# Reconciling forecasts of age distribution of death counts: An application to annuity pricing

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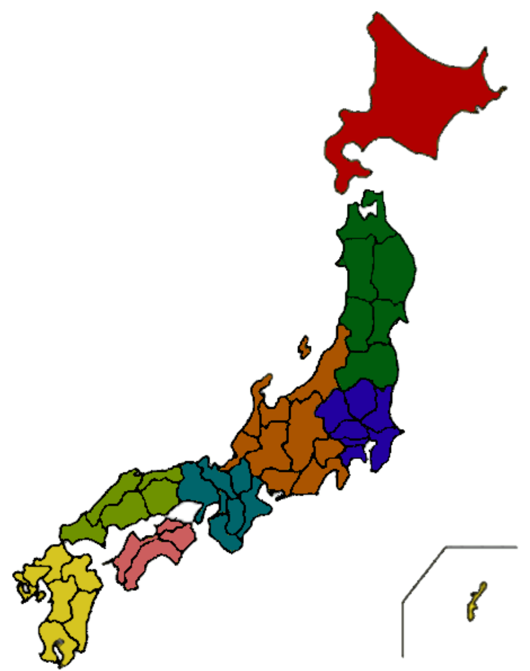
# Japanese age-specific lifetable

1. Japanese national and sub-national age-specific life-table death counts from 1975 to 2014 from *Japanese Mortality Database*
2. Period life-table radix is fixed at 100,000 at age 0 for each year group
3. 8 five-year year groups, 1975-1979, 1980-1984, $\cdots$, 2010-2014
4. 24 age groups, age 0, 1-4, 5-9, 10-14, $\cdots$, 105-109, 110+
5. Due to zero counts for age 110+ for some years, merge this age group with age group 105-109



# Japanese group structure

Have national and sub-national mortality rates, data structure is displayed below where each row denotes a level of disaggregation

| Group level | Number of series |
|---|---|
| Japan | 1 |
| Sex | 2 |
| Region | 8 |
| Region × Sex | 16 |
| Prefecture | 47 |
| Prefecture × Sex | 94 |
| Total | 168 |

# Compositional data-analytic approach

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2. Sample space of compositional data is the simplex

$$S^K = \left\{ \boldsymbol{D} = (d_1, \dots, d_K)^\top, \quad d_x > 0, \quad \sum_{x=1}^{K} d_x = c \right\}$$

where $c$ is a fixed constant (such as, radix in period life table), $^\top$ denote vector transpose, simplex sample space is $K-1$ dimensional subset of $R^{K-1}$

# CoDa in action

1. Begin from a data matrix $\boldsymbol{D}$ of size $n \times K$ of life-table deaths $(d_{t,x})$ with $n$ rows representing the number of years and $K$ columns representing the age $x$. Sum of each row adds up to life-table radix, such as 100,000

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2. Compute geometric mean at each age, given by

$$\alpha_x = \exp^{\frac{1}{n}\sum_{t=1}^{n}\ln(d_{t,x})}, \qquad x = 1, \dots, K$$

For a given year $t$, divide $(d_{t,1}, \dots, d_{t,K})$ by corresponding geometric means $(\alpha_1, \dots, \alpha_K)$,

$$C\left[\frac{d_{t,1}}{\alpha_1}, \frac{d_{t,2}}{\alpha_2}, \dots, \frac{d_{t,K}}{\alpha_K}\right]$$

# CoDa in action

$C[\cdot]$ represents a closure operation, performing standardization

$$f_{t,x} = \frac{\frac{d_{t,x}}{\alpha_x}}{\frac{d_{t,1}}{\alpha_1} + \frac{d_{t,2}}{\alpha_2} + \cdots + \frac{d_{t,K}}{\alpha_K}}, \qquad x = 1, \dots, K$$

where $f_{t,x}$ is a non-negative value

# CoDa in action

3. Log-ratio transformation: Aitchison (1982, 1986) showed that compositional data are represented in a restricted space where components can only vary between 0 and positive constant, proposed centered log-ratio transformation

$$h_{t,x} = \ln\left(\frac{f_{t,x}}{g_t}\right)$$

where $g_t$ are the geometric means over age at time $t$

$$g_t = \exp^{\frac{1}{K}\sum_{x=1}^{K}\ln(f_{t,x})}.$$

Transformed data matrix is $\boldsymbol{H}$ with elements $h_{t,x} \in R$ real-valued ☺

# CoDa in action

4. Principal component analysis: applied to the matrix $\boldsymbol{H}_x = \{h_{t,1}, \dots, h_{t,K}\}$ to obtain the estimated principal components and their associated scores,

$$h_{t,x} = \sum_{\ell=1}^{\min(n,K)} \beta_{t,\ell}\phi_{\ell,x} \approx \sum_{\ell=1}^{L} \beta_{t,\ell}\phi_{\ell,x}$$

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- $L$ denotes number of retained components

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7. Transform back to compositional data: take inverse centered log-ratio transformation

$$\widehat{f}_{n+h|n,x} = C\left[\exp^{\widehat{h}_{n+h|n,x}}\right]$$

# CoDa in action

7. $C[\cdot]$ is closure operator, performing standardization

$$\widehat{f}_{n+h|n,x} = \frac{\exp^{\widehat{h}_{n+h|n,x}}}{\exp^{\widehat{h}_{n+h|n,1}} + \cdots + \exp^{\widehat{h}_{n+h|n,K}}}$$

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8 Add back the geometric means, to obtain forecasts of life-table death matrix $\widehat{d}_{n+h|n,x}$:

$$\begin{aligned}\widehat{d}_{n+h|n,x} &= C\left[\widehat{f}_{n+h|n,x} \times \alpha_x\right] \\ &= \left[\frac{\widehat{f}_{n+h|n,1} \times \alpha_1}{\sum_{x=1}^{K} \widehat{f}_{n+h|n,x} \times \alpha_x}, \dots, \frac{\widehat{f}_{n+h|n,K} \times \alpha_K}{\sum_{x=1}^{K} \widehat{f}_{n+h|n,x} \times \alpha_x}\right]\end{aligned}$$

where $\alpha_x$ denotes age-specific geometric mean of $d_{t,x}$

# Selecting the number of components

To determine number of components $L$, determine the value of $L$ as the minimum number of components that reaches a certain level of proportion of total variance explained by $L$ leading components

$$L = \operatorname*{arg\,min}_{L: L \geq 1} \left\{ \sum_{\ell=1}^{L} \widehat{\lambda}_\ell \Big/ \sum_{\ell=1}^{\min\{n,K\}} \widehat{\lambda}_\ell \mathbb{1}_{\{\widehat{\lambda}_\ell > 0\}} \right\}$$

where $\delta = 95\%$, $\mathbb{1}\{\cdot\}$ denotes binary indicator function excluding possible zero eigenvalues. The chosen $L = 1$.

# Bootstrapped forecasts

1. Bootstrapped functional time series can be obtained

$$\widehat{h}^b_{t,x} = \sum_{\ell=1}^{L} \widehat{\beta}^b_{t,\ell}\widehat{\phi}_{\ell,x}, \qquad t = 1,\dots,n,$$

where $\widehat{\beta}^b_{t,\ell}$: bootstrapped $\ell^{\text{th}}$ principal component scores, for $b = 1,\dots,B$ and $B$ is the number of bootstrap replications

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2. For each bootstrap replication, we obtain the forecast of $h_{n+h,x}$ as

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$\widehat{\beta}^b_{n+h,\ell}$: forecast of the bootstrapped principal component scores

3. By randomly sampling with replacement the observations corresponding to the year index of the in-sample fitted errors, we obtain a set of bootstrapped model residuals

# Forecast reconciliation of death count

Japanese data follow a three-level hierarchy, coupled with sex grouping variable (S. & Hyndman, 2017, JCGS; S. & Haberman, IME)

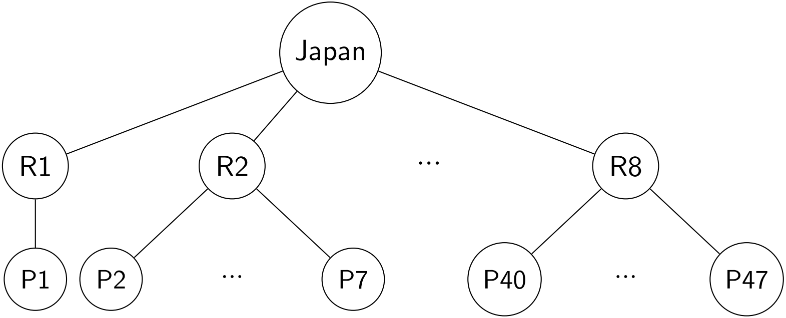


**Figure:** *Japanese geographical hierarchy tree diagram*

Refer to a disaggregated series using notation $X \times S$; $X$ is geographical area and $S$ is sex

$$
\underbrace{\begin{bmatrix}
\text{d}_{\text{Japan*T},t} \\
\text{d}_{\text{Japan*F},t} \\
\text{d}_{\text{Japan*M},t} \\
\text{d}_{\text{R1*T},t} \\
\vdots \\
\text{d}_{\text{R8*T},t} \\
\text{d}_{\text{R1*F},t} \\
\vdots \\
\text{d}_{\text{R8*F},t} \\
\text{d}_{\text{R1*M},t} \\
\vdots \\
\text{d}_{\text{R8*M},t} \\
\text{d}_{\text{P1*T},t} \\
\vdots \\
\text{d}_{\text{P47*T},t} \\
\text{d}_{\text{P1*F},t} \\
\text{d}_{\text{P1*M},t} \\
\vdots \\
\text{d}_{\text{P47*F},t} \\
\text{d}_{\text{P47*M},t}
\end{bmatrix}}_{\boldsymbol{d}_t}
=
\underbrace{\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}}_{\boldsymbol{S}:=\text{summing matrix}}
\underbrace{\begin{bmatrix}
\text{d}_{\text{P1*F},t} \\
\text{d}_{\text{P1*M},t} \\
\text{d}_{\text{P2*F},t} \\
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\text{d}_{\text{P47*M},t}
\end{bmatrix}}_{\boldsymbol{b}_t}
$$

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3. Performs well when there is a strong *signal-to-noise* ratio

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- $\boldsymbol{\epsilon}_{n+h}$ denotes reconciliation errors

# Estimating regression coefficient

To estimate regression coefficient, Hyndman et al. (2011) and Hyndman et al. (2016) proposed a weighted least-squares solution

$$\widehat{\boldsymbol{\beta}}_{n+h} = \left( \boldsymbol{S}^\top \underbrace{\boldsymbol{W}_h^{-1}}_{\text{pain☹}} \boldsymbol{S} \right)^{-1} \boldsymbol{S}^\top \boldsymbol{W}_h^{-1} \widehat{\boldsymbol{D}}_{n+h}$$

where $\boldsymbol{W}_h$ is a diagonal matrix

# How to estimate $\boldsymbol{W}_h$?

1. Assuming error terms follow same group structure, $\boldsymbol{W}_h = k_h \boldsymbol{I}$ and $\boldsymbol{I}$ is identity matrix. Revised forecasts are

$$\overline{\boldsymbol{D}}_{n+h} = \boldsymbol{S}\widehat{\boldsymbol{\beta}}_{n+h} = \boldsymbol{S}(\boldsymbol{S}^\top \boldsymbol{S})^{-1}\boldsymbol{S}^\top \widehat{\boldsymbol{D}}_{n+h},$$

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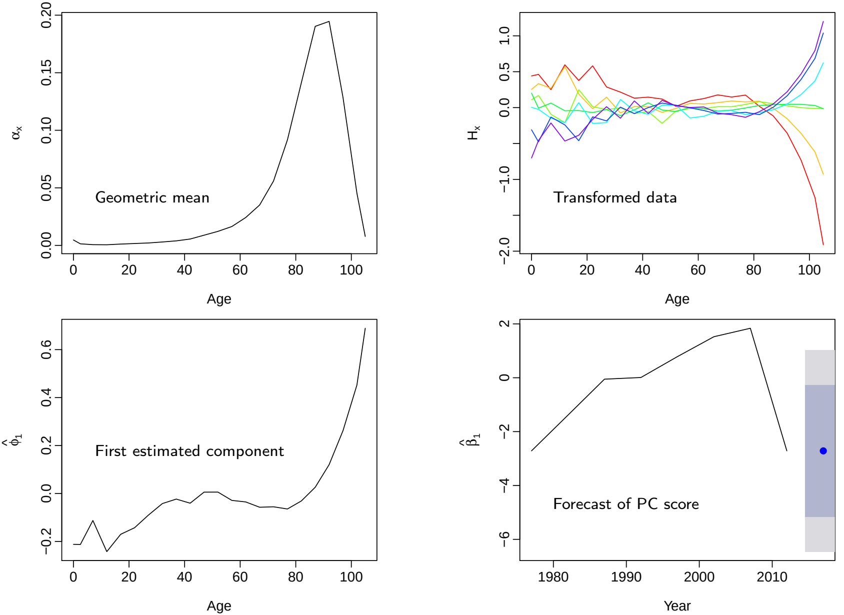
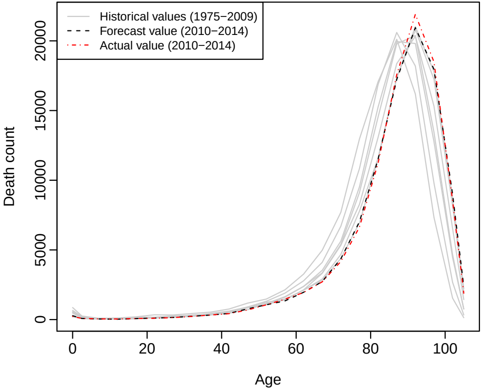
2. Assuming $\boldsymbol{W}_h = k_h \times \boldsymbol{W}_1$, we approximate $\boldsymbol{W}_1$ by its diagonal using in-sample fitted errors. Assigning weights as inverse proportion to variance, so places smallest weights where we have largest residual variance (WLS)

# Model fitting (Okinawa female data)



# Forecast death counts

Based on historical death from 1975 to 2009, produce one-step-ahead point forecasts of age-specific life-table death between 2010 and 2014




Age distribution of death counts continues to be negative skewed with more deaths occurring at older ages

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3. With two one-step-ahead forecasts, evaluate out-of-sample forecast accuracy

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$$\text{MAPE}_k = \frac{1}{23 \times 2} \sum_{\xi=1}^{2} \sum_{x=1}^{23} \left| \frac{d_{n+\xi,x}^k - \widehat{d}_{n+\xi,x}}{d_{n+\xi,x}^k} \right| \times 100,$$

where $d_{n+\xi,x}^k$ denotes actual holdout sample for age $x$ and forecasting year $\xi$ in $k^{\text{th}}$ series

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   where $d_{n+\xi,x}^k$ denotes actual holdout sample for age $x$ and forecasting year $\xi$ in $k^{\text{th}}$ series
3. By averaging $\text{MAPE}_m$ across number of series within each level of disaggregation, obtain an overall assessment of point forecast accuracy for each level within collection of series

$$\text{MAPE} = \frac{1}{M_i} \sum_{m=1}^{M_i} \text{MAPE}_m$$

   where $M_i$ denotes number of series at $i^{\text{th}}$ level of disaggregation

# Interval forecast evaluation

1. Consider the common case of symmetric $100(1-\gamma)\%$ prediction intervals, with lower and upper bounds that are predictive quantiles at $\gamma/2$ and $1-\gamma/2$, denoted by $\widehat{d}^{l}_{n+\xi,x}$ and $\widehat{d}^{u}_{n+\xi,x}$

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2. A scoring rule for the interval forecasts at time point $d_{\xi+h,x}$ is

$$S^{k}_{\gamma,\xi}\left[\widehat{d}^{l}_{n+\xi,x}, \widehat{d}^{u}_{n+\xi,x}, d_{n+\xi,x}\right] = \left(\widehat{d}^{u}_{n+\xi,x} - \widehat{d}^{l}_{n+\xi,x}\right) + \frac{2}{\gamma}\left(\widehat{d}^{l}_{n+\xi,x} - d_{n+\xi,x}\right)$$

$$\mathbb{1}\left\{d_{n+\xi,x} < \widehat{d}^{l}_{n+\xi,x}\right\} + \frac{2}{\gamma}\left(d_{n+\xi,x} - \widehat{d}^{u}_{n+\xi,x}\right)\mathbb{1}\left\{d_{n+\xi,x} > \widehat{d}^{u}_{n+\xi,x}\right\}$$

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where $\mathbb{1}\{\cdot\}$: binary indicator function, $\gamma$: level of significance

3. For different ages and years in the forecasting period, mean interval score is

$$\overline{S}^k_{\gamma} = \frac{1}{23\times 2}\sum_{\xi=1}^{2}\sum_{x=1}^{23} S^k_{\gamma,\xi}\left[\widehat{d}^{l}_{n+\xi,x}, \widehat{d}^{u}_{n+\xi,x}; d_{n+\xi,x}\right], \quad \overline{S}_{\gamma}(h) = \frac{1}{M_i}\sum_{k=1}^{M_i}\overline{S}^k_{\gamma}$$

# Point forecast evaluation for forecasting death counts

| | MAPE | | number of smaller errors / number of series at each level | |
|---|---|---|---|---|
| Level | CoDa | RW | CoDa | RW |
| Total | **6.8831** | 8.0765 | 100% | 0% |
| Sex | **7.6630** | 8.2054 | 100% | 0% |
| Region | **8.4605** | 9.3633 | 87.50% | 12.50% |
| Region + Sex | **9.5975** | 10.0833 | 68.75% | 31.25% |
| Prefecture | **10.1161** | 11.5056 | 91.49% | 8.51% |
| Prefecture + Sex | **12.4527** | 13.8352 | 81.91% | 18.09% |

# Point forecast evaluation (reconciliation methods)

| Level | BU | OLS | WLS |
|---|---|---|---|
| Total | 7.6064 | 7.3324 | **7.2925** |
| Sex | 7.8143 | 7.5184 | **7.4846** |
| Region | 9.3641 | 9.0333 | **9.0323** |
| Region + Sex | 9.4131 | **9.1015** | 9.1639 |
| Prefecture | 11.1255 | 10.7811 | **10.7494** |
| Prefecture + Sex | 12.4527 | **12.1693** | 12.2157 |
| Overall Mean | 9.6294 | **9.3227** | 9.3231 |

# Interval forecast evaluation

| Level | CoDa | BU | OLS | WLS |
|---|---|---|---|---|
| Total | 1108.76 | 900.91 | **848.53** | 857.71 |
| Sex | 1089.50 | **947.71** | 962.52 | 991.92 |
| Region | 1145.86 | 815.33 | **772.02** | 780.80 |
| Region + Sex | 1123.92 | 771.30 | 724.75 | **719.90** |
| Prefecture | 1201.80 | 900.82 | **791.10** | 792.98 |
| Prefecture + Sex | 1187.09 | 1187.09 | 1110.32 | **1081.02** |
| Overall Mean | 1142.82 | 920.53 | **868.21** | 870.72 |

# Life annuity

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2. Apply forecasts of death counts to calculation of single-premium term immediate annuities
3. $\tau$ year survival probability of a person aged $x$ currently at $t = 0$ is determined by

$$
\begin{aligned}
{}_\tau p_x &= \prod_{j=1}^{\tau} p_{x+j-1} \\
&= \prod_{j=1}^{\tau} (1 - q_{x+j-1}) = \prod_{j=1}^{\tau} \left(1 - \frac{d_{x+j-1}}{l_{x+j-1}}\right)
\end{aligned}
$$

where $d_{x+j-1}$ denotes number of death counts between ages $x + j - 1$ and $x + j$; $l_{x+j-1}$ denotes number of lives alive at age $x + j - 1$

# Annuity price calculation

Price of an annuity with maturity $T$ year, written for a $x$-year-old with benefit \$1 per year, is given

$$
\begin{aligned}
a_x^T\left(d_{1:T}^x\right) &= \sum_{\tau=1}^{T} B(t=0,\tau) \times \mathsf{E}\left(\mathbb{1}_{T_x>\tau} | d_{1:\tau}^x\right) \\
&= \sum_{\tau=1}^{T} \underbrace{B(t=0,\tau)}_{\text{bond price}} \times \underbrace{{}_{\tau}p_x(d_{1:\tau}^x)}_{\text{survival probability}}
\end{aligned}
$$

1. $B(t=0,\tau)$ is $\tau$-year bond price, where $\tau < T$

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1. $B(t=0,\tau)$ is $\tau$-year bond price, where $\tau < T$
2. $d_{1:\tau}^x$ is first $\tau$ elements of $d_{1:T}^x$
3. ${}_\tau p_x(d_{1:\tau}^x)$ denotes survival probability given a random $d_{1:\tau}^x$

# Comparison of life annuity premium calculation

1. Compare annuity price estimates for different ages and maturities between methods for a female policyholder living in Japan
2. Assume a constant interest rate at $\eta = 3\%$ and $B(t = 0, \tau) = \exp^{-\eta\tau}$

# Fixed-term annuity price (age = 60) for Japan (F, M, T)

| Series | | $T = 5$ | $T = 10$ | $T = 15$ | $T = 20$ | $T = 25$ | $T = 30$ |
|---|---|---|---|---|---|---|---|
| Female | LB | 4.5255 | 8.3311 | 11.4895 | 14.0474 | 16.0071 | 17.3350 |
| | Mean | 4.5288 | 8.3448 | 11.5274 | 14.1288 | 16.1626 | 17.6018 |
| | UB | 4.5370 | 8.3830 | 11.6356 | 14.3754 | 16.6576 | 18.5063 |
| Male | LB | 4.4540 | 8.0646 | 10.9043 | 13.0075 | 14.4030 | 15.1543 |
| | Mean | 4.4602 | 8.0897 | 10.9659 | 13.1356 | 14.6187 | 15.4637 |
| | UB | 4.4729 | 8.1467 | 11.1276 | 13.4911 | 15.2772 | 16.4944 |
| Total | LB | 4.4912 | 8.2011 | 11.2047 | 13.5497 | 15.2536 | 16.3333 |
| | Mean | 4.4958 | 8.2223 | 11.2618 | 13.6700 | 15.4712 | 16.6753 |
| | UB | 4.5056 | 8.2714 | 11.4018 | 13.9851 | 16.0845 | 17.7222 |

# Thank you

1. A draft paper is available upon request from hanlin.shang@anu.edu.au

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2. Follow me at Research Gate https://www.researchgate.net/profile/Han_Lin_Shang

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