Forecasting mortality using statistical moments

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Max-Planck Odense Center on the Biodemography of Aging
Institute of Public Health
University of Southern Denmark

International Longevity Risk and Capital Markets Solutions Conference
Taipei, Taiwan

September 21, 2017
Which indicators have been used to forecast mortality?

Life table indicators:

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Cairns et al. (2006), Debon et al. (2008), Cairns et al. (2009), King and Soneji (2011),...
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Brass (1971), Scherbov and Ediev (2016), ...
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Oeppen (2008), Bergeron-Boucher et al. (2017), Basellini and Camarda (forthcoming), ...
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Torri and Vaupel (2012), Raftery et al. (2013), Pascariu et al. (forthcoming 2018), ...
Mortality evolution: $q_x$
Mortality evolution: $d_x$
What defines a distribution?

A distribution is defined by the totality of its moments.

**Moment**: Quantitative measure of the shape of a set of points or probability density → the expected value of the $n$-th power of a random variable.

\[
M(n) = \int_a^\omega (x - c)^n f(x)dx, \quad \text{where} \quad n = 0, 1, 2 \ldots
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for example:

\[
M(0) = 1 \quad M(1) = \text{Mean} \quad M(2) = \text{Variance} \quad M(3) = \text{Skewness} \quad M(4) = \text{Kurtosis} \quad \ldots
\]
Problem!

All the moments up to infinity are required in order to uniquely identify a density function.

The problem of reconstructing a pdf from a limited number of moments is known as the finite moment problem (Hausdorff 1921).
Maximum entropy method (*MaxEnt*)

**Objective:** reconstruct a distribution given a number of known moments that satisfy the maximum entropy condition (Mead & Papanicolaou, 1984).

Construct specific sequences of functions $f_N(x)$ which eventually converge to the true distribution $f(x)$ as $N$ approaches infinity

$$ M(n) = \int_a^\omega (x - c)^n f_N(x) \, dx, \quad n = 0, 1, 2 \ldots, N. $$
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$$M(n) = \int_{a}^{\omega} (x - c)^n f_N(x) \, dx, \quad n = 0, 1, 2 \ldots, N.$$ 

**Strategy for finding the local maxima** - the method of Lagrange multipliers:

$$\mathcal{L} = H + \sum_{n=0}^{N} \lambda_n \left[ \hat{M}(n) - M(n) \right].$$
**Entropy**: the amount of information, $I(x)$, produced by a probabilistic stochastic source of data, $x$ (Shannon, 1951):

$$H = E[I(x)] = E[- \log_b f(x)] = - \int_a^\omega f(x) \log_b f(x) \, dx$$

The entropy is measured in *bits*, *nats* and *bann*.

Zero entropy $\implies$ Certain event  
Maximum entropy $\implies$ Uncertain event. We have no way of learning about the outcome unless we see it!
MaxEnt density reconstruction (USA, 1990, Male population)
MaxEnt density reconstruction (USA, 1990, Male population)

Estimated vs. Observed density for different sample sizes:

- Density (N = 2): 77.52%
- Density (N = 3): 90.46%
- Density (N = 5): 92.87%
- Density (N = 7): 97.36%

Age (x) vs. f(x) for observed and estimated densities.
New approach to mortality forecasting

1. Extrapolate the observed raw/central moments of the observed distribution of deaths ($D_x$ or life table $d_x$) using multivariate time-series models: VAR, VAR-X, VARMA ...

2. Estimate the future distribution using $MaxEnt$ method;

3. Convert the estimated $f_x$ into death probabilities $q_x$ either by working a life table starting from $d_x$ or by employing a Gauss-Newton algorithm (for higher accuracy).
Forecasting statistical moments

Forecast of central moments of the distribution of deaths together with 95% prediction intervals, using a VAR(1) model

Distributions of deaths forecast ($D_x$)

Observed empirical distribution of deaths

Data source: www.mortality.org: England & Wales, Male population
Distributions of deaths forecast ($D_x$)

**Observed empirical distribution of deaths**

Data source: www.mortality.org: England & Wales, Male population
Distributions of deaths forecast \((D_x)\)

**Observed and forecast of the distribution using the first 7 moments**

Data source: www.mortality.org: England & Wales, Male population
Distributions of deaths forecast ($D_x$)

Observed and forecast of the distribution using the first 7 moments

<table>
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<tr>
<th>Year</th>
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<th>Skewness</th>
<th>Kurtosis</th>
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<tbody>
<tr>
<td>1980</td>
<td>68.46</td>
<td>264.46</td>
<td>-1.87</td>
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<tr>
<td>2013</td>
<td>74.86</td>
<td>250.44</td>
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<td>2030</td>
<td>77.19</td>
<td>256.92</td>
<td>-1.46</td>
<td>6.15</td>
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<td>2050</td>
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Data source: www.mortality.org: England & Wales, Male population
Mortality rates forecast ($q_x$)

Data: 
- Forecast
- Observed

Data source: www.mortality.org: England & Wales, Male population

Pascariu, Lenart & Canudas-Romo
MaxEnt $\rightarrow dx$ forecast
September 21, 2017
The source of the longevity risk has become identifiable;
Conclusion & Discussion

- The source of the longevity risk has become identifiable;
- No direct/indirect assumption of constant changes in mortality;
The source of the longevity risk has become identifiable;

No direct/indirect assumption of constant changes in mortality;

Ability to included exogenous variables in the time series analysis. (information on smoking or obesity, trends in life expectancy, modal age at death, etc.)
mpascariu@health.sdu.dk