Optimal Management of Individual Pension Plans

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Abstract

We consider optimal investment for an individual pension savings plan in receipt of gradual contributions against which one cannot borrow, using expected power utility as the optimality criterion. It is well known that in the presence of credit constraints the Samuelson paradigm of investment in constant proportions out of total wealth (including current savings and future contributions) no longer applies. Instead, the optimal investment gives rise to so-called stochastic lifestyling, whereby for low levels of accumulated capital it is optimal to invest fully in stocks and then gradually switch to safer assets as the level of savings increases. In stochastic lifestyling not only does the leverage between risky and safe assets change but also the actual mix of the risky assets varies over time. While the existing literature relies on complex numerical computations to quantify optimal lifestyling the present paper provides a simple formula that captures the main essence of the lifestyling effect.

Keywords: optimal investment, stochastic lifestyling, credit constraint, defined contribution, power utility

Mathematics Subject Classification (2010): 90C20, 90C39, 35K55, 49J20

JEL Classification Code: G11, C61, G23

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1 Introduction

Consider a model with $d$ risky assets whose dynamics are given by the SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad (1.1)$$

where $B$ are $d$ uncorrelated Brownian motions, $\mu \in \mathbb{R}^d$, and $\sigma \in \mathbb{R}^{d \times d}$ is regular. In addition there is a risk-free asset with value $S^0 = e^{rt}$. An individual who starts working at time 0 and retires at time $T$ makes pension contributions at the rate $y_t$ per unit of time. The task of the pension fund manager is to invest these contributions on behalf of the individual so as to maximize the expected utility of the terminal value of the pension plan. There are many pension plans that allow for individual investment policies, such as the 401k plans in the U.S. or the Swedish "premium pensions", to mention but a few.

To aid tractability it is customary to consider utility functions of the form

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1. \quad (1.2)$$

The analysis can be extended to $\gamma = 1$ with $U(x) = \ln x$ but in order to keep the notation simple we shall not do so here. When the individual savings plans can borrow as well as invest at the risk-free rate $r$ [Samuelson (1969)], and more explicitly [Hakansson (1970)], have pointed out that the presence of contributions does not change the optimal strategy in the following sense. The optimal risky investment is always in constant proportions $\bar{\pi}$ with

$$\bar{\pi} = (\mu - r)\Upsilon (\sigma \Upsilon)^{-1}, \quad (1.3)$$

provided that the investment is made out of the combined value of the cash in hand and the present value (PV) of all future contributions.

Such strategy may lead to short positions in some of the risky assets. When the short sales of risky assets are prohibited the constant proportions strategy changes to

$$\hat{\pi} = \arg \max_{\pi \geq 0} \pi (\mu - r) - \frac{1}{2} \pi (\sigma \Upsilon) \pi^\top, \quad (1.4)$$

but the basic story remains the same in that the optimal risky investment is in fixed proportions scaled linearly by risk tolerance, $\bar{\pi} / \gamma$, and the optimal strategy is computed as if one lived in a world with no contributions and the initial wealth were equal to the present value of all future contributions.

Here we are concerned with yet another type of situation prevailing in practice where, in addition to the shortsale constraints on risky assets, risk-free borrowing is possible at a rate substantially higher than the deposit rate $r$. Defined contribution pension plans are a typical point in case since it is difficult for an individual, or the fund manager on their behalf, to borrow against the value of the future pension contributions. This situation calls for a model
with a shortsale constraint on the risk-free as well as the risky assets. This means that in addition to \( \pi_t \geq 0 \) we also require

\[
\pi_t 1 \leq \alpha_t, \tag{1.5}
\]

where

\[
\alpha_t = \frac{W_t}{PV_t + W_t} \tag{1.6}
\]

is the ratio of the already accumulated savings \( W_t \) to the entire capital composed of the accumulated savings plus present value \( PV_t \) of all future contributions, and \( 1 \) stands for a \( d \)-dimensional column vector of ones.

It is clear that any constant proportion strategy will violate the borrowing constraint \((1.5)\) for low values of accumulated funds \( W_t \). Naively, one may be tempted to use the Samuelson’s strategy \( \hat{\pi} \) adjusted for the credit constraint, that is to employ the strategy

\[
\pi^{(1)}(\alpha_t) := \min \left( \frac{\alpha_t}{\hat{\pi} 1}, \frac{1}{\gamma} \right) \hat{\pi}. \tag{1.7}
\]

However, it turns out that the character of the optimal investment strategy changes much more dramatically in the face of the constraint \((1.5)\) than the equation \((1.7)\) might suggest. Not only does the optimal leverage change over time with changes in \( \alpha_t \), but simultaneously it is optimal to vary the composition of the risky investment. Secondly, the optimal weights for \( \alpha_t = 1 \) are no longer in fixed proportions \( \hat{\pi} \) multiplied by risk tolerance, but for low levels of risk aversion there is substitution towards the riskier assets.

It transpires from the preceding discussion that the stochastic constraint \( \pi_t 1 \leq \alpha_t \) provides a simple framework capturing a phenomenon known in pension finance as stochastic lifestyling, a term coined by Cairns et al. (2006), whereby it is optimal early on to invest the accumulated savings into stocks and then gradually switch the investment into bonds and safe deposits as the retirement approaches and the total amount of savings increases. Because the fully optimal strategy is a function of both time \( t \) and the accumulated savings \( W_t \) and because it has to be computed numerically by dynamic programming, at first sight it is difficult to identify the dependence of the lifestyling effect on the credit constraint \( \alpha_t \). However, we show that there is a related quasi-optimal strategy, which depends only on \( \alpha_t \) and which can be computed simply from a static constrained quadratic programme, namely

\[
\pi^{(2)}(\alpha_t) = \arg \max_{\pi \geq 0, \pi 1 \leq \alpha_t} \pi (\mu - r) - \frac{\gamma}{2} \pi (\sigma \sigma^T) \pi^T. \tag{1.8}
\]

We show that strategy \( \pi^{(2)}(\alpha_t) \) is an excellent approximation to the fully optimal strategy and can therefore serve as a simple rule of thumb for pension plan providers who wish to offer a choice of lifestyling strategies to their clients, while also specifying the sense in which such lifestyling is optimal. To reduce the barriers to application even further we analyze the explicit dependence of \( \pi^{(2)} \) on \( \alpha_t \) for a given set of binding constraints. For
example, assuming that the constraints $\pi \geq 0$ are not binding, the quasi-optimal strategy $\pi^{(2)}$ is of the form

$$
\pi^{(2)}(\alpha_t) = \frac{\hat{\pi}}{\gamma} + \frac{1}{1^\gamma(\sigma\sigma^\top)^{-1}} \min(\alpha_t - \frac{\hat{\pi}1}{\gamma}, 0).
$$

(1.9)

Formula (1.9) captures the main essence of the lifestyling effect, representing in a nutshell the main contribution of our paper. It not only shows the change in portfolio composition as a function of $\alpha_t$ for fixed risk aversion, but it also neatly demonstrates that the portfolio composition will change with decreasing $\gamma$ when there are no future contributions to consider ($\alpha_t = 1$).

The article is organized as follows. Section 2 introduces what we call the “Samuelson transform”, linking a model with gradual contributions to an equivalent model where all capital is paid up-front but there are additional constraints on how the capital can be invested. In Section 3 we review the mathematical theory guaranteeing existence of an optimal strategy in a world without contributions and via the Samuelson link also in a world with contributions and credit constraints. In Section 4 we provide a simple numerical example to contrast the fully optimal strategy with a naive rescaling of Samuelson’s fixed weights strategy, both in terms of welfare impact and portfolio weights. In Section 5 we propose a simple quasi-optimal strategy computable by means of a static constrained quadratic programme and construct an explicit formula that captures the essence of the lifestyling effect. Section 6 examines the role of leverage in alleviating welfare losses stemming from the credit constraint. Section 7 concludes.

## 2 Samuelson transform

We denote by $Y_t = \int_0^t y(u)du$ the cumulative pension contribution up to and including time $t$. Function $y$ is assumed to be non-negative and integrable on $[0, T]$. The price process of all assets, including the risk-free asset, is denoted by $S \equiv (S^0, S^{1:d})$. We assume $S^{1:d}$ is a geometric Brownian motion with drift as described in equation (1.1), while $S^0_t := e^{rt}$ represents a bank account with risk-free deposit rate $r$. Risk-free borrowing is excluded.

The process

$$
PV_t := \int_t^T e^{-r(u-t)}dY_u,
$$

is the present value at time $t$ of all contributions in the period $(t, T]$.

**Definition 2.1** We say that $\varphi$ is an admissible self-financing strategy for price process $S$ and cumulative contributions $Y$, writing $\varphi \in \Theta(S, Y)$, if

$$
\varphi_0S_0 + \int_0^t \varphi_udS_u + Y_t = \varphi_tS_t,
$$

4
and
\[ \int_0^T \sum_{i=1}^d (\varphi^S_{it} S_i^t)^2 du < \infty, \] almost surely.

We denote by \( \Theta_x(S, Y) \) the set of all admissible self-financing strategies with initial capital \( x \),
\[ \Theta_x(S, Y) := \{ \varphi \in \Theta(S, Y) : \varphi_0 S_0 = x \}. \]

Consider the following transformation of trading strategies \( \varphi \mapsto \overline{\varphi} \):
\[
\overline{\varphi}^{1:d}_t = \varphi_t^{1:d}, \\
\overline{\varphi}^0_t = \varphi^0_t + e^{-rt} PV_t.
\]

We call (2.1, 2.2) the Samuelson transform. Using the numeraire change technique of Ge-\( \text{man et al.} \) (1995) it is readily seen that the Samuelson transform is a one-to-one mapping between \( \Theta_x(S, Y) \) and \( \Theta_{x+PV_0}(S, 0) \).

We can now turn our attention to a situation where borrowing against future contributions is no longer possible.

**Definition 2.2** Consider an arbitrary self-financing strategy \( x \in \Theta_x(S, Y) \) with an arbitrary contribution process \( Y \). Assume that \( \varphi \geq 0 \) and \( S \geq 0 \). We define the vector of proportions, \( \pi(\varphi) \), invested in available risky assets by
\[
\pi_i(\varphi) := \frac{\varphi^i S^i}{\varphi S} \quad \text{for} \quad i = 1, \ldots, d,
\]
using the convention \( 0/0 = 0 \).

**Proposition 2.3** Suppose \( S \geq 0 \). The Samuelson transform is a one-to-one mapping between
\[
\{ \varphi \in \Theta_x(S, Y) : \pi(\varphi) \geq 0, \pi(\varphi) \mathbf{1} \leq 1 \},
\]
and
\[
\{ \overline{\varphi} \in \Theta_{x+PV_0}(S, 0) : \pi(\overline{\varphi}) \geq 0, \pi(\overline{\varphi}) \mathbf{1} \leq 1 - PV/\varphi S \}. \]

**Proof.** \( \pi(\varphi) \geq 0 \land \pi(\varphi) \mathbf{1} \leq 1 \iff \varphi^0 S^0 \geq 0 \land \varphi^{1:d} \geq 0 \iff \overline{\varphi}^0 S^0 \geq PV, \overline{\varphi}^{1:d} \geq 0 \iff \pi(\overline{\varphi}) \geq 0 \land \pi(\overline{\varphi}) \mathbf{1} \leq 1 - PV/\varphi S \). \( \square \)

The parallel between the classical Samuelson paradigm and the situation where the risk-free borrowing against future contributions is precluded is now fully clarified. While in the classical case the sum of risky proportions is unconstrained, now in (2.4) there is a stochastic constraint on the total proportion invested in the risky assets which must never exceed \( 1 - PV/\varphi S \) in Samuelson’s world without contributions. In economic terms this says that risky investment can only be financed from past contributions and from past capital gains. The new constraint not only influences the leverage of the optimal portfolio but it also has a significant impact on the relative proportions invested in risky assets. We analyze this intriguing phenomenon in detail below.
3 Optimal investment

**Theorem 3.1** The following statements hold:

1) The equation

\[
0 = \max_{\pi \geq 0, \pi \leq 1 - \PV/\pi} v_t + xv_x(r + \pi(\mu - r)) + \frac{x^2}{2}v_{xx}\pi\sigma^T\pi^T,
\]

\(v(T, x) = x^{1-\gamma}/(1 - \gamma),\)

has a unique continuous solution \(v\), concave in \(x\). This solution belongs to \(C^{1,2}([0, T] \times (0, \infty))\). The optimal policy \(x(t, x)\) is Lipschitz-continuous in \(x\), uniformly in \(t\).

2) The solution \(v\) in (3.1) satisfies

\[
v(0, \PV_0) = \max_{\pi \in \Theta \Pi_0(S, 0)} E\left[ \frac{1}{1 - \gamma} (\pi_T S_T)^{1-\gamma} \right].
\]

3) There is a unique process \(W\) satisfying

\[
\begin{align*}
dW_t &= rW_t dt + W_t \pi^*(t, W_t)(dS_t^{1:d} - S_t^{1:d} - r dt), \\
W_0 &= \PV_0.
\end{align*}
\]

4) The optimal strategy \(\phi\) in (3.2) satisfies

\[
\begin{align*}
\phi^i_t &= \pi^*_t(t, W_t)W_t/S_t^i, \\
\phi^0_t &= e^{-rt}W_t(1 - \pi^*(t, W_t)1),
\end{align*}
\]

and

\(\phi S = \overline{W}.\)

**Proof.** The link between the HJB equation (3.1) and the optimization (3.2) has been analyzed thoroughly in Vila and Zariphopoulou (1998) and Zariphopoulou (1994). The difference with the present paper is that we additionally face shortselling constraints and the credit constraint has an explicit time dependence via \(\PV_t\), while the constraints in the work of Zariphopoulou are time-homogeneous. However, one can check that all the arguments go through in the present setting.

**Corollary 3.2** The optimal strategy \(\phi\) in

\[
\max_{\phi \in \Theta_0(S, Y), \pi(\phi) \in \A(1)} E\left[ \frac{1}{1 - \gamma} (\phi_T S_T)^{1-\gamma} \right],
\]

satisfies the equality \(\phi S + \PV = \overline{W}\) and it is given by

\[
\begin{align*}
\phi^i &= \phi^i, \\
\phi^0 &= e^{-rt}(\overline{W}_t(1 - \pi^*(t, \overline{W}_t)1) - \PV).
\end{align*}
\]

**Proof.** By Proposition 2.3 the optimizers in (3.2) and (3.3) are linked via the Samuelson transform and the statement follows immediately from its properties.
Table 1: Certainty equivalents and internal rates of return for the optimal and naive strategies, \( \pi^* \) and \( \pi^{(1)} \), respectively.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>CE(^*)</th>
<th>IRR(^*)</th>
<th>CE(^{(1)})</th>
<th>IRR(^{(1)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.65</td>
<td>5.50%</td>
<td>2.26</td>
<td>3.64%</td>
</tr>
<tr>
<td>5</td>
<td>2.18</td>
<td>3.49%</td>
<td>1.97</td>
<td>3.08%</td>
</tr>
<tr>
<td>8</td>
<td>1.82</td>
<td>2.74%</td>
<td>1.75</td>
<td>2.58%</td>
</tr>
</tbody>
</table>

### 4 Numerical illustration

Consider the log-normal model of asset returns described in the introduction. Numerically, we will take risk-free return of \( r = 1\% \) and two risky assets with drifts \( \mu_1 = 2\% \) (representing bond returns), \( \mu_2 = 10\% \) (representing stock returns), volatilities 5%, 25% respectively and correlation -0.05, yielding the covariance matrix

\[
\sigma \sigma^\top = \begin{bmatrix}
0.0025 & -0.000625 \\
-0.000625 & 0.0625
\end{bmatrix}.
\]

The investment horizon has been set to \( T = 40 \) years. We have used the cumulative contribution process \( Y_t = t/T \) so that the cumulative contribution is normalized to 1. The present framework provides methodology capable of analyzing and comparing results for various non-linear contribution profiles, but in the interest of brevity we do not consider them here. Some indication of the welfare implications of contribution timing is given in Section 6.

We examine three levels of relative risk aversion: low \( (\gamma = 2) \), moderate \( (\gamma = 5) \) and high \( (\gamma = 8) \). We report the utility of competing strategies both in terms of certainty equivalent wealth and in terms of (certainty equivalent) internal rate of return\(^1\).

Let us begin by comparing the performance of the optimal strategy \( \pi^* \), computed numerically from Theorem 3.1, and of the rescaled Samuelson strategy \( \pi^{(1)} \), computed explicitly from equation (1.7). Table 1 shows that \( \pi^* \) significantly outperforms the naive strategy for low and medium levels of risk aversion, while with high risk aversion the outperformance is relatively modest.

To gain better understanding of where the outperformance originates from we first analyze the case \( \gamma = 8 \), where the welfare loss is relatively small. We report in Table 2 the optimal portfolio weights out of the cash-in-hand \( \pi^*(t, \text{PV}_t + W_t) \frac{\text{PV}_t + W_t}{W_t} \) as a function of time \( t \) and accumulated savings \( W_t \). The corresponding naive weights \( \pi^{(1)}(t, \frac{W_t}{\text{PV}_t + W_t}) \frac{\text{PV}_t + W_t}{W_t} \) are shown in Table 3. We observe that for high levels of cash in hand there is a good agreement between the optimal and the naive strategy, with both sets of weights tending towards

\(^1\)The certainty equivalent is computed from the formula \( CE = (E((\sigma_T S_T)^{1-\gamma})^{1/(1-\gamma)} \). The certainty equivalent internal rate of return is given as the interest rate \( \rho \) satisfying \( CE = \int_0^T e^{\rho(T-t)} dY_t \).
\[ \pi^*(t, PV_t + W_t)(1 + PV_t/W_t) \]

Table 2: Optimal proportions of risky investment out of the cash in hand \( \pi^*(t, PV_t + W_t)(1 + PV_t/W_t) \) as a function of \( t \) and \( W_t \) with \( \gamma = 8 \).

<table>
<thead>
<tr>
<th>( W_t )</th>
<th>( t = 0 )</th>
<th>( t = 10 )</th>
<th>( t = 20 )</th>
<th>( t = 30 )</th>
<th>( t = 39.975 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00001</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
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<td>0.8800</td>
<td>0.1200</td>
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<td>0.1200</td>
</tr>
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<td>0.4604</td>
</tr>
<tr>
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<td>0.5740</td>
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<td>0.6817</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
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<td>0.7041</td>
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<td>0.6770</td>
</tr>
<tr>
<td>20</td>
<td>0.5689</td>
<td>0.1931</td>
<td>0.5641</td>
<td>0.1915</td>
<td>0.5588</td>
</tr>
</tbody>
</table>

Table 3: Naive proportions of risky investment out of the cash in hand \( \pi^{(1)}(W_t/(W_t + PV_t))(1 + PV_t/W_t) \) as a function of \( t \) and \( W_t \) with \( \gamma = 8 \).

<table>
<thead>
<tr>
<th>( W_t )</th>
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<th>( t = 20 )</th>
<th>( t = 30 )</th>
<th>( t = 39.975 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00001</td>
<td>0.7466</td>
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<td>0.7466</td>
<td>0.2534</td>
<td>0.7466</td>
</tr>
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<td>20</td>
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<td>0.5641</td>
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</table>

\[ \hat{\pi}/\gamma = (54.64\%, 18.55\%) \] as \( W_t \to \infty \). For low level of accumulated savings the difference is substantial, however, with the optimal portfolio being invested fully in stocks while the naive strategy is suggesting bond-dominated weights \( \hat{\pi}/(\hat{\pi}1) = (74.66\%, 25.34\%) \).

The discrepancy between \( \pi^* \) and \( \pi^{(1)} \) is even more pronounced for \( \gamma = 2 \). The naive portfolio weights out of the cash in hand are now fixed at \( \hat{\pi}/(\hat{\pi}1) = (74.66\%, 25.34\%) \) for all levels of accumulated savings \( W_t \). The optimal investment is shown in Table 4. We now observe substantial differences between the naive and the optimal weights also for high level of accumulated savings \( W_t \) where the constraint \( \alpha = W_t/(PV_t + W_t) \) tends to 1. We show later in Section 5 that for high values of \( W_t \) the optimal weights \( \pi^*(t, PV_t + W_t)(PV_t + W_t)/W_t \) tend to the
Table 4: Optimal proportions of risky investment out of the cash in hand $\pi^*(t, PV_t + W_t)(1 + PV_t/W_t)$ as a function of $t$ and $W_t$ with $\gamma = 2$.

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<th>$t = 39.975$</th>
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<tr>
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<td>0.3475</td>
<td>0.6525</td>
</tr>
<tr>
<td>1</td>
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<td>1.0000</td>
<td>0.9999</td>
<td>0.0749</td>
<td>0.3483</td>
</tr>
<tr>
<td>2</td>
<td>0.1004</td>
<td>0.8996</td>
<td>0.1532</td>
<td>0.2048</td>
<td>0.3483</td>
</tr>
<tr>
<td>20</td>
<td>0.3242</td>
<td>0.6758</td>
<td>0.3295</td>
<td>0.3418</td>
<td>0.6510</td>
</tr>
</tbody>
</table>

Expression

$$\arg \max_{\pi \geq 0, \pi^T 1 \leq 1} \pi (\mu - r) - \frac{\gamma}{2} \pi (\sigma \sigma^T) \pi^T$$

$$= \frac{\hat{\pi}}{\gamma} + \frac{1^T (\sigma \sigma^T)^{-1}}{1^T (\sigma \sigma^T)^{-1} 1} \min(1 - \frac{\hat{\pi} 1}{\gamma}, 0). \quad (4.1)$$

In the present example we have $\hat{\pi} = (437\%, 148\%), \hat{\pi} 1 = 5.85$ and

$$\zeta := \frac{1^T (\sigma \sigma^T)^{-1}}{1^T (\sigma \sigma^T)^{-1} 1} = (95.28\%, 4.72\%).$$

Thus as the risk aversion falls below 5.85 there is a strong substitution away from bonds towards stocks. The substitution continues until the risk aversion reaches the level of $1.27 = \hat{\pi} 1 - \hat{\pi} 1/\zeta 1$ below which all accumulated savings are to be invested in stocks only.

For $\gamma = 2$ the limiting portfolio weights (4.1) are (34.91\%, 65.09\%) while the naive strategy uses almost the opposite ratio $\pi^{(1)}(1) = \hat{\pi}/(\hat{\pi} 1) = (74.66\%, 25.34\%)$. Thus, in addition to the discrepancy between $\pi^*$ and $\pi^{(1)}$ for low values of $W_t$ which was present already for $\gamma = 8$, we now face additional discrepancy of the portfolio weights for high level of accumulated savings. The combined effect makes the strategy $\pi^{(1)}$ substantially suboptimal for low levels of risk aversion.

5 Analytic formula for stochastic lifestyling

The previous section has highlighted that the optimal trading strategy $\pi^*$ substantially outperforms the strategy based on mechanical rescaling of fixed Samuelson’s portfolio weights $\hat{\pi}$. This happens for two reasons: firstly, the relative mix of stocks and bonds in the optimal portfolio varies with the value of the accumulated savings, moving progressively from
stocks to bonds as the value of the savings increases over time. Secondly, for high savings levels the relative weights in stocks and bonds do depend on the risk aversion when risk aversion falls below the sum of unconstrained weights $\hat{\pi}1$. In the present section we will examine this "lifestyling" phenomenon in more detail, with the view to providing an analytic approximation of the switching formula.

On inspection of the Hamilton-Jacobi-Bellman PDE (3.1) one notes that the optimal portfolio is given by

$$\pi^*(t, W_t) = \arg \max_{\pi \geq 0, \pi 1 \leq \alpha_t} \pi(\mu - r) - \frac{1}{2} R(t, W_t)\pi(\sigma\sigma^\top)\pi^\top$$

where $R(t, W_t)$ is the state-dependent coefficient of the relative risk aversion of the indirect utility function and $\alpha_t = 1 - PV_t/W_t$. From a purely engineering point of view it makes sense to examine a suboptimal strategy where we replace state-dependent value $R(t, W_t)$ with the constant $R(T, W_T) \equiv \gamma$.

$$\pi^{(2)}(\alpha_t) = \arg \max_{\pi \geq 0, \pi 1 \leq \alpha_t} \pi(\mu - r) - \frac{\gamma}{2} \pi(\sigma\sigma^\top)\pi^\top. \quad (5.1)$$

The obvious advantage of (5.1) is that it dispenses with the need to solve a dynamic programming problem and leaves us with a much simpler task of constrained quadratic programming (CQP). Whether $\pi^{(2)}$ is a good approximation to the optimal strategy now depends on how close the actual risk aversion $R(t, W_t)$ is to the fixed value $\gamma$.

Note that we have encountered the quantity $\pi^{(2)}(1)$ previously in the analytic formula (4.1) as the portfolio weight that $\pi^*$ tends to for high savings levels $W_t$. It therefore makes sense to consider an even simpler suboptimal strategy that lies somewhere between $\pi^{(2)}$ and $\pi^{(1)}$, namely

$$\pi^{(3)}(\alpha_t) := \min \left( \frac{\alpha_t}{\pi^{(2)}(1)}, 1 \right) \pi^{(2)}(1).$$

We remark that for low levels of risk aversion the sum of constrained weights $\pi^{(2)}(1)1$ typically equals 1, which means the new portfolio weights out of cash in hand are fixed at $\pi^{(3)}(\alpha_t)/\alpha_t \equiv \pi^{(2)}(1)$ all the time.

The certainty equivalents of strategies $\pi^{(2)}$ and $\pi^{(3)}$ for the three different levels of risk aversion are shown in Table 5. For high levels of risk aversion strategy $\pi^{(3)}$ coincides with the naive strategy $\pi^{(1)}$, as discussed earlier. However, by virtue of its construction $\pi^{(3)}$ is guaranteed to copy the values of $\pi^*$ and $\pi^{(2)}$ for high values of $W_t$ at any level of risk aversion. In contrast, $\pi^{(1)}$ leads to incorrect portfolio weights $\hat{\pi}/(\hat{\pi}1)$ out of the cash in hand when $W_t$ is high and risk aversion is low enough. This is illustrated in Tables 1 and 5 where for $\gamma = 8$ the performance of $\pi^{(3)}$ is identical to that of $\pi^{(1)}$, but for risk aversions of 5 and 2 the strategy $\pi^{(3)}$ outperforms $\pi^{(1)}$ by some margin. We conclude that the welfare difference between the optimal strategy $\pi^*$ and the less naive strategy $\pi^{(3)}$ reflects precisely the economic value of correct lifestyling strategy at low levels of accumulated capital.
Table 5: Certainty equivalents and internal rates of return of strategies $\pi^*$, $\pi^{(2)}$ and $\pi^{(3)}$ for different levels of risk aversion.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>CE*</th>
<th>IRR*</th>
<th>CE$^{(2)}$</th>
<th>IRR$^{(2)}$</th>
<th>CE$^{(3)}$</th>
<th>IRR$^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.6503</td>
<td>5.50%</td>
<td>3.6502</td>
<td>5.50%</td>
<td>3.3360</td>
<td>5.16%</td>
</tr>
<tr>
<td>5</td>
<td>2.1771</td>
<td>3.49%</td>
<td>2.1771</td>
<td>3.49%</td>
<td>2.0155</td>
<td>3.17%</td>
</tr>
<tr>
<td>8</td>
<td>1.8161</td>
<td>2.74%</td>
<td>1.8160</td>
<td>2.74%</td>
<td>1.7511</td>
<td>2.58%</td>
</tr>
</tbody>
</table>

Table 6: Constrained quadratic programming (CQP) proportions of risky investment out of the cash in hand $\pi^{(2)}(t, PV_t + W_t)(1 + PV_t/W_t)$ as a function of $t$ and $W_t$ with $\gamma = 8$.

<table>
<thead>
<tr>
<th>$W_t$</th>
<th>$t = 0$</th>
<th>$t = 10$</th>
<th>$t = 20$</th>
<th>$t = 30$</th>
<th>$t = 39.975$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00001</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>1</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
<tr>
<td>20</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

One observes in Table 5 that the CQP investment strategy $\pi^{(2)}$ is for all practical purposes indistinguishable from the fully optimal investment $\pi^*$. On inspection of portfolio weights in Tables 2 and 6 we note the largest discrepancy between the two strategies occurs for $t = 0$ at the savings level of $W = 0.2$ (recall that $PV_0 = 0.82$) and it amounts to about 6 percentage points shift towards stocks for the CQP strategy. Thus the CQP weights tend to be slightly riskier than the fully optimal investment for middling savings levels.

Generally speaking, the agreement between $\pi^*$ and $\pi^{(2)}$ is guaranteed to be excellent for very low and very high savings levels, since in the former case both strategies invest the entire cash in hand in stocks, while in the latter case we have already seen the optimal weights of both strategies tend to the value $\pi^{(2)}(1)$ given in (4.1).

Let us now take a closer look at formula (5.1). By completing the square we have

$$\pi^{(2)}(\alpha) = \arg \min_{\pi \geq 0, \pi^T \mathbf{1} \leq \alpha} \left\{ \frac{1}{\gamma} (\mu - r)^T \sigma^{-1} \right\}^2. $$

(5.2)

Since the expression on the right-hand side of (5.2) is strictly convex in $\pi$, those constraints in (5.2) that are not binding can be safely removed and the binding constraints applied with equality. Therefore, if some constraints in (5.2) are active, (5.2) is equivalent to

$$\pi^{(2)}(\alpha) = \arg \min_{A_2 \pi^T = b_2} \| A_1 \pi^T - b_1 \|^2, $$

(5.3)
where $A_1 = \sigma^\top$, $b_1 = \sigma^{-1}(\mu - r)/\gamma$ and $A_2, b_2$ represent the active constraints. Assuming that at least one constraint is active, the solution of (5.3) is given by

$$
\pi^{(2)}(\alpha) = A_1^{-1}b_1 + (A_1^\top A_1)^{-1}A_2^\top(A_2(A_1^\top A_1)^{-1}A_2^\top)^{-1}(b_2 - A_2A_1^{-1}b_1).
$$

(5.4)

Suppose that the only active constraint in (5.1) is

$$\pi_1 = \alpha.
$$

(5.5)

In this case $A_2 = 1 = (1, 1, \ldots, 1) \in \mathbb{R}^d$, $b_2 = \alpha$ and (5.4) takes the form

$$
\pi^{(2)}(\alpha) = \frac{\hat{\pi}}{\gamma} + \frac{1^\top (\sigma \sigma^\top)^{-1}}{1^\top (\sigma \sigma^\top)^{-1}1}(\alpha - \frac{\hat{\pi}1}{\gamma}),
$$

(5.6)

where

$$\hat{\pi} = (\mu - r)^\top(\sigma \sigma^\top)^{-1}
$$

(5.7)

represents the optimal unit risk-aversion weights without credit constraint.

Recall that in our numerical illustration the lifestyleing correction vector is

$$\zeta = \frac{1^\top (\sigma \sigma^\top)^{-1}}{1^\top (\sigma \sigma^\top)^{-1}1} = (95.28\%, 4.72\%).
$$

For high level of risk aversion $\gamma = 8$ the constraint $\pi 1 \leq \alpha$ becomes binding below $\hat{\alpha} = \frac{5.85}{8} = 73\%$. The optimal investment switches 100% to stocks below $\alpha = 15.7\%$. For low level of risk aversion $\gamma = 2$ the constraint $\pi 1 \leq \alpha$ binds for all values of $\alpha \in [0, 1]$ and the investment switches fully into stocks for all $\alpha$ below 63.4\%. For risk aversion below $1.27 = \hat{\pi}1 - \hat{\pi}1/\zeta_1$ it is optimal to invest the entire cash in hand in stocks at all times.

6 The impact of leverage and contribution timing

In this section we will investigate the economic impact of the credit constraint faced by a savings plan in receipt of gradual contributions. As an unattainable benchmark we first consider the situation where all contributions are paid up-front in the amount of $PV_0$. In the second instance we look at the effect of leverage in alleviating the credit constraint by allowing higher volume of early investment in the riskier but more profitable stocks.

We do this by replacing the matrix $\sigma$ with $L\sigma$ and the parameter $\mu - r$ with $L(\mu - r)$, where $L = 2, 3$ stands for leverage. The certainty equivalents of the strategy with gradual contributions ($CE^*$), with higher leverage and with up-front payment are shown in Table [7].

The corresponding levels of certainty equivalent internal rate of return are given in Table [8].

The economic value of early contributions is clearly significant. Higher leverage is only an imperfect substitute for early contributions. The effect of contribution timing is relatively modest for high risk aversion, but for low aversion the gain from using leverage of 2 is 63 basis points annually over 40 years and for leverage of 3 the gain goes up to 81 basis points. With up-front contribution the gain is as high as 122 basis points per annum.
Table 7: Certainty equivalent wealth for models with higher leverage or upfront investment.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>CE*</th>
<th>Leverage 2</th>
<th>Leverage 3</th>
<th>Up-front</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.6503</td>
<td>4.3232</td>
<td>4.5423</td>
<td>5.1016</td>
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<tr>
<td>5</td>
<td>2.1771</td>
<td>2.2654</td>
<td>2.2875</td>
<td>2.4813</td>
</tr>
<tr>
<td>8</td>
<td>1.8161</td>
<td>1.8428</td>
<td>1.8480</td>
<td>1.9151</td>
</tr>
</tbody>
</table>

Table 8: Internal rates of return for models with higher leverage or upfront investment.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>IRR*</th>
<th>Leverage 2</th>
<th>Leverage 3</th>
<th>Up-front</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5.50%</td>
<td>6.13%</td>
<td>6.31%</td>
<td>6.72%</td>
</tr>
<tr>
<td>5</td>
<td>3.49%</td>
<td>3.65%</td>
<td>3.69%</td>
<td>4.02%</td>
</tr>
<tr>
<td>8</td>
<td>2.74%</td>
<td>2.80%</td>
<td>2.81%</td>
<td>2.96%</td>
</tr>
</tbody>
</table>

7 Conclusions

We have considered optimal investment for an individual savings plan receiving gradual contributions against which it cannot borrow. Using expected power utility as an optimality criterion we have shown that in the presence of credit constraints the Samuelson paradigm of investment in constant proportions out of total wealth including current savings and present value of future contributions changes substantially in two important respects. Firstly, for high levels of accumulated savings the relative investment in risky bonds and stocks becomes a function of investor’s risk aversion, with strong substitution from bonds towards stocks for lower values of risk aversion. Secondly, for low levels of accumulated savings it becomes optimal to switch entirely to stocks, in a strategy that resembles stochastic lifestyling of Cairns et al. (2006).

Since the computation of the fully optimal strategy is prohibitively technical for practitioners, we have proposed a quasi-optimal strategy involving only a static constrained quadratic programme, easily implementable in a spreadsheet. We have shown numerically that the CQP strategy is practically indistinguishable from the optimal investment in terms of its welfare implications. We have provided an explicit formula (5.6) which helps visualize the lifestyling effect and further lowers the technical barrier towards its implementation.

In the concluding part of the paper we have examined the welfare effect of the credit constraint and the role of leverage in potentially alleviating the effects of borrowing restrictions. We have found that the welfare effect of credit constraint is moderate for high levels of risk aversion, but it is highly significant for moderate and low risk aversion. In the latter case we have found that plausible levels of leverage are only a partial substitute to investing the present value of all future contributions up-front.

The main message of the paper is that naive rescaling of the classical Samuelson strategy
to meet the credit constraint is a poor investment advice. Instead, our research, together with several previous studies in optimal lifecycle portfolio allocation, recommends strategies that switch from stocks to bonds as the amount of accumulated capital increases. Importantly, this paper provides a simple formula that captures the main essence of the lifestyling effect.

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References


