f_A + (1 - p_0) f_{A^c}
\]

\[
P_0[A \mid C] = \frac{p_0 \lambda_0}{1 - p_0 + p_0 \lambda_0}.
\]

- Hence \( P_0 \mid_C \) is a mixture distribution. This suggests estimation of \( P^*_1[A] \) by a **mixture model** approach.
Example: Moody’s 2008 rating distributions
The Kullback-Leibler estimator

- Assume that $P_1$ has density $g > 0$ with respect to $\mu$. Minimise the Kullback-Leibler (KL) distance between $g$ and $p f_A + (1 - p) f_{Ac}$:

$$KL(p) = \int g \log \left( \frac{g}{p f_A + (1 - p) f_{Ac}} \right) d\mu$$

$$= E_1 \left[ \log(g/f_{Ac}) \right] - E_1 \left[ \log(p \lambda_0 + 1 - p) \right]. \quad (2)$$

- If $E_1$ is an empirical measure, minimising the KL distance gives a maximum likelihood estimator of $P_1^*[A]$.

- **First order condition** for minimum:

$$KL'(p) = 0 \iff E_1 \left[ \frac{\lambda_0 - 1}{1 - p + p \lambda_0} \right] = 0. \quad (3)$$

- A solution of (3) is called **KL estimator** of $P_1^*[A]$.
Exact fit for the KL estimator

- Suppose that $P_1[\lambda_0 = 1] < 1$. Then there is a unique solution $0 < p_1 < 1$ to (3) if and only if

$$E_1[\lambda_0] > 1 \quad \text{and} \quad E_1[1/\lambda_0] > 1. \quad (4)$$

- If there is a solution $0 < p_1 < 1$ to (3) then there is a probability measure $P_1^*$ on $\sigma(\{A\} \cup C)$ such that
  1) $P_1^*|_C = P_1$,
  2) $P_1^*[A] = p_1$, and
  3) $P_1^*[C | A] = \int_C \frac{g\lambda_0}{1-p_1+p_1\lambda_0} \, d\mu$ and $P_1^*[C | A^c] = \int_C \frac{g}{1-p_1+p_1\lambda_0} \, d\mu$ for $C \in C$.

- Property 1) is called exact fit.

- $P_1^*$ is the only probability measure on $\sigma(\{A\} \cup C)$ with 1) and density ratio $\lambda_0$. $P_1^*$ is called KL extension of $P_1$.

- The measure extension result still holds if $g$ is a density of $P_1$ with respect to some measure $\nu \neq \mu$. 
Comments

▶ In the multi-class case, there is no similarly simple condition like (4) for the existence of a solution to the first order equations for the KL minimisation.

▶ The criterion (4) seems to be satisfied most of the time.

▶ Is there another way to assess ex ante (before column “DR 2009” on slide 5 is observed) whether Total Probability or KL approach (or none of the two) is better?

▶ A partial response comes from studying prior probability shift (Moreno-Torres et al., 2012):
  ▶ In general, it holds that \( g_A = \frac{g \lambda_0}{1-p_1+p_1 \lambda_0} \neq f_A \) and
  \( g_{A^c} = \frac{g}{1-p_1+p_1 \lambda_0} \neq f_{A^c} \).
  ▶ Prior probability shift denotes special case \( g = q f_A + (1-q) f_{A^c} \).
  Then it follows that \( f_A = g_A \) and \( f_{A^c} = g_{A^c} \).
Example: Best vs. exact fit for Moody’s 2009 data

- **Observed vs. best fit 2009 unconditional rating distribution**
  - Probability vs. Rating categories (Caa-C to Aaa)
  - Observed and best fit distributions

- **Best fit vs. exact fit 2009 conditional rating distributions**
  - Probability vs. Rating categories (Caa-C to Aaa)
  - Best fit default, exact fit default, best fit survival, exact fit survival

Diagram showing the comparison between observed and best fit distributions for Moody’s 2009 data, with conditional probability distributions for different rating categories.
Forecasting as a measure extension problem

Prior probability shift

- Let $q \in (0, 1)$ and assume that $P_1$ is given by

$$\frac{dP_1}{d\mu} = g = q f_A + (1-q) f_{A^c}. \quad (5)$$

- Then $p_1 = q$ is the unique solution of (3) in $(0, 1)$.

- Moreover, it holds that

$$E_1[P_0[A|C]] - q = (p_0 - q) \frac{E_0[P_0[A|C] (1-P_0[A|C])]}{p_0 (1-p_0)}.$$

- For the regression of $1_A$ on $C$ under $P_0$ we have that

$$1 - R^2 = \frac{E_0[P_0[A|C] (1-P_0[A|C])]}{p_0 (1-p_0)}.$$

- Hence the Total Probability and KL estimates of $q$ are the less different the better the forecast of $A$ by $P_0[A|C]$ is on the training set.
Some thoughts

- Under assumption (5), the KL estimator is an unbiased estimator of the class probability.
- Hofer and Kreml (2013) analyse a credit dataset that seems to fulfil (5).
- On Moody’s data (Moody’s, 2013), KL performs worse than Total Probability on average.
- Clearly, if historical records are available a decision between KL and Total Probability should be based on time series analysis.
- For non-credit applications, sometimes a rationale based on causality can be helpful.
Causality in classification problems

- Classification problem: Infer class $Y$ of an observation based on covariates $X$.
- Fawcett and Flach (2005) distinguish two types of 'classification domains':
  - (i) $X \rightarrow Y$ where the class is causally dependent on the covariates $X$.
  - (ii) $Y \rightarrow X$ where different classes cause different outcomes of $X$.
- Fawcett and Flach (2005) describe two examples of (ii):
  - Infection status with regard to a disease and illness symptoms.
  - Manufacturing fault status and properties of the produced goods.
- (ii) is considered a justification of assumption (5).
- There is no clear causality in credit classification problems.
A prudent approach to probability of default quantification

Let $g > 0$ be a $\mu$-density of $P_1$ on $(\Omega, C)$. If (4) holds, $g$ has the following decomposition:

$$g = p_1 g_A + (1 - p_1) g_{A^c},$$

with $g_A$ and $g_{A^c}$ as on Slide 14.

Define $P_0^*$ on $(\Omega, \sigma(\{A\} \cup C))$ by

$$\frac{dP_0^*|_C}{d\mu} = p_0 g_A + (1 - p_0) g_{A^c}$$


Moreover, with $R_*^2 = 1 - \frac{E_0^*[P_0[A|C] (1 - P_0[A|C])]}{p_0 (1-p_0)}$ we obtain

$$p_1 R_*^2 + (1 - R_*^2) p_0 = E_1 [P_0[A|C]].$$
When to deploy the Kullback-Leibler estimator?

A prudent approach to probability of default quantification II

► Hence, it holds that

\[
\begin{align*}
  p_0 \leq p_1 & \Rightarrow p_0 \leq E_1[P_0[A \mid C]] \leq p_1, \\
  p_0 \geq p_1 & \Rightarrow p_0 \geq E_1[P_0[A \mid C]] \geq p_1.
\end{align*}
\]

► This observation suggests the following prudent estimation method for \( P_1^*[A] \):

► Determine \( p_1 \) according to (3).
► If \( p_0 \leq p_1 \) choose \( P_1^*[A] = p_1 \).
► If \( p_0 \geq p_1 \) choose \( P_1^*[A] = E_1[P_0[A \mid C]] \).

► With this approach, there is an incentive to optimise the accuracy of the conditional probabilities of default \( P_0[A \mid C] \) (see slide 16).
The problem

- For sake of illustration, suppose that on slide 4
  - ’issuers’ is replaced by ’% of exposure’ and
  - ’default rate’ is replaced by ’loss rate’.
- Are then the Total Probability and KL forecast methods applicable?
  - Clearly, ’yes’ for Total Probability because then it is simply assumed that the grade-level loss rates in 2009 are the same as the ones observed in 2008.
  - Less clear for KL because its derivation is heavily based on probability calculus.
- Two interpretations of model (slide 7):
  - Individual: $p_0$ is one issuer’s probability of default.
  - Collective: $p_0$ is the proportion of all issuers that default.
The finite measure approach

▶ With the collective interpretation of the model (slide 7), it is applicable to the ’exposure – loss rate’ problem:
  ▶ Probabilities are understood as proportions.
  ▶ Probability calculus is calculus of proportions in terms of finite measures.
  ▶ Conditional probabilities are relative proportions.
  ▶ Bayes’ formula is a re-engineering tool without interpretation of causality.
▶ Limited practical application to a retail credit loss estimation problem was inconclusive with regard to suitability of approach.
▶ Suggestion to use the prudent approach described before.
We have studied the problem of forecasting prior class probabilities in the presence of a changed covariates distribution. Straight-forward forecasts based on Law of Total Probability (TP) may underestimate the amount of change of the prior probabilities. Alternative simple finite mixture model approach is promising:

- Deploying the Kullback-Leibler (KL) estimator provides exact fit of the changed covariates distribution.
- In the binary classification case, the KL estimator always forecasts more change of the prior probabilities than the TP.
- In credit risk, this can be used to obtain conservative estimates of probability of default and expected loss.

This approach may reduce dependence on macroeconomic data and assumptions of similarities of portfolios.

Loss provisioning and stress testing are potential applications.


