Dynamic Longevity Hedging in the Presence of Population Basis Risk: A Feasibility Analysis from Technical and Economic Perspectives

Kenneth Q. Zhou and Johnny S.-H. Li

University of Waterloo

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Outline

Figure 1: The outline of the proposed dynamic hedging strategy.
Overview

1. Dynamic Longevity Hedging
   - Constructing the dynamic hedging strategy
   - A two-population example

2. Residual Risk Transferring
   - Constructing the customized surplus swap
   - A multi-population example
Model and Approximation

The augmented common factor model (Li and Lee, 2005):

$$\ln(m_{x,t}^{(i)}) = a_x^{(i)} + B_x K_{t+1} + b_x^{(i)} k_{t+1}^{(i)} + e_x, \quad i = 1, 2, \ldots$$

where

- $K_t$ follows a random walk:
  $$K_t = c + K_{t-1} + \epsilon_t;$$

- $k_t^{(i)}$ follows an AR(1) process:
  $$k_t^{(i)} = c^{(i)} + \phi^{(i)} k_{t-1}^{(i)} + \epsilon_t^{(i)}.$$
The approximation method - an extension of the work of Cairns (2011):

\[
f_{x,t}^{(i)}(T, K_t, k_t^{(i)}) = \Phi^{-1}(p_{x,t}^{(i)}(T, K_t, k_t^{(i)}))
\]

\[
\approx D_{x,t,0}^{(i)}(T) + D_{x,t,1}^{(i)}(T) \cdot (K_t - \hat{K}_t)
+ D_{x,t,2}^{(i)}(T) \cdot (k_t^{(i)} - \hat{k}_t^{(i)})
\]

where

- \(\Phi^{-1}\) is the probit function;
- \(p_{x,t}^{(i)}(T, K_t, k_t^{(i)})\) is the time-\(t\) spot survival probability for \(T\) years;
- \(\hat{K}_t = E(K_t|K_0)\);
- \(\hat{k}_t^{(i)} = E(k_t^{(i)}|k_0^{(i)})\);

- \(D_{x,t,0}^{(i)}(T) = f_{x,t}^{(i)}(T, \hat{K}_t, \hat{k}_t^{(i)})\);
- \(D_{x,t,1}^{(i)}(T) = \left. \frac{\partial f_{x,t}^{(i)}(T,K_t,\hat{k}_t^{(i)})}{\partial K_t} \right|_{K_t=\hat{K}_t}\);
- \(D_{x,t,2}^{(i)}(T) = \left. \frac{\partial f_{x,t}^{(i)}(T,\hat{K}_t,k_t^{(i)})}{\partial k_t^{(i)}} \right|_{k_t^{(i)}=\hat{k}_t^{(i)}}.\)
To construct the dynamic hedging strategy, the following quantities are calculated using the approximation method:

- The time-\(t\) present value of the pension liabilities.
- The time-\(t\) present value of the hedging instruments.
- The first derivative with respect to \(K_t\) of the time-\(t\) present value of the pension liabilities, \(\Delta_{K_t}^{liab}\).
- The first derivative with respect to \(K_t\) of the time-\(t\) present value of the hedging instruments, \(\Delta_{K_t}^{hedge}\).
Dynamic Hedging

To determine the optimal hedge ratio, $h_t$, the first derivatives of the pension liabilities and the hedging instruments with respect to $K_t$ are matched:

$$\Delta^{liab}K_t = h_t \cdot \Delta^{hedge}K_t.$$ 

At time $t$, $h_t$ units of the hedging instruments are purchased to hedge the uncertainty of the pension liabilities from time $t$ to $t + 1$.

At time $t + 1$, the hedging instruments purchased at time $t$ are closed out to compensate for the shortfall of the pension plan from the longevity risk.
Hedge Effectiveness

The hedge effectiveness at time $t$ is calculated as:

$$HE_t = 1 - \frac{\text{Var}(H_t - L_t)}{\text{Var}(L_t)}$$

where

- $L_t$ is the time-0 present value of the realized liabilities from time 0 to $t$ and the future liabilities after time $t$;
- $H_t$ is the time-0 present value of the payoffs of the hedging instruments up to time $t$ with an initial reserve of $H_0 = L_0$. 

Kenneth Q. Zhou and Johnny S.-H. Li University of Waterloo
A Two-Population Example

- Two populations: Continuous Mortality Investigation (CMI) and England and Wales (EW)
- Liability: 30 years of $1 pension liabilities subject to the mortality experience of a male individual aged 60 in year 2005 from the CMI population.
- Hedging instrument: 10-year age-75 q-forward contracts linked to the EW population, which are unlimitedly available and liquidly traded.
Figure 2: The time-0 present value of the liabilities over time.
Figure 3: The 10-year age-75 q-forward rate over time.
Figure 4: The optimal hedge ratio over time.
Figure 5: The hedge result over time.
Motivations

- Population basis risk exists if standardized index-linked hedging instruments are used.
- The degree of the population basis risk varies according to the populations involved in the hedge.
- Population basis risk cannot be diversified within a pension plan, but may be diversifiable across different pension plans.
The Customized Surplus Swap

- The swap is a cash exchange agreement between a pension plan and a reinsurer.
- It transfers population basis risk and sampling risk from one party to another.
- The swap exchanges the economic surplus of a pension plan after the implementation of a dynamic longevity hedging strategy.
- It eventually offloads the residual risk from the pension plan to the counterparty.
The Surplus of a Pension Plan

- The total liability of the pension plan at time $t$ is

$$L_t = CL_t + FL_t,$$

where $CL_t$ and $FL_t$ are the time-$t$ values of the current and future liabilities, respectively.

- The hedging portfolio for the pension plan at time $t$ is

$$H_t = (1 + r)(H_{t-1} - CL_{t-1}) + P_t,$$

where $P_t$ is the payoff of the hedging instruments at time $t$, and $H_0 = L_0$ is the initial reserve.

- Finally, the surplus of the pension plan at time $t$ is defined as

$$SP_t = H_t - L_t.$$
The Cash Flow of the Swap

The goal is to have $SP_t = 0$. Hence, the cash flow of the swap at time $t$ is defined as $CF_t = -SP_t$. It follows that

$$CF_t = L_t - (1 + r)(H_{t-1} - CL_{t-1}) - P_t.$$ 

If the swap is set up every year, then

$$CF_t = L_t - (1 + r)FL_{t-1} - P_t.$$ 

Both $L_t$ and $FL_{t-1}$ can be determined by whichever valuation methods agreed to by the two parties. The payoff of the hedging instruments will be determined by the market price at time $t$ and the number of holdings at time $t-1$. 

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University of Waterloo
A Multi-Population Example

- 20 national populations.
- Liability: 30 years of $1 pension liabilities subject to the mortality experience of a male individual aged 60 in year 2008 from each population.
- Hedging instrument: 10-year age-75 q-forward contracts linked to the EW population, which are unlimitedly available and liquidly traded.
- Sampling risk: a binomial frequency model with an assumed population size of 10,000 for each population.
Figure 6: The hedge result over time for the 20 populations.
Figure 7: The surplus over time for the 20 populations.
A multi-population example

Figure 8: The cash flow over time for the 20 populations.
Figure 9: The mean cash flow of the 20 populations.
Conclusion

Figure 10: The outline of the proposed dynamic hedging strategy.