Dividend Discount Approach to Equity Derivatives Modelling

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There are various directions of research on the topic of dividends:

- **Empirical properties, forecasting:**
  - van Binsbergen et al. [vBBK10]

- **Dividends within equity dynamics:**
  - Bos-Vandermark [BV02] (closed form formula with discrete dividends)
  - Overhaus et al. [OFK+02] and [OBB+07] (affine model)
  - Vellekoop, Nieuwenhuis [VN06] (model survey)

- **Dividend discount models:**
  - Korn-Rogers [KR05] (dividend announcement)
  - Bernhart, Mai [BM12] (general framework)

- **Dividend derivatives modelling:**
  - Tunaru [Tun14] (uncertain timing, cum dividend process)
  - Bühler et al. [BDS10] (discrete dividends with stochastic yield)
Equity dynamics

When modelling equity dynamics there are two popular ways of incorporating dividends:

- **Continuous dividend payment stream**, proportional to spot level $S$, namely **continuous dividend yield** $q$:

  $$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma_t dW_t$$

- **Discrete dividends** with known future ex-dividend dates $0 < t_1 < t_2 < \ldots$, namely

  $$D_{t_i}^i = f_i(S_{t_i -})$$

with deterministic $f_i$. In practice one often assumes

- deterministic payments in the near future: $f_i = c_i$
- proportional dividends for the distant future: $f_i(S_{t_i -}) = c_i S_{t_i -}$
- Positivity of $S$ is preserved if $f_i$ is capped at $S_{t_i -}$ or $f_i(x) \leq x$. 
Dividend discount models assume that the stock price $S_t$ can be viewed as the present value of all future dividends, namely

$$S_t = \sum_{t_i > t} D_t^i$$

(1)

where $D_t^i$ denotes the value discounted to $t$ of the $i$-th dividend going ex at time $t_i$. Also, for any given stock model $(S_t, t \geq 0)$ with discrete dividends one has

$$S_{t_i^-} - S_{t_i} = D_{t_i}^i$$

(2)

In this presentation we address the following questions:

1. Given a stock model $(S_t, t \geq 0)$ with discrete dividends, are there processes $D^i$ such that (1) holds?
2. Do these processes $D^i$ have ‘reasonable’ properties?
3. What is the dynamics of $(S_t, t \geq 0)$ if one starts with ‘reasonable’ processes $D^i$?
Calibration

Let $F_t$ and $P_t$ denote equity forward and discount factor, respectively. One has the calibration condition

$$P_t F_t = S_0 - \sum_{t_i \leq t} D^i_0 = \sum_{t_i > t} D^i_0$$

(3)

If a stock does not pay dividends then one may assume a single dividend payment at infinity with value $D_t = S_t$ at time $t$. Otherwise one can assume $D^i_0$ to be known for all $i$ up to some index $k$. With exponential growth thereafter, namely

$$D^i_0 = D^k_0 e^{-q \Delta (i-k)} \quad \text{and} \quad t_i = t_k + \Delta (i-k)$$

for all $i \geq k$ and some time increment $\Delta$ one computes

$$q \Delta = \log \frac{S_0 - \sum_{i<k} D^i_0}{S_0 - \sum_{i \leq k} D^i_0} = \log \frac{P_{t_{k-1}} F_{t_{k-1}}}{P_{t_k} F_{t_k}}$$

from (1) or (3) with $t = 0$. Growth parameter $q$ can be interpreted as long term dividend yield.

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In a famous article Bos and Vandermark [BV02] present a variant of the Black Scholes formula, namely

$$C^{BV}(K, t) = P_t \text{Black} \left( P_t^{-1}(S_0 - X^n_t), K + P_t^{-1}X^f_t, \sigma^2 t \right)$$

(4)

with deterministic functions

$$X^n_t = \sum_{t_i \leq t} \frac{(t - t_i)^+}{t} D^i_0$$

$$X^f_t = \sum_{t_i \leq t} \frac{t_i \wedge t}{t} D^i_0$$

and volatility $\sigma$.

This formula is a good approximation for a European option price with deterministic dividend drops and volatility $\sigma$ in between. Additionally, the formula is continuous both as valuation date moves across a dividend and maturity moves across a dividend. The model as defined in their paper depends on the maturity $t$ of the European option.
We now fix $t$ independently of the product, namely

$$X_{t}^{n,T} = \sum_{t_{i} \leq t} \frac{(T - t_{i})^{+}}{T} D_{0}^{i} \quad X_{t}^{f,T} = \sum_{t_{i} \leq t} \frac{t_{i} \wedge T}{T} D_{0}^{i}$$

Inserting $X_{t}^{n,T}, X_{t}^{f,T}$ rather than $X_{t}^{n}, X_{t}^{f}$ into formula (4) implies a stock process $S$ given as

$$P_{t}S_{t} = (S_{0} - X_{t}^{n,T})M_{t} - X_{t}^{f,T}$$

where $M$ is a geometric Brownian motion with mean 1 and volatility $\sigma$. This gives rise to affine dividends, namely

$$S_{t_{i}-} - S_{t_{i}} = P_{t_{i}}^{-1}D_{0}^{i} \left( \frac{(T - t_{i})^{+}}{T} M_{t_{i}} + \frac{t_{i} \wedge T}{T} \right)$$

$$= P_{t_{i}}^{-1}D_{0}^{i} \left( \frac{(T - t_{i})^{+}}{T} \frac{P_{t_{i}}S_{t_{i}-} + X_{t_{i}-}^{f,T}}{S_{0} - X_{t_{i}-}^{n,T}} + \frac{t_{i} \wedge T}{T} \right) = \alpha_{i} + \beta_{i}S_{t_{i}-}$$

Note that $\alpha_{i} \downarrow 0$ for $t_{i} \downarrow 0$ and $\beta_{i} \downarrow 0$ for $t_{i} \uparrow T$. 
It is now standard to model dividends as affine functions of the spot at the ex-date

\[ D_{t_i}^i = \alpha_i + \beta_i S_{t_i} \]  

(5)

with for example \( \beta_i = 0 \) for \( t_i < 1y \) and \( \alpha_i = 0 \) if \( t_i \geq 5y \).

This is equivalent to the stock process being an affine function in the sense that

\[ S_t = (F_t - C_t) M_t + C_t \]  

(6)

holds for some martingale \( M \) and a deterministic function \( C \).

This approach dates back to [BFF⁺00] p7f. More recent references include Overhaus et al. [OFK⁺02] p116ff as well as [OBB⁺07] p6ff.
More formally, one has (compare Overhaus et al. [OBB$^+$07] p7):

**Proposition (affine stock model):** Assume a stock process $S$ with jumps due to dividends only. The following two statements are equivalent:

1. The stock process $S$ is positive and pays positive affine dividends (5) satisfying

\[
\alpha_i = 0 \quad \beta_i \in [0, 1) \quad P_t \alpha_i \geq -\beta_i \sum_{j \geq i} P_{t_j} \alpha_j \prod_{k=i}^j (1 - \beta_k)^{-1} \quad \text{for all } i \geq n \text{ for some } n
\]

2. The stock process $S$ is an affine function (6) of a positive continuous martingale $M$ with $M_0 = 1$ where $C$ satisfies

\[
C_t = 0 \quad \Delta F_t \leq \Delta C_t \leq 0 \quad dC_t = rC_t
\]

with notation $\Delta f_t \equiv f_t - f_{t-}$ for a given càdlàg function $f$. 
(2) ⇒ (1): Substituting (6) into the dividend drop $S_{t_i} - S_{t_i}$ yields

$$S_{t_i} - S_{t_i} = (-\Delta F_{t_i} + \Delta C_{t_i}) M_{t_i} - \Delta C_{t_i}$$

$$= (-\Delta F_{t_i} + \Delta C_{t_i}) \frac{S_{t_i} - C_{t_i}}{F_{t_i} - C_{t_i}} - \Delta C_{t_i} \geq 0$$

and hence

$$\alpha_i = \frac{C_{t_i} - \Delta F_{t_i}}{F_{t_i} - C_{t_i}}$$

$$\beta_i = -\frac{\Delta(F_{t_i} - C_{t_i})}{F_{t_i} - C_{t_i}}$$

One computes

$$\alpha_i + \beta_i C_{t_i} = -\Delta C_{t_i} = C_{t_i} - \frac{P_{t_{i+1}}}{P_{t_i}} C_{t_{i+1}}$$

and hence

$$P_{t_i} C_{t_i} = \frac{P_{t_i} \alpha_i}{1 - \beta_i} + \frac{P_{t_{i+1}} C_{t_{i+1}}}{1 - \beta_i} = \ldots = \sum_{j \geq i} P_{t_j} \alpha_j \prod_{k=i}^{j} (1 - \beta_k)^{-1}$$

The properties of $C$ imply the required properties of $\alpha_i$ and $\beta_i$ for all $i$ as well as positivity of $S$. 

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Affine stock process

(1) ⇒ (2): Both \( C \) and \( M \) can be constructed by induction on \([t_{i-1}, t_i)\) if given on \( t \geq t_i \) such that (6) holds:

- For \( t \geq t_n \) define \( C_t = 0 \) as well as \( M_t \) through \( S_t = F_t M_t \).
- If \( C \) is defined for \( t \geq t_i \) set

\[
- \Delta C_{t_i} = \alpha_i + \beta_i C_{t_i} = \frac{\alpha_i + \beta_i C_{t_i}}{1 - \beta_i} \tag{7}
\]

This defines \( C \) and hence \( M \) through (6) on \( t \geq t_i - 1 \), and one has

\[
\Delta S_{t_i} = (\Delta F_{t_i} - \Delta C_{t_i}) \frac{S_{t_i} - C_{t_i}}{F_{t_i} - C_{t_i}} + \Delta C_{t_i} = \ldots = -\beta_i S_{t_i} - \alpha_i
\]

where we used

\[
-\Delta F_{t_i} = \alpha_i + \beta_i F_{t_i} = \frac{\alpha_i + \beta_i F_{t_i}}{1 - \beta_i}
\]

Note that as seen above (7) yields via induction

\[
P_{t_i} C_{t_i} = \sum_{j \geq i} P_{t_j} \alpha_j \prod_{k=i}^{j} (1 - \beta_k)^{-1}
\]

Hence \( C \) has the required properties. Continuity of \( M \) at \( t_i \) can also be established.
Examples

Special cases include

- Proportional dividends: \( C_t = 0 \)
- Deterministic dividends up to some final date \( T \):
  \[
P_t C_t = \sum_{T \geq t_i > t} D_0^i
  \]
  Without \( T \) one has the deterministic case \( S_t = F_t \) due to (3).
- Blend model:
  \[
P_t C_t = \sum_{t_i > t} (1 - \tau_i) D_0^i
  \]  \tag{8}
  with some increasing sequence \( \tau \) in \([0, 1]\).
- We have seen above that Bos-Vandermark is a model with affine dividends. Since \( C_t \leq 0 \) the stock process can be negative.
Affine dividend dynamics

So far we only referred to expected discounted dividend $D_0^i$ as well as dividend drop $D_{t_i}^i$. The next step is to define $D_t^i$ for $t < t_i$.

We obtain for the affine model (6) from (2)

$$D_{t_i}^i = \left( P_t^{-1} D_0^i - \Delta C_{t_i} \right) M_{t_i} + \Delta C_{t_i}$$

For $t \leq t_i$ we may replace $D_{t_i}^i, P_{t_i}, M_{t_i}$ by $D_t^i, P_t, M_t$, namely

$$P_t D_t^i = \left( D_0^i - P_t \Delta C_{t_i} \right) M_t + P_t \Delta C_{t_i}$$

Summation over all $i$ with $t_i > t$ yields

$$P_t S_t = \left( P_t F_t - \sum_{t_i > t} P_t \Delta C_{t_i} \right) M_t + \sum_{t_i > t} P_t \Delta C_{t_i}$$

$$= P_t (F_t - C_t) M_t + P_tC_t$$

We summarize:
Proposition (affine dividends): Dividend dynamics

\[
\begin{align*}
\frac{dD^i_t}{t} &= r_tD^i_t dt + P_t^{-1} \left(D^i_0 - P_t \Delta C_t \right) dM_t \\
&= r_tD^i_t dt + P_t^{-1}D^i_0 \tau_i dM_t
\end{align*}
\]

(9)

together with (1) implies the affine stock model (6).

Remarks:

1. Dividend volatility is homogenous in \( t \) in the sense that volatility does not decrease as \( t \) approaches \( t_i \) (assuming \( M \) has constant volatility).

2. Dividend dynamics is not homogenous in \( i \) since dividend volatility scales with \( \tau_i \). The effect is non-homogenous equity returns: near returns experience deterministic dividends while far returns have proportional dividends.
Homogenous dividend dynamics

**Definition:** Homogenous dividend dynamics assumes an increasing function $\tau$ with values in $[0, 1]$ and $\tau_0 = 0$ as well as

$$dD^i_t = r_tD^i_t dt + P_{t}^{-1}D_0^i\tau_{t_i-t}dM_t$$

(10)

The difference with the blend model (9) is that the impact $dM_t$ is scaled down as the ex-dividend date approaches.

The solution is given as

$$P_tD^i_t = D_0^i \left(1 + \int_0^t \tau_{t_i-s}dM_s \right)$$

In case of continuous $\tau$ integration by parts yields

$$P_tD^i_t = D_0^i \left(1 + \tau_{t_i-t}M_t - \tau_{t_i}M_0 - \int_0^t M_s d\tau_{t_i-s} \right)$$

Note that $M \geq 0$ and $M_0 = 1$ guarantees positive dividends.
Proposition (homogenous dividends): With homogenous dividend dynamics the stock process is given as sum of an affine function and an integral of the martingale $M$. More specifically:

$$S_t = F_t + P_t^{-1} \left( \mathcal{T}_t^t M_t - \mathcal{T}_0^t M_0 - \int_0^t M_s d\mathcal{T}_s^t \right)$$

with $\mathcal{T}_s^t$ for $0 \leq s \leq t$ defined as

$$\mathcal{T}_s^t = \sum_{t_i > t} D_0^i \tau_{t_i - s}$$

Proof: One has

$$P_t S_t = P_t \sum_{t_i > t} D_t^i$$

$$= \sum_{t_i > t} D_0^i \left( 1 + \tau_{t_i - t} M_t - \tau_{t_i} M_0 - \int_0^t M_s d\tau_{t_i - s} \right)$$
Examples: indicator function

Dividend announcement at a fixed time interval $a$ before the ex-date is represented by the indicator function

$$\tau_t = 1_{\{t > a\}}$$

Assuming $M_0 = 1$ one has

$$P_t D_t^i = D_0^i \left( 1 + \int_0^t 1_{\{t_i - s > a\}} dM_s \right)$$

$$= D_0^i M_{t \wedge (t_i - a)^+}$$

and

$$S_t = \sum_{t_i > t} D_t^i = P_t^{-1} \left( \sum_{t_i - a > t} D_0^i M_t + \sum_{t_i > t \geq t_i - a} D_0^i M_{(t_i - a)^+} \right)$$  (11)
The situation discussed by Korn and Rogers [KR05] is obtained by further specializing

\[ t_i = ih, \quad a = (1 - \varepsilon)h \geq 0, \quad M_t = e^{-\mu t} X_t / X_0, \quad P_t = e^{-rt} \]

where \( X \) is an exponential Lévy process with mean

\[ E [ X_t ] = X_0 e^{\mu t} \]

as well as

\[ D_0^i = \lambda X_0 P_{t_i - a} e^{\mu (t_i - a)} \]
Formula (11) yields

\[ S_t = \lambda \sum_{(i-1+\varepsilon)h > t} e^{(\mu - r)((i-1+\varepsilon)h - t)} X_t \]

\[ + \lambda \sum_{ih > t \geq (i-1+\varepsilon)h} e^{r(t-(i-1+\varepsilon)h)} X_{(i-1+\varepsilon)h} \]

The second sum is either empty if the next dividend after \( t \) has not been announced yet or has exactly one term.

Note that formula (11) also covers the case of longer announcement periods, namely \( a > h \) or \( \varepsilon < 0 \). In that case the second sum consisting of all dividends announced at \( t \) may have more than one term.
An alternative to the indicator function is the piecewise linear function

\[ \tau_t = \frac{t}{a} \wedge 1 \]

where \( a \) is the time distance beyond which a given dividend has maximal volatility. One obtains for \( t \geq a \)

\[
P_t S_t = \left( P_{a+t} F_{a+t} + \sum_{a+t \geq t_i > t} D_0^i \frac{t_i - t}{a} \right) M_t + a^{-1} \sum_{a+t \geq t_i > t} D_0^i \int_{(t_i-a)^+}^{t} M_s ds
\]

For \( t < a \) one has to add the term

\[
\sum_{a \geq t_i > t} D_0^i \left( 1 - \frac{t_i}{a} \right)
\]

representing the announced part of near dividends.
Examples: exponential

With

$$\tau_t = 1 - e^{-\lambda t}$$

one obtains

$$\mathcal{T}_s^t = P_t \left( F_t - e^{\lambda s} C_t \right)$$

$$d\mathcal{T}_s^t = \lambda P_t C_t e^{\lambda s} ds$$

where $C_t$ is defined through

$$P_t C_t = \sum_{t_i > t} D_0^i e^{-\lambda t_i}$$

Thus

$$S_t = C_t + \left( F_t - e^{\lambda t} C_t \right) M_t + \lambda C_t \int_0^t e^{\lambda s} M_s ds$$

The corresponding non-homogenous affine dividend model (6) is

$$S_t = C_t + (F_t - C_t) M_t$$
Examples: exponential

$\lambda = 0.5$, $\sigma = 20\%$, $r = 5\%$, annual dividends $d = 10\%$ paid before maturity.
Affine stock models (including Bos-Vandermark) can be viewed as dividend discount models where all dividends are driven by the same martingale factor. The volatility level of each dividend does not change through time.

Homogenous dividend models behave like bonds in the sense that their volatility level decreases through time, reaching zero at the ex-date.

The stock return distribution of homogenous dividend models is more reasonable as one does not shift from affine for near returns to proportional for far returns.

Modelling homogenous dividend models is more involved since the stock process is a time average of a positive martingale.
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Appendix: stochastic dividend yield

Bühler [BDS10] suggests an Ornstein-Uhlenbeck process \( y \) driving dividend yield. The resulting dynamics can be written as

\[
S_t = F_t N_{t_i} M_t \quad t_i \leq t < t_{i+1}
\]

where \( N \) is an adapted process of the form

\[
N_t = \frac{S_0}{F_t} e^{-\sum_{t_i \leq t} d_i} = \frac{S_0}{F_t} e^{-\sum_{t_i \leq t} C_i + D_i + E_i y_t} = e^{-\sum_{t_i \leq t} C_i + E_i y_t}
\]

with \( E \left[ M_{t_i} N_{t_i} \right] = 1, \ P_{t_i} F_{t_i} = e^{-D_i} P_{t_{i-1}} F_{t_{i-1}} \) as well as \( E_i \in \{1, D_i\} \).

Remark: It is not required that \( N \) or \( NM \) is a martingale and indeed

\[
E_{t_{i-1}} \left[ N_{t_i} \right] = N_{t_{i-1}} e^{-C_i} E_{t_{i-1}} \left[ e^{-E_i y_{t_i}} \right] \neq N_{t_{i-1}}
\]
Appendix: stochastic dividend yield

The dividend drop $D_{t_i}^i$ satisfies

$$P_{t_i}D_{t_i}^i = -P_{t_i}\Delta S_{t_i} = (P_{t_i-1}F_{t_i-1}N_{t_i-1} - P_{t_i}F_{t_i}N_{t_i}) M_{t_i}$$

A consistent dividend dynamics for $t \leq t_i$ can be defined as

$$P_t D_t^i = (P_{t_i-1}F_{t_i-1}N_{t_i-1} \wedge t - P_{t_i}F_{t_i}N_{t_i} \wedge t) M_t$$

For $t \leq t_{i-1}$ this simplifies to

$$P_t D_t^i = D_0^i N_t M_t$$

One observes

$$P_t S_t = P_t \sum_{t_i > t} D_t^i = P_{t_i-1}F_{t_i-1}N_{t_i-1} M_t \quad \text{for} \quad t_{i-1} \leq t < t_i$$

as required. Bühler [BDS10] also discusses affine dividends

$$D_{t_i}^i = \alpha_i + \beta_i S_{t_i-} = \alpha_i + \left(1 - \frac{N_{t_i}}{N_{t_i-1}}\right) S_{t_i-}$$

Analysis of a version of this model with homogenous dividend dynamics is left for future research.