
Semi-Coherent Multi-Population Mortality Modeling: The Impact on Longevity Risk Securitization

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Developing a Large, Liquid Longevity Market: A Key Challenge

- ▶ The key challenge on the modeling front is to develop multi-population stochastic mortality models (Cairns, 2013).
- ▶ The roles of multi-population stochastic mortality models in securitization:
 - ▶ Quantification of population basis risk
 - ▶ Pricing and risk management for securities that are associated with multiple populations (e.g., the Kortis deal)
- ▶ Other applications of multi-population models:
 - ▶ Modeling both genders simultaneously
 - ▶ Adding credibility to projections for smaller populations

Recent Developments in Multi-Population Mortality Modeling

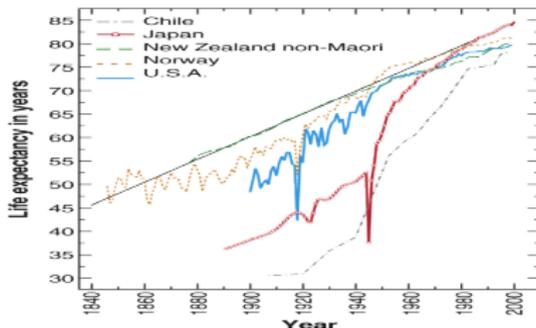
- ▶ Most of the recent developments are built upon a concept called **coherence** (Li and Lee, 2005).
- ▶ The coherence hypothesis:
 - ▶ The projected future mortality rates of two or more related populations do not diverge over the long run.
 - ▶ E.g., the augmented common factor model:
 - ▶ The main time-trend – a random walk with drift.
 - ▶ The population-specific time trends – AR(1).
 - ▶ Similar methods are used by, for example, Cairns et al. (2011), Dowd et al. (2011), Hatzopoulos and Haberman (2013), Jarner and Kryger (2011) and Zhou et al. (2014).

Grounds for the Coherence Hypothesis

- ▶ Mortality forecasts that diverge indefinitely are deemed biologically unreasonable.
- ▶ Empirical observations:
 - ▶ Wilson (2001) found a global convergence in mortality levels by comparing the distributions of the global population by life expectancy over three different time periods, 1950-55, 1975-80 and 2000.
 - ▶ A similar study by White (2002) using data from 21 high-income countries over the period of 1955 to 1996 also led to the same conclusion.

Evidence against the Coherence Hypothesis

- ▶ Oeppen and Vaupel (2002): The U.S. has been moving away from the best practice life expectancy.



- ▶ Waldron (2007) and Mackenbach et al. (2003): Individuals with higher incomes experienced faster mortality improvement.

Research Objectives

- ▶ To introduce a new concept called **semi-coherence**:
 - ▶ The mortality trajectories of two related populations are permitted to diverge, as long as the divergence does not exceed a specific tolerance corridor, beyond which mean-reversion will come into effect.
- ▶ To implement the concept of semi-coherence using a vector threshold autoregressive (VETAR) process.
- ▶ To address the impact of **hypothesis risk** on pricing and risk management.

Three Different Forms of Coherence

Non-Coherence

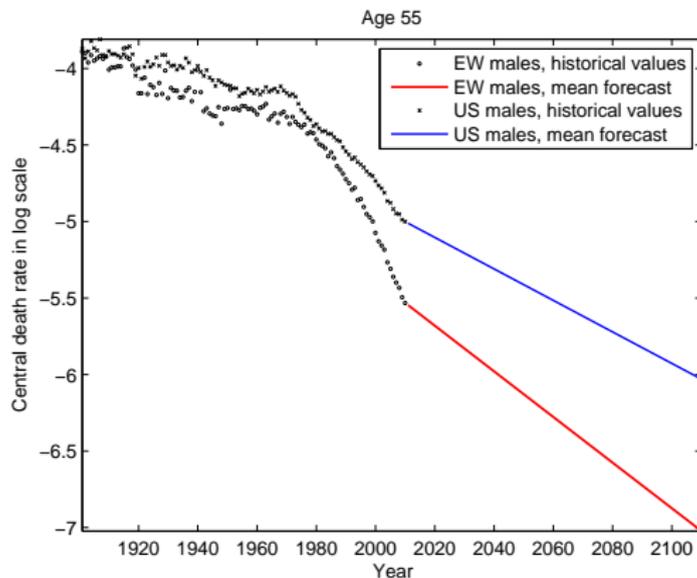
- ▶ An arbitrary combination of single-population models is generally non-coherent.
- ▶ E.g., a combination of two unrelated single-population Lee-Carter models:

$$\begin{aligned}\ln(m(x, t, i)) &= \alpha_x^{(i)} + \beta_x^{(i)} \kappa_t^{(i)}, & i = 1, 2, \\ \kappa_t^{(i)} &= c^{(i)} + \kappa_{t-1}^{(i)} + \epsilon_t^{(i)}, & i = 1, 2,\end{aligned}$$

where

- ▶ $m(x, t, i)$ is population i 's central death rate at age x and in year t ;
- ▶ $\alpha_x^{(i)}$, $\beta_x^{(i)}$ and $c^{(i)}$ are constants;
- ▶ $\epsilon_t^{(1)}$ and $\epsilon_t^{(2)}$ are independent normal random variables.

Non-Coherence



Forecasts of the central death rates at age 55 derived from two independent Lee-Carter models

Full-Coherence

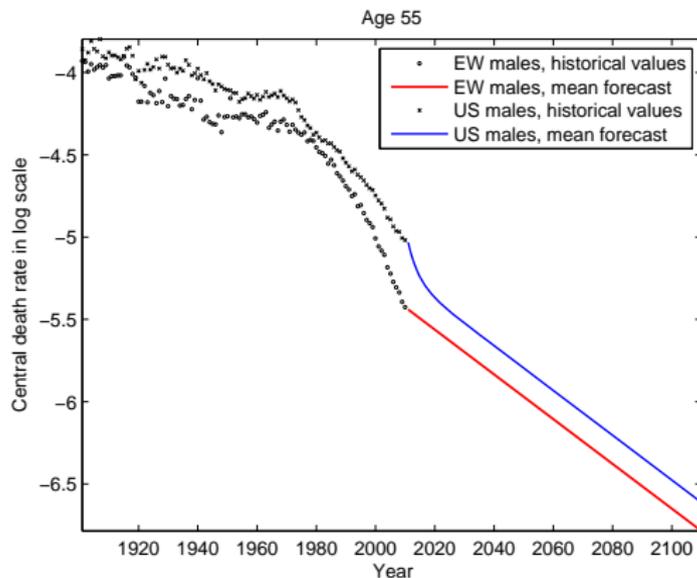
- ▶ Mathematically, full coherence implies that $m(x, t, i)/m(x, t, j)$ reverts to a constant as $t \rightarrow \infty$ for $i \neq j$.
- ▶ E.g., the two-population Lee-Carter model by Cairns et al. (2011):

$$\begin{aligned} \ln(m(x, t, i)) &= \alpha_x^{(i)} + \beta_x \kappa_t^{(i)}, \quad i = 1, 2, \\ \kappa_{t+1}^{(1)} &= \kappa_t^{(1)} + \mu_\kappa + Z_\kappa(t+1), \\ \Delta_\kappa(t) &= \kappa_t^{(1)} - \kappa_t^{(2)}, \\ \Delta_\kappa(t+1) &= \mu_{\Delta_\kappa} + \phi_{\Delta_\kappa} \Delta_\kappa(t) + Z_{\Delta_\kappa}(t+1), \end{aligned}$$

where

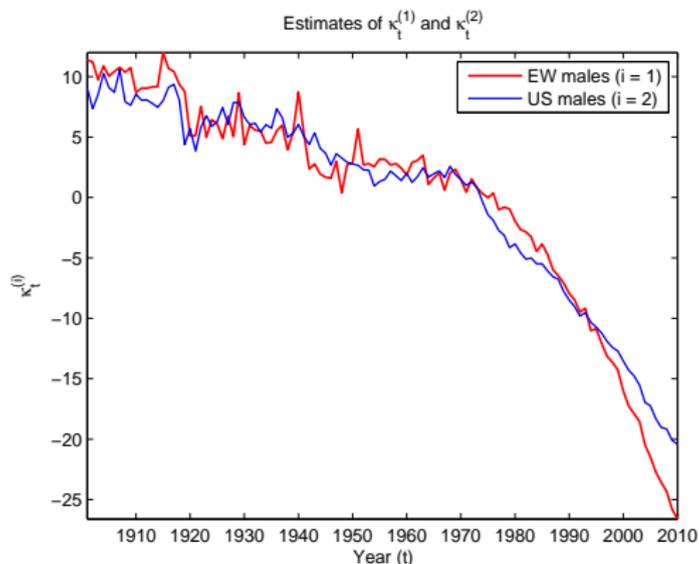
- ▶ $\mu_\kappa, \mu_{\Delta_\kappa}, \phi_{\Delta_\kappa}$ and β_x are constants;
- ▶ ϕ_{Δ_κ} such that $|\phi_{\Delta_\kappa}| < 1$;
- ▶ $(Z_\kappa(t), Z_{\Delta_\kappa}(t))'$ is a zero-mean bivariate normal random vectors.

Full-Coherence



Forecasts of the central death rates at age 55 derived from the fully-coherent two-population Lee-Carter model

Limitations of Full-Coherence



The estimates of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ in the Lee-Carter model, $t = 1901, \dots, 2010$

Semi-Coherence

- ▶ Less restrictive than full-coherence, but still includes some sort of mean-reversion.
- ▶ A tolerance corridor is imposed on the mortality differential between two populations.
 - ▶ If the tolerance corridor is not exceeded, the expected mortality trajectories of the two populations can diverge.
 - ▶ Otherwise, the system would switch to another state, in which the expected paces of mortality reduction for would be adjusted so that the differential (in absolute value) is expected to shrink over time.

Properties of Semi-Coherent Mortality Forecasts

- ▶ Mean-reversion does not necessarily begin at the forecast origin.
- ▶ The expected mortality trajectories of the two populations being modeled can diverge over certain periods of time, thereby covering a wider range of possible mortality differentials.
- ▶ Mean-reversion comes into effect at random time points when the tolerance corridor is exceeded. The intermittent mean-reversions prevent the expected trajectories from diverging indefinitely.

Using the VETAR Process for Semi-Coherent Mortality Forecasting

The Set-up

- ▶ The Lee-Carter with common age response parameters:
 $\ln(m(x, t, i)) = \alpha_x^{(i)} + \beta_x \kappa_t^{(i)}, \quad i = 1, 2.$
- ▶ Model $Z_t = (\Delta \kappa_t^{(1)}, \Delta \kappa_t^{(2)})'$ with a VETAR process:

$$Z_t = \begin{cases} \phi^{(1)} + \sum_{j=1}^{p_1} \Phi_j^{(1)} Z_{t-j} + a_t^{(1)}, & y_{t-d} \leq r_1 \\ \phi^{(2)} + \sum_{j=1}^{p_2} \Phi_j^{(2)} Z_{t-j} + a_t^{(2)}, & r_1 < y_{t-d} \leq r_2 \\ \phi^{(3)} + \sum_{j=1}^{p_3} \Phi_j^{(3)} Z_{t-j} + a_t^{(3)}, & r_2 < y_{t-d} \end{cases},$$

where

- ▶ $\phi^{(g)}$ is a constant 2×1 vector, $g = 1, 2, 3$;
- ▶ $\Phi_j^{(g)}$ is a 2×2 constant matrix, $g = 1, 2, 3, j = 1, \dots, p_g$;
- ▶ $a_t^{(g)}$ is a 2×1 normal random vector, $g = 1, 2, 3$;
- ▶ r_1 and r_2 are constants (the threshold parameters).

The Set-up

- ▶ We define threshold random variable as

$$y_{t-d} = \frac{1}{5} \sum_{i=0}^4 (\kappa_{t-d-i}^{(1)} - \kappa_{t-d-i}^{(2)}),$$

where d is the delay parameter.

- ▶ We can view y_t as the average difference in the general mortality levels between the two populations over a lookback period of 5 years.
- ▶ The interval $[r_1, r_2)$ can be seen as the tolerance corridor.

Intermittent Mean Reversions

- ▶ Define

$$(\mu^{(g)}(1), \mu^{(g)}(2))' = (I - \sum_{j=1}^{p_g} \Phi_j^{(g)})^{-1} \phi^{(g)},$$

where I is the 2×2 identity matrix.

- ▶ We can regard $\mu^{(g)}(i)$ as the steady-state expected rate of change in $\kappa_t^{(i)}$ in the g th regime.
- ▶ We expect $\mu^{(g)}(i) < 0$ for $g = 1, 2, 3$ and $i = 1, 2$.
- ▶ We also expect $\hat{\mu}^{(1)}(1) > \hat{\mu}^{(1)}(2)$ and $\hat{\mu}^{(3)}(1) < \hat{\mu}^{(3)}(2)$, which jointly ensures the forecasts are semi-coherent.

The Estimated Parameters

- ▶ The optimal model:

$$d = 2, \rho_1 = \rho_2 = \rho_3 = 2, (r_1, r_2] = (-0.02997, 0.12544].$$

The first regime ($g = 1$)

$$\phi^{(1)} = \begin{pmatrix} -0.043 \\ -0.027 \end{pmatrix} \quad \Phi_1^{(1)} = \begin{pmatrix} 0.000 & -0.752 \\ 0.323 & -0.487 \end{pmatrix} \quad \Phi_2^{(1)} = \begin{pmatrix} 0.000 & 0.000 \\ 0.247 & 0.000 \end{pmatrix}$$

The second regime ($g = 2$)

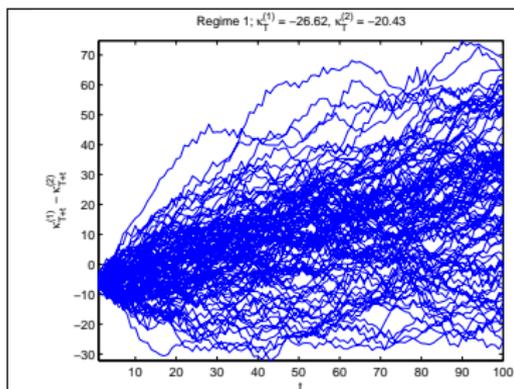
$$\phi^{(2)} = \begin{pmatrix} -0.041 \\ -0.027 \end{pmatrix} \quad \Phi_1^{(2)} = \begin{pmatrix} -0.615 & 0.481 \\ 0.000 & -0.354 \end{pmatrix} \quad \Phi_2^{(2)} = \begin{pmatrix} -0.313 & 0.000 \\ 0.134 & 0.000 \end{pmatrix}$$

The third regime ($g = 3$)

$$\phi^{(3)} = \begin{pmatrix} -0.054 \\ -0.007 \end{pmatrix} \quad \Phi_1^{(3)} = \begin{pmatrix} 0.000 & 0.000 \\ 0.000 & 0.000 \end{pmatrix} \quad \Phi_2^{(3)} = \begin{pmatrix} 0.000 & 0.000 \\ 0.493 & 0.000 \end{pmatrix}$$

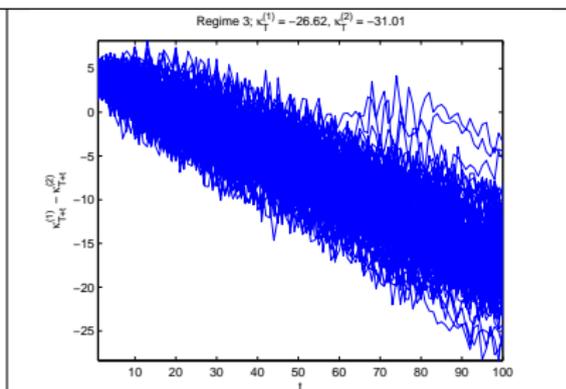
- ▶ The inequalities $\hat{\mu}^{(1)}(1) > \hat{\mu}^{(1)}(2)$ and $\hat{\mu}^{(3)}(1) < \hat{\mu}^{(3)}(2)$ hold.

Intermittent Mean Reversion : An Illustration



Sample paths simulated from the VAR in **Regime 1**

(when $y_t = \kappa_t^{(1)} - \kappa_t^{(2)}$ is too small)



Sample paths simulated from the VAR in **Regime 3**

(when $y_t = \kappa_t^{(1)} - \kappa_t^{(2)}$ is too large)

The Candidate Models

All candidate models are built on the same base model structure:

$$\ln(m(x, t, i)) = \alpha_x^{(i)} + \beta_x \kappa_t^{(i)}, \quad i = 1, 2.$$

▶ **Model NC**

The evolutions of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are modeled by two independent random walks with drift.

▶ **Model SC**

The evolutions of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are modeled by the VETAR process (the proposed approach).

▶ **Model FC-A**

The evolutions of $\kappa_t^{(1)}$ and $\kappa_t^{(1)} - \kappa_t^{(2)}$ are modeled by a random walk with drift and an AR(1), respectively (Cairns et al., 2011).

▶ **Model FC-B**

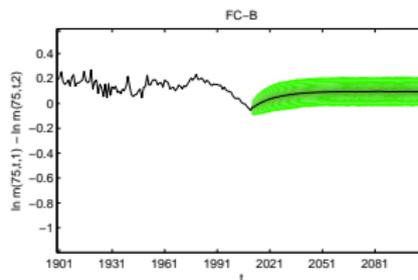
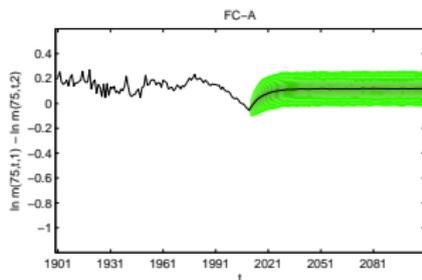
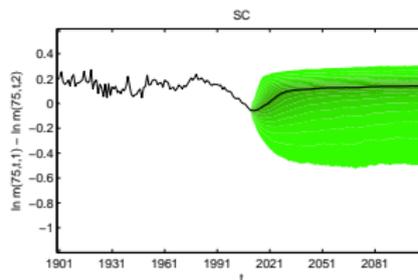
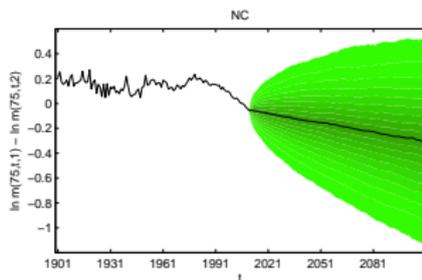
The evolutions of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are modeled by a special VECM that is configured to ensure full-coherence (Zhou et al., 2014).

Goodness-of-fit

	Period effect process	AIC
Model NC	Two independent random walks	42.741
Model SC	A VETAR process	-38.772
Model FC-A	A random walk and an AR(1)	35.266
Model FC-B	A special VECM	15.237

The AIC values of the estimated processes for the period effects $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$

Properties of the Resulting Mortality Forecasts



Forecasts of $\ln m(75, t, 1) - \ln m(75, t, 2)$, generated from Models NC, SC, FC-A and FC-B

The Impact on Longevity Risk Management

The Set-up

- ▶ The liability being hedged
 - ▶ A life annuity sold to EW males who are currently aged 60
 - ▶ Payable at the end of each year until death or age 85, whichever the earliest

- ▶ Two static hedges
 1. Two S-forwards
 - ▶ Payoffs linked to the realized survival rates of US males currently aged 60; mature 5 and 25 years from now
 2. Five q-forwards
 - ▶ Payoffs linked to the realized death probabilities of US males currently aged 60; mature 6, 11, 16, 21 and 26 years from now

The Best Achievable Hedge Effectiveness

- ▶ The minimum attainable variance

$$\min_{w_1, \dots, w_k} \text{Var} \left(L - \sum_{i=1}^k w_i H_i \right),$$

where w_i is the weight on the i th hedging instrument, and L and H_i are the random present values of the liability and the i th hedging instrument, respectively.

- ▶ The best achievable hedge effectiveness (% reduction in portfolio variance):

	NC	SC	FC-A	FC-B
Two S-forwards	0.05%	76%	90%	81%
Five q-forwards	0.06%	77%	90%	81%

Hedge Effectiveness Produced from a Duration-Matching Strategy

- ▶ The notional amounts of the hedging instruments are computed by “key q-durations”, a duration-matching strategy (Li and Luo, 2012).
- ▶ Adjustments for population basis risk are made, on the basis of the augmented common factor model.
- ▶ The hedge effectiveness achieved by the strategy:

	NC	SC	FC-A	FC-B
Five q-forwards	-14%	22%	51%	49%

The Impact on Pricing Mortality-Linked Securities

The Security Being Priced

- ▶ Similar to the Kortis deal launched by Swiss Re in 2011.
- ▶ A 10-year bond sold at par, pays quarterly coupons at 3-month LIBOR plus a spread.
- ▶ The principal is at risk, linked to the survival rate divergence index (SRDI), defined by

$$SRDI_t = S_t^{(1)} - S_t^{(2)},$$

where $S_t^{(1)}$ and $S_t^{(2)}$ are the realized t -year survival rate for EW and US males aged 65 at inception, respectively.

- ▶ The principal reduction is calculated as follows:

$$\min \left(\frac{\max(SRDI_{10} - ap, 0)}{ep - ap}, 1 \right),$$

where $ap = 0.05$ and $ep = 0.06$ are the attachment and exhaustion points, respectively.

The Pricing Method

- ▶ The bondholders are compensated by a spread over LIBOR (a risk premium).
- ▶ To determine the risk premium, the risk cubic pricing method (Lane, 2000) is used.
- ▶ The risk premium is the sum of the expected losses (EL) and the expected excess return (EER):

$$\log(EER) = \gamma + \alpha \log(PFL) + \beta \log(CEL),$$

where PFL is annualized probability of the first dollar loss and $CEL = EL/PFL$ is the conditional expected loss.

- ▶ Parameters γ , α and β are estimated by a linear regression fitted to the data associated with the transactions of mortality-linked securities from 2003 to 2010-Q1.

Pricing Results

- ▶ Risk-cubic pricing parameters:

γ	α	β
1.1075	1.0661	1.4119

- ▶ Pricing results:

Model	<i>PFL</i>	<i>PE</i>	<i>EL</i>	<i>CEL</i>	Spread
NC	0.86%	0.32%	0.55%	63.69%	740 bps
SC	0.49%	0.13%	0.28%	58.22%	357 bps
FC-A	0	0	0	—	—
FC-B	0	0	0	—	—

Conclusion

Concluding Remarks

- ▶ The concept of semi-coherence is proposed.
- ▶ The concept of semi-coherence is implemented with the VETAR process.
- ▶ The impact of hypothesis risk is demonstrated.
- ▶ Directions for further research:
 - ▶ Develop other semi-coherent multi-population mortality models
 - ▶ Develop statistical tests for various coherence hypotheses
 - ▶ Develop dynamic, state-dependent longevity hedging strategies