Modeling Stochastic Mortality for Joint Lives through Subordinators

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Introduction

- We propose a novel approach to model mortality of dependent lives.
- Stochastic Mortality - We model the hazard rate process of an individual through a time changed Brownian motion, and introduce the dependence through dependent subordinators.
- Define the death time as a stopping time of the hazard rate process.

Review of Existing Models

- Copula-based joint life models.
  - Use Copula function to describe the correlation of the survival rate of the couple (Frees et al., 1996).
  - Mixed frailty copula (Carroere, 2000).
  - Conditional law of mortality through copula (Spreeuw, 2006).
  - Archimedian copula (Luciano et al., 2007).
- Stochastic mortality.
  - CIR process for mortality rate (Lorenzo et al., 2006).
  - Cox process that allows "jumps" on death arrival (Luciano et al., 2007).

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Our Model

- We focus on Stochastic Mortality.
- Use time changed Brownian Motion with correlated subordinators to model hazard rate process.
- "Internal clock" v.s. calendar time.
- Dependence through the subordinators.

Our Model

- Conceptually, we have very flexible assumption.
  - Our model allows the non-monotonicity of the hazard rate process.
  - Allows the association level between joint lives to be changed with time, which captures the fact that individuals' internal characteristics could play an increasingly more important role in determining the probability of death as they age.
  - Allows jumps in the hazard rate process.
- Empirically, we exploit a famous Canadian insurance data set.

Our Model

- Use time changed Brownian Motion with correlated subordinators to model hazard rate process.
- Subordinator: "Internal clock"
- Dependence of the mortality processes is modeled through the dependence of the subordinators.
- A common and an idiosyncratic components.
- Introduce a common time changing factor which reflects the dependence within each couple. By including this common factor in the “internal clock” of both members, we introduce dependence into their mortality processes.

Our Model

- For individual m, let
  - $X^m = (X_t^m)_{t \geq 0}$ denote a "base" stochastic process, and
  - $G^m = (G_s^m)_{s \geq 0}$ be a non–negative, non–decreasing RCLL stochastic process with $\lim_{s \to \infty} G_s^m = \infty$.
- The death time is defined as the stopping time $t^m = inf\{t | G_t^m \geq t^{*m}\}$, with $t^{*m} = inf\{X_t^m \geq 0\}$, $m = \{M, F\}$.

Our Model

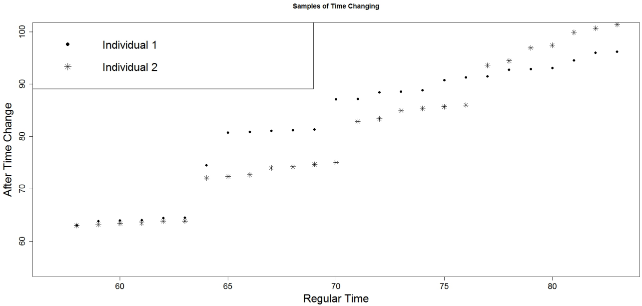
- Consider the male partner ($m = M$) and the female partner ($m = F$).
  - Denote X_t^M and X_t^F as the “base” stochastic processes for the male and the female, and G_t^M and G_t^F as their time changing respectively.
  - Introduce G_t as the common time changing factor, and H^M, H^F as the unique time changing factors for the male and the female. Here, G_t, H_t^M, H_t^F are all non–negative, non–decreasing stochastic processes, with G_t, H_t^M, H^{tF}, X_t^M, X_t^F being mutually independent processes.
  - Let $G_t^M = \alpha^M G_t + (1 - \alpha^M) H_t^M$, and $G_t^F = \alpha^F G_t + (1 - \alpha^F) H_t^F$, with $0 \leq \alpha \leq 1^M$ and $0 \leq \alpha \leq 1^M$.

Our Model

- α^M and α^F model the dependence level between a couple. Two mortality processes are completely independent if $\alpha^M = 0$ and $\alpha^F = 0$, and reach the highest dependence level if $\alpha^M = 1$ and $\alpha^F = 1$.
- α^M and α^F need not to be constants. α^M and α^F can be modeled as functions of time, i.e. $\alpha^M = \alpha^M(t)$ and $\alpha^F = \alpha^F(t)$. In our model, $\alpha^M(t)$ and $\alpha^F(t)$ are built as deterministic functions of time.

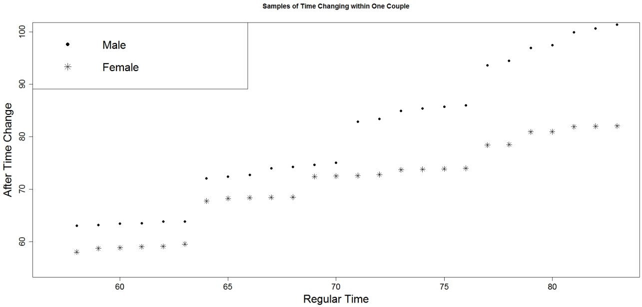
Our Model

- Samples of Subordinator



Our Model

- Samples of Subordinators of a pair



Our Model

- We find that NIG (Normal Inverse Gaussian) process can well describe stochastic mortality.
- The subordinators take the form of

$$\begin{aligned} G_t &= IG(\ t\ ,\ b\ ) \\ G_t^{M0} &= IG(\ \frac{1-\sqrt{\alpha^M}}{\sqrt{1-\alpha^M}}t\ ,\ \frac{b\times\sqrt{1-\alpha^M}}{\sqrt{\alpha^M}}\ ) \\ G_t^{F0} &= IG(\ \frac{1-\sqrt{\alpha^F}}{\sqrt{1-\alpha^F}}t\ ,\ \frac{b\times\sqrt{1-\alpha^F}}{\sqrt{\alpha^F}}\ ) \end{aligned} \quad (1)$$

, and the Brownian motion takes the form of

$$\begin{aligned} \beta^M &= \sqrt{\alpha^{M^2} - b^2/(\alpha^M \sigma^{M^2})} \\ \beta^F &= \sqrt{\alpha^{F^2} - b^2/(\alpha^F \sigma^{F^2})}\ . \end{aligned} \quad (2)$$

Data

- Source: A famous Canatian insurance data set[1].
- Taking into consideration the mortality changing between generations, the impact of age difference, and also the sample size, we select samples with the male and female both born between 1910 and 1925 and whose age differences are not greater than 5. This narrow down to a subset of 7,270 pairs of observations.
- However, the same method can be applied to any other generations, age differences, and to same sex marriage.

[1]We wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and EmilianoValdez, for allowing use of the data in this paper." The Society of Actuaries was the one who purchasedthis data and must therefore be duly acknowledged

Data

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Table: Summary of Birth Years (Female by Male)

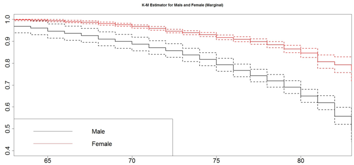
Canadian Insurance Data Set

		Year of Birth (F)															
		1910	1911	1912	1913	1914	1915	1916	1917	1918	1919	1920	1921	1922	1923	1924	1925
Year of Birth (M)	1910	13	11	14	9	20	15	8	0	0	0	0	0	0	0	0	0
	1911	12	19	25	20	26	24	14	11	0	0	0	0	0	0	0	0
	1912	4	16	18	34	26	40	23	23	10	0	0	0	0	0	0	0
	1913	10	12	23	27	36	56	37	56	26	13	0	0	0	0	0	0
	1914	1	15	6	23	45	48	52	51	59	56	22	0	0	0	0	0
	1915	1	6	19	19	40	66	84	60	64	67	74	27	0	0	0	0
	1916	0	2	14	10	44	47	71	51	76	74	68	56	42	0	0	0
	1917	0	0	0	14	15	25	44	72	76	86	83	76	47	30	0	0
	1918	0	0	0	1	10	16	39	57	68	77	112	104	71	61	38	0
	1919	0	0	0	0	5	18	28	31	36	64	84	116	76	95	71	26
	1920	0	0	0	0	0	6	17	29	51	85	118	136	105	96	101	83
	1921	0	0	0	0	0	0	10	15	26	35	83	114	128	89	119	101
	1922	0	0	0	0	0	0	0	7	15	28	55	78	110	129	87	99
	1923	0	0	0	0	0	0	0	0	7	14	34	50	49	98	105	107
	1924	0	0	0	0	0	0	0	0	0	11	23	36	27	60	91	96
	1925	0	0	0	0	0	0	0	0	0	0	9	24	22	39	48	102

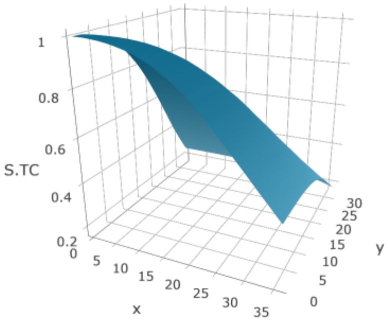
Data

- Kaplan-Meier Estimation of Marginal Survival Probability

Age	Male	Female
63	0.968	0.998
64	0.96	0.996
65	0.946	0.994
66	0.936	0.989
67	0.926	0.986
68	0.91	0.98
69	0.898	0.975
70	0.886	0.967
71	0.87	0.959
72	0.856	0.946
73	0.837	0.938
74	0.817	0.93
75	0.792	0.917
76	0.766	0.908
77	0.742	0.898
78	0.718	0.884
79	0.69	0.864
80	0.65	0.846
81	0.618	0.806
82	0.558	0.791
83	0.492	0.767



Results



Results

Table: Parameter Estimation - Fixed α^M and α^F

	α^M	α^F	b	σ^M	σ^F
Estimated	0.673	0.663	0.193	0.660	0.698

$\alpha^M(t)$ and $\alpha^F(t)$ as functions of time

	α^M	α^F	b	σ^M	σ^F	C_M	C_F
Estimated	0.673	0.663	0.193	0.660	0.698	0.990	1.000

Table: L^1 Distance - Fixed α^M and α^F

	Mean	Median	Std.
Average	0.035	0.032	0.019

$\alpha^M(t)$ and $\alpha^F(t)$ as functions of time

	Mean	Median	Std.
Average	0.028	0.024	0.017

Discussion

- Easy to follow and easy to implement.
- Allows the association level between joint lives to be changed with time.
- Allows the non-monotonicity of the hazard rate process.

Discussion

- Implications: risk and insurance practice.
- Life insurance and annuity pricing - more accurate with joint life model;
- Insurance pricing - can be extended to other relationships (e.g., owner and pet), even including non-health related relationships (e.g. auto and house); multiple household members;
- Guide household financial management and retirement planning

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Thank you for your listening!

Questions