Delta and Gamma hedging of mortality and interest-rate risk

Elisa Luciano\textsuperscript{1}, Luca Regis\textsuperscript{2}, Elena Vigna\textsuperscript{3}

\textsuperscript{1}University of Torino, Collegio Carlo Alberto, ICER
\textsuperscript{2}University of Torino
\textsuperscript{3}University of Torino, Collegio Carlo Alberto, CERP

Longevity VII Conference
Frankfurt, September 2011
Outline

1. Introduction
2. Main Assumptions
3. Change of measure
4. Delta-Gamma Exposure and Hedging of reserves
5. Examples
6. Conclusions
GENERAL PROBLEM

Price and hedge life contracts in the presence of systematic mortality risk

- starting from a continuous-time description of stochastic mortality which can be handled analytically and is RELIABLE under the historical measure
- WITHOUT IMPOSING no arbitrage
- with a manageable, PARSIMONIOUS model
- which integrates INTEREST-RATE RISK, still in a PARSIMONIOUS way
SPECIFIC AIM

Obtain in closed-form Delta and Gamma sensitivities and hedges for the reserves.

WHY? in order to

- get an intuitive representation of mortality risk as difference between forecasted and actual mortality intensity
- get an hedge easy to compute and monitor
- easily incorporate budget constraints (linear systems)
- include Delta and Gamma coverage of interest-rate risk
- foster liquidity and develop a secondary market for longevity bonds
SOLUTION

- for each generation, we use an affine stochastic intensity which has the Gompertz law as non-stochastic counterpart (under the historical measure)
- we prove that there exist measure changes which permit to adopt an Heath Jarrow and Morton (HJM) –like framework for pricing/reserving, without imposing no arbitrage
- we characterize prices/reserves and Greeks under such measures
- we solve with both riskless and risky Hull–White interest rates
WHAT ABOUT APPLICATIONS?

As an example we

- calibrate mortality to UK insured males (historical measure)
- calibrate interest rate to the UK Government–bond market (risk neutral measure)
- compute sensitivity and hedges of pure endowments
MORTALITY RISK under historical measure $\mathbb{P}$

- Death arrival is modelled as the first jump time of a doubly stochastic process.

- Let $\lambda_x(t)$ be the mortality intensity of generation $x$ at time $t$. We assume that

$$d\lambda_x(t) = a(t, \lambda_x(t))dt + \sigma(t, \lambda_x(t))dW_x(t)$$  \hspace{1cm} (1)

with $a$ and $\sigma$ affine in $\lambda_x$ (Assumptions 1 and 2)

- Let $S_x(t, T)$ be the probability for a head of generation $x$, alive at time $t$, to survive from $t$ to $T$. Then

$$S_x(t, T) = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ solve appropriate Riccati equations.
MORTALITY RISK II

The forward death intensity is defined as

\[ f_x(t, T) = -\frac{\partial}{\partial T} \ln(S_x(t, T)). \]

It represents the probability of dying right after \( T \), as forecasted at \( t \). It is the "best forecast" of the actual one, \( \lambda \), since

\[ f_x(T, T) = \lambda_x(T) \]
TWO SPECIAL CASES

- Ornstein-Uhlenbeck (OU) process without mean reversion
  \[ d\lambda_x(t) = a\lambda_x(t)dt + \sigma dW_x(t) \]

- Feller (FEL) process without mean reversion
  \[ d\lambda_x(t) = a\lambda_x(t)dt + \sigma \sqrt{\lambda_x(t)}dW_x(t) \]
WHY?

- conditions for $\lambda_x$ to be positive and for the survival probability to be decreasing in $T$ are specified and/or verified
- both have the Gompertz law as expectation
- parsimonious models, good for analytical tractability
- although they are very simple, they proved to fit accurately historical and projected mortality tables (Luciano and Vigna, 2008; Luciano, Spreeuw and Vigna, 2008), better than their mean reverting counterparts (*).
- all in all, the requirements for a good mortality model listed by Cairns, Blake and Dowd (2006) seem to be satisfied

(*) for four generations from 1885 to 1945 the m.s.e. with respect to Human Mortality Database and IML92 data range from 0.00012 to 0.00085.
Let $F(t,T)$ be the time-$t$ forward interest rate for maturity $T$, so that $B(t,T) = \exp\left(-\int_t^T F(t,u)du\right)$.

We assume that

$$dF(t,T) = A(t,T)dt + \Sigma(t,T)dW_F(t)$$

with $W_F$ independent of all $W_x$ (Assumption 3)
Let the SYSTEMATIC MORTALITY RISK premium be

\[ \theta_x(t) := \frac{p(t) + q(t)\lambda_x(t)}{\sigma(t, \lambda_x(t))} \]

with \(p(t)\) and \(q(t)\) continuous functions of time (Assumption 4). There exists an equivalent measure \(\mathbb{Q}\) under which \(\lambda\) is still affine

\[ d\lambda_x(t) = [a(t, \lambda_x(t)) + p(t) + q(t)\lambda_x(t)] dt + \sigma(t, \lambda_x(t))dW'_x. \]

For OU and FEL we choose \(p = 0\) and \(q \in \mathbb{R}\) (constant risk premium), so that the mortality intensity is still OU and FEL.

This implies that \(\mathbb{Q}\) is not only EQUIVALENT, but also RISK NEUTRAL, that is arbitrages are ruled out (Theorem 1).
We assume no risk premium for the IDIOSYNCRATIC MORTALITY RISK (Assumption 5)

As customary, we assume that no arbitrage holds in the FINANCIAL market. For simplicity, we let the market be complete (Assumption 6). Then

\[ dF(t, T) = A'(t, T)dt + \Sigma(t, T)dW'_F(t) \]

where \( A' \) satisfies the HJM relationship:

\[ A'(t, T) = \Sigma(t, T) \int_t^T \Sigma(t, u)du \]
Consider a pure endowment (Arrow Debreu security) with expiration $T$, on an individual of generation $x$. Its price – or fair value of the obligation or reserve – is

$$P_x(t, T) = S_x(t, T)B(t, T) = \exp \left( - \int_t^T [f_x(t, u) + F(t, u)] du \right)$$

where $f_x$ and $F$ are measure-changed.

Before $t$, $P_x$ is stochastic: $\tilde{P} = \tilde{S}_x(t, T)\tilde{B}(t, T)$. 
MORTALITY RISK EXPOSURE

Under Assumption 4

\[
\tilde{S}(t, T) = \frac{S(0, T)}{S(0, t)} \exp \left[ -\int_t^T \int_0^z \left[ v(u, T)du + w(u, T)dW'(u) \right] dz \right]
\]

In the OU case

\[
\tilde{S}(t, T) = \frac{S(0, T)}{S(0, t)} \exp \left[ -X(t, T)I(t) - Y(t, T) \right]
\]

where

\[
a' := a + q
\]

\[
X(t, T) := \frac{\exp(a'(T - t)) - 1}{a'}
\]

\[
Y(t, T) := -\sigma^2 [1 - e^{-2a't}]X(t, T)^2 / (4a')
\]

and \(I(t)\) is the mortality risk factor or forecast error:

\[
I(t) := \tilde{\lambda}(t) - f(0, t)
\]

Notice that the hedge depends on the risk premium \(q\), but the risk factor is independent of the horizon of the survival probability, \(T\).
SENSITIVITY to mortality risk

If \( F(t, T) = 0 \) for all \( t \) and \( T \), then \( S = P \) and the sensitivity of the reserve to the mortality risk factor is

\[
dP = dS = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial I} dI + \frac{1}{2} \frac{\partial^2 S}{\partial I^2} (dI)^2
\]

In the OU case

\[
\Delta^M = \frac{\partial S}{\partial I} = -SX \leq 0
\]

\[
\Gamma^M = \frac{\partial^2 S}{\partial I^2} = SX^2 \geq 0
\]
If $F(t,T)$ satisfies Assumption 3 and is Hull-White under $\mathbb{Q}$, namely

$$\Sigma(t,T) = \Sigma \exp(-g(T-t)), \quad \Sigma > 0, g > 0$$

then

$$\tilde{B}(t,T) = \frac{B(0,T)}{B(0,t)} \exp \left[ -\tilde{X}(t,T)K(t) - \tilde{Y}(t,T) \right]$$

where $\tilde{X}$ and $\tilde{Y}$ are defined similarly to $X$ and $Y$ of the mortality risk and $K(t)$ is the financial risk factor or forecast error, measured by the difference between the short and forward rate:

$$K(t) := \tilde{r}(t) - F(0,t)$$
SENSITIVITY to mortality and financial risk

If $F(t, T)$ is not identically null, $P = SB$ and

$$dP = BdS + SdB$$

For fixed $t$

$$dP = B \left[ \Delta^M dI + \frac{1}{2} \Gamma^M (dI)^2 \right] + S \left[ \Delta^F dK + \frac{1}{2} \Gamma^F (dK)^2 \right]$$

where

$$\Delta^F = \frac{\partial B}{\partial K} = -B\overline{X} \leq 0$$

$$\Gamma^F = \frac{\partial^2 B}{\partial K^2} = B\overline{X}^2 \geq 0$$
DELTA GAMMA HEDGING

Given $n$ endowments, we can hedge them using $m$ additional hedging contracts with different expiry.

- we build the portfolio

$$\Pi(t) = nP + \sum_{i=1}^{m} n_i P(t, T_i)$$

- Then, the numbers of hedging contracts $n_i$ can be chosen so as to make the portfolio deltas and gammas null (linear systems):

$$\Delta^M_\Pi = \Gamma^M_\Pi = \Delta^F_\Pi = \Gamma^F_\Pi = 0$$

- $n_i < 0$ means a net sale of pure endowments, $n_i > 0$ a net purchase of longevity bonds
- a self-financing portfolio requires $\Pi(0) = 0$
- can be extended to other insurance policies/assets
EXAMPLE

- Take an insurance company which sold $n$ pure endowments with maturity $T$, i.e. a portfolio short $n$ contracts with value $P(0, T)$.
- They can fix two tenors $T_1$ and $T_2$ and choose $n_1, n_2$ so that the portfolio made up of $n, n_1, n_2$ endowments/longevity bonds is Delta and Gamma hedged.
- Or they can choose $n, n_1, n_2$ so that it is self financed and Delta and Gamma hedged.
CALIBRATED EXAMPLE

- we calibrate OU intensity to the survival probabilities of 65-years old UK males using insured data (IML tables), i.e. under the $\mathbb{P}$ measure. The ML estimates are $a = 10.94\%$, $\sigma = 0.07\%$

- we switch from $\mathbb{P}$ to $\mathbb{Q}$ using Assumption 4, which makes the intensity still OU under $\mathbb{Q}$. In this application we select $q = 0$

- we calibrate Hull-White interest rates to UK Government-bond quotes, i.e. under the $\mathbb{Q}$ measure: $g = 2.72\%$, $\Sigma = 0.65\%$
CALIBRATED EXAMPLE II

For a pure endowment with maturity $T$, we obtain

<table>
<thead>
<tr>
<th>Maturity $T$</th>
<th>$\Delta^M$</th>
<th>$\Gamma^M$</th>
<th>$\Delta^F$</th>
<th>$\Gamma^F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-6.274$</td>
<td>$41.757$</td>
<td>$-4.299$</td>
<td>$20.096$</td>
</tr>
<tr>
<td>15</td>
<td>$-27.192$</td>
<td>$1034.084$</td>
<td>$-6.96$</td>
<td>$85.721$</td>
</tr>
<tr>
<td>25</td>
<td>$-41.771$</td>
<td>$5501.92$</td>
<td>$-4.56$</td>
<td>$82.713$</td>
</tr>
</tbody>
</table>

Notice that $|\Delta^M| > |\Delta^F|$ and $\Gamma^M > \Gamma^F$.

However, under realistic hypothesis on the shocks – or risk factor realizations – $\Delta I$ and $\Delta K$ the effect of mortality and financial risk have the same order of magnitude, i.e.

$$\Delta^M \Delta I \simeq \Delta^F \Delta K \quad \text{and} \quad \Delta^M \Delta I + \frac{1}{2} \Gamma^M \Delta I^2 \simeq \Delta^F \Delta K + \frac{1}{2} \Gamma^F \Delta K^2$$

Take for instance $T = 25$, $\Delta I = -5$ bp, $\Delta K = -50$ bp. Then,

$$\Delta^M \Delta I = 0.0209 \quad \Delta^F \Delta K = 0.0228$$
To finish, suppose an insurance company sold a pure endowment expiring in 15 years. It can Delta and Gamma hedge its reserve using pure endowments/longevity bonds, as follows.

**Mortality risk**

1. purchase 1.1 and 0.26 longevity bonds expiring in 10 and 20 years; cost of the hedge: 0.37
2. purchase 0.48 and 0.60 longevity bonds expiring in 10 and 20 years, issue 0.1 pure endowments with maturity 30 years; this is a self financing strategy

**Mortality and financial risk**

1. take also a short position in 0.6 zero coupon bonds with maturity 5 and 0.1 long positions in zcbs with maturity 20 years; cost of the hedge: -0.14
CONCLUSIONS

The paper introduces a hedging tool for mortality and interest rate risk that:

- is easy-to-handle
- is based on a reliable mortality model
- is based on a standard interest-rate model
- leads to solving linear systems
- is very well-known and widely used when restricted to financial risk only
- last but not least, it can be extended to other insurance contracts (death assurances, annuities...) and mortality derivatives
EXTENSIONS

- explicit treatment of market incompleteness by recognizing the correspondence between the measure selection and the hedging criterion (see He and Pearson, 1991)
- dynamic assessment of hedge effectiveness (hedging error, as in option pricing)
- two population extension in order to include basis risk (Dahl et al., 2008, Cairns et al. 2011a, 2011b)
REFERENCES

- Cairns, A.J.G., 2011b, Robust hedging of longevity risk, working paper, Heriot-Watt University and Longevity VII.
Theorem

Let $\lambda$ be a purely diffusive process which satisfies Assumption 4. Let its forward intensity under $Q$ be

$$df(t, T) = v(t, T)dt + w(t, T)dW'(t).$$

Then, the HJM condition

$$v(t, T) = w(t, T) \int_t^T w(t, s)ds$$

is satisfied if and only if:

$$\frac{\partial m(t, T)}{\partial T} = n(t, t) \frac{\partial n(t, T)}{\partial T}$$

where $m(\cdot)$ and $n(\cdot)$ are the drift and diffusion of $S(t, T)$. This condition is satisfied by the OU and the FEL processes with $p = 0$ and $q$ constant.