Price Bounds of Mortality-Linked Security in Incomplete Insurance Market

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Introduction

• Longevity and mortality catastrophe have increased insurers’ risk.

• Mortality securitization or mortality-linked securities (MLS) are regarded as a prescription to mitigate these risks.

• Two seminal innovations: the Swiss Re mortality bond issued in 2003 and the EIB/BNP longevity bond in 2004

• The MLS pricing issue also draws much attentions.
Purposes of this article

• Seek reasonable price bounds for MLS in incomplete market and take the Swiss Re mortality bond as an example.

• The gain-loss ratio is imposed as a subjective constraint on the potential MLS return to preclude too good investment opportunities.

• The price bounds will be useful for setting bid-asked spreads and making trading strategies for insurers.
Valuation approaches on MLS

• The risk-adjusted or no-arbitrage pricing method

• The Wang transform method
  – Lin and Cox (2008), Denuit et al. (2007) and Cox et al. (2006)

• Instantaneous Sharpe ratio
  – Milevsky et al. (2005), Bayraktar and Young (2007), Young (2008) and Bayraktar et al. (2009)

• Utility-based valuation
MLS Valuation Limitations 1

• The no-arbitrage pricing method
  – Use less model specifications on the economy, such as individual preference and wealth.
  – Require underlying transaction price.

• The utility-based method
  – Use restrictive assumptions such as specific utility function and aggregated wealth distribution
  – Suffering larger model risks and specification errors
Our Solutions 1

• Propose a framework that can accommodate model risk of the utility-based method when the transaction prices are unavailable, and that can keep the no-arbitrage pricing rule hold.

• No-arbitrage pricing still can be applied when we cave the MLS transaction prices.
MLS Valuation Limitations 2

• MLS market is an incomplete market
  —The underlying of MLS are usually not traded in financial markets
  —Insurers only have a limited hedging ability to replicate (can’t perfectly hedge) the MLS payoffs.
  —The no-arbitrage pricing method can only obtain a price range or a \textbf{price bound} instead of a single value.
Our Solution 2

• We explore rational price bounds of MLS under the insurers with a limited hedging ability.
• Try to narrow the no-arbitrage price bound by the gain-loss ratio method proposed by Bawa and Lindenberger (1977) and Bernardo and Ledoit (2000).
Intuitions of the gain-loss ratio 1

- The excess payoff

\[ y = x - e^{r^T P^*} \]  

– where \( x \) is the random payoff of MLS

– \( P^* \) is the benchmark price given by the benchmark model.

- Define

\[ \text{Gain-loss ratio} = \frac{E^Q[y^+]}{E^Q[y^-]} \]  

– where \( Q \) is risk-adjusted probability according to the benchmark model

– \( y^+ = \max(y, 0) \) the positive part of the excess payoff

– \( y^- = \max(-y, 0) \) the negative part of the excess payoff
Intuitions of the gain-loss ratio 2

• The benchmark price, $P^*$, is correct or fair, if and only if $E^Q[y] = 0$, or equivalently, the gain-loss ratio

$$E^Q[y^+]/E^Q[y^-] = 1.$$ 

• To recognize the mispricing from the benchmark model (model risk or specification errors) and still keep the price in a reasonable range, we allow the gain-loss ratio to deviate from one but not exceed the ceiling value $L$:

$$E^Q[y^+]/E^Q[y^-] \leq L$$
Theoretical parts

• Bernardo and Ledoit (2000) show a duality between the admissible pricing kernels and the gain-loss ratio as

\[
\begin{align*}
\text{Min} & \quad \left\{ m_j > 0, E[m_j x_j^{bi}] = p^{bi} \right\} \quad \sup_{j=1,\ldots,J} \left( \frac{m_j}{m_j^*} \right) \\
\text{inf}_{j=1,\ldots,J} \left( \frac{m_j}{m_j^*} \right) & = \text{Max} \quad \frac{E^Q[y^+]}{E^Q[y^-]} \\
& \quad \left(3\right)
\end{align*}
\]

where

- \( m_j \) is individual’s pricing kernels over state \( j \)
- \( m_j^* \) is benchmark pricing kernels over state \( j \)

• Therefore, imposing a restriction on the gain-loss ratio is equal to imposing a restriction on the pricing kernel.
The upper bounds

• Define the excess payoff for the upper bound

\[ y^u = x^b - x. \]

where \( x^b = \sum_i w_i x^{bi} \) is payoff of the basis asset portfolio formed by basis asset \( b_i \) with weight \( w_i \).

• The upper price bound for the MLS is obtained by solving the minimisation problem:

\[
\begin{align*}
\text{Min} & \quad \sum_{w_{bi}} w_{bi} p^{bi} - P^* \geq 0 \\
\text{s.t} & \quad \frac{E^Q[(y^u)^+]}{E^Q[(y^u)^-]} \geq L \quad \text{and} \quad x^b = \sum w_{bi} x^{bi},
\end{align*}
\]

where \( \sum_{i} w_{bi} p^{bi} \) is the replication cost and the lowest upper bound.
The lower bound

• Define the excess payoff for the lower bound

\[ y^l = x - x^b \]

where \( x^b = \sum_i w_i x^{bi} \) is payoff of the basis asset portfolio formed by basis asset \( b_i \) with weight \( w_i \).

• The **lower price bound** for the MLS is obtained by solving the minimisation problem:

\[
\text{Max}_{w_{bi}} \sum_i w_{bi} p^{bi} - P^* \leq 0
\]

s.t. \( \frac{E^Q[(y^l)^+]}{E^Q[(y^l)^-]} \geq L \) and \( x^b = \sum w_{bi} x^{bi} \),

where \( \sum_i w_{bi} p^{bi} \) is the replication cost and highest lower bound.
Example: The Swiss Re Mortality Bonds

• Swiss Re issues the mortality bond, at an amount of $400 million, with the payoff at maturity day:

\[
\text{loss}_t = \begin{cases} 
1 & \text{if } q_t > K_2q_0 \\
\left(q_t - K_1q_0\right)/(K_2 - K_1)q_0 & \text{if } K_1q_0 \leq q_t \leq K_2q_0 \\
0 & \text{if } q_t < K_1q_0.
\end{cases}
\]

where \( t = 1, 2, 3 \) for years 2004, 2005 and 2006

• The proportion of principal returned to the bondholders on the maturity date is

\[ x = \text{Max}\left(1 - \sum_t \text{loss}_t, 0\right). \]

• Following Equation (1), the mortality bond price at time 0 is

\[ P = E^Q[x]e^{-rT}. \]
Numerical Study Assumptions

• Selected Mortality Projection Model
  – The Chen and Cox (2009) model

• Basis assets
  – Three basis assets: pure life insurances, pure annuities and risk-free bonds
The Numerical Programming

• Under the three-basis-asset framework, the upper price bound for the Swiss Re mortality bond is

\[
\text{Min } w_{b1}e^{-rT} + w_{b2}P^{b2} + w_{b3}P^{b3} - P^* \geq 0
\]

\[
\frac{E^Q[(x^b_j - x_j)^+]}{E^Q[(x^b_j - x_j)^-]} \geq L \quad \text{and } x^b_j = w_{b1} + w_{b2}(1-\sum_t q_{t,j}) + w_{b3} \sum_t q_{t,j}
\]

• The replicating cost, \( w_1e^{-rT}+w_2P^{b2}+w_3P^{b3} \), is the upper price bound with three basis assets.
The Numerical Programming

• The three-basis-asset framework, the lower price bound for the Swiss Re mortality bond is

\[
\begin{align*}
\text{Max} & \quad \left( w_{b1} e^{-rT} + w_{b2} P^b_2 + w_{b3} P^b_3 - P^* \right) \leq 0 \\
\text{s.t.} & \quad \frac{E^Q[(x_j - x_j^b)^+]}{E^Q[(x_j - x_j^b)^-]} \geq L \quad \text{and} \quad x_j^b = w_{b1} + w_{b2} (1 - \sum_t q_{t,j}) + w_{b3} \sum_t q_{t,j}.
\end{align*}
\]

• The replicating cost, \( w_1 e^{-rT} + w_2 P^b_2 + w_3 P^b_3 \), is the lower price bound with three basis assets.
Numerical procedures

1. Simulate the mortality projections by 15,000 times from 2004 to 2006. The initial mortality index $q_0$ starts from 0.007 to 0.014.

2. Specify the benchmark pricing model and determine the benchmark price $P^*$ and basis asset prices $P^{b1}$, $P^{b2}$, and $P^{b3}$.

3. The strike prices are $K_1=1.3 \times 0.008453$ and $K_2=1.5 \times 0.008453$. 
Numerical procedures

3. Substitute $P^*$, $P^{b1}$, $P^{b2}$, and $P^{b3}$ into the optimisation programs.

4. Choose $L=2$ to 5 to solve the optimal weights and obtain the upper and lower bounds for $P^*$.

5. Repeat the above procedures for the three benchmark pricing models.
Numerical Results 1

• Price Bounds under Lin and Cox Model and Wang Transform Valuation
• Figure 1
• Table 1

Price bounds at $q_0=0.008453$ under the Lin and Cox (2008) model, $L=2$ to 5

[Lower bound, Upper bound]

<table>
<thead>
<tr>
<th></th>
<th>$L=2$</th>
<th>$L=3$</th>
<th>$L=4$</th>
<th>$L=5$</th>
<th>Benchmark price</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\lambda = 0$</td>
<td>[0.986, 0.996]</td>
<td>[0.979, 0.998]</td>
<td>[0.972, 0.998]</td>
<td>[0.966, 0.999]</td>
<td>0.988</td>
</tr>
<tr>
<td>Range width</td>
<td>0.011</td>
<td>0.019</td>
<td>0.026</td>
<td>0.033</td>
<td></td>
</tr>
<tr>
<td>(b) $\lambda = -0.45$</td>
<td>[0.971, 0.992]</td>
<td>[0.960, 0.995]</td>
<td>[0.946, 0.996]</td>
<td>[0.934, 0.997]</td>
<td>0.983</td>
</tr>
<tr>
<td>Range width</td>
<td>0.021</td>
<td>0.035</td>
<td>0.051</td>
<td>0.063</td>
<td></td>
</tr>
<tr>
<td>(c) $\lambda = -1$</td>
<td>[0.943, 0.985]</td>
<td>[0.919, 0.990]</td>
<td>[0.897, 0.992]</td>
<td>[0.877, 0.994]</td>
<td>0.969</td>
</tr>
<tr>
<td>Range width</td>
<td>0.041</td>
<td>0.070</td>
<td>0.095</td>
<td>0.117</td>
<td></td>
</tr>
<tr>
<td>(d) $\lambda = -1.3603$</td>
<td><strong>[0.926, 0.978]</strong></td>
<td>[0.897, 0.985]</td>
<td>[0.871, 0.989]</td>
<td>[0.848, 0.991]</td>
<td><strong>0.960</strong></td>
</tr>
<tr>
<td>Range width</td>
<td><strong>0.053</strong></td>
<td><strong>0.088</strong></td>
<td><strong>0.118</strong></td>
<td><strong>0.143</strong></td>
<td></td>
</tr>
</tbody>
</table>
Analysis

• The higher gain-loss ratio, $L$, the wider the bound.

• Bounds range from the precise price implied by the benchmark pricing model, to the loosest price bounds implied by the pure no-arbitrage method.

• More kinds of basis assets, the tighter the bound.
Numerical Results 2

- Price Bounds under the Chen and Cox Model and no-arbitrage Pricing
- Figure 2
- Table 2

Price bounds at \( q_0 = 0.008453 \) under Chen and Cox model, \( L = 2 \) to \( 5 \)

<table>
<thead>
<tr>
<th>( \lambda_1 = \lambda_2 = \lambda_3 = 0 )</th>
<th>( L = 2 )</th>
<th>( L = 3 )</th>
<th>( L = 4 )</th>
<th>( L = 5 )</th>
<th>Benchmark price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range width</td>
<td>[1.000, 1.000]</td>
<td>[1.000, 1.000]</td>
<td>[1.000, 1.000]</td>
<td>[1.000, 1.000]</td>
<td>1.000</td>
</tr>
<tr>
<td>Range width</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 = \lambda_2 = \lambda_3 = 0.83 )</td>
<td>[0.996, 0.999]</td>
<td>[0.994, 0.999]</td>
<td>[0.992, 0.999]</td>
<td>[0.990, 1.000]</td>
<td>0.998</td>
</tr>
<tr>
<td>Range width</td>
<td>0.003</td>
<td>0.005</td>
<td>0.007</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 = \lambda_2 = \lambda_3 = 1 )</td>
<td>[0.983, 0.996]</td>
<td>[0.976, 0.997]</td>
<td>[0.968, 0.998]</td>
<td>[0.962, 0.998]</td>
<td>0.990</td>
</tr>
<tr>
<td>Range width</td>
<td>0.012</td>
<td>0.021</td>
<td>0.029</td>
<td>0.036</td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 = \lambda_2 = \lambda_3 = 1.5 )</td>
<td>[0.933, 0.981]</td>
<td>[0.908, 0.987]</td>
<td>[0.887, 0.990]</td>
<td>[0.868, 0.992]</td>
<td>0.960</td>
</tr>
<tr>
<td>Range width</td>
<td>0.047</td>
<td>0.078</td>
<td>0.103</td>
<td>0.124</td>
<td></td>
</tr>
</tbody>
</table>
Numerical Results 3

- The Price Bound with Additional Basis Assets
- Figure 4
- Table 4

Price bounds with an additional basis asset, from $L=2$ to $5$.

<table>
<thead>
<tr>
<th></th>
<th>$L=2$</th>
<th>$L=3$</th>
<th>$L=4$</th>
<th>$L=5$</th>
<th>Benchmark price</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) The Lin and Cox model with $\lambda = -1.3603$</td>
<td>[0.947, 0.974]</td>
<td>[0.932, 0.980]</td>
<td>[0.918, 0.984]</td>
<td>[0.905, 0.987]</td>
<td>0.960</td>
</tr>
<tr>
<td>Range width</td>
<td>0.028</td>
<td>0.049</td>
<td>0.066</td>
<td>0.082</td>
<td></td>
</tr>
<tr>
<td>(b) The Chen and Cox model with $\lambda_1 = \lambda_2 = \lambda_3 = 1.5$</td>
<td>[0.941, 0.974]</td>
<td>[0.927, 0.980]</td>
<td>[0.914, 0.983]</td>
<td>[0.902, 0.986]</td>
<td>0.960</td>
</tr>
<tr>
<td>Range width</td>
<td>0.033</td>
<td>0.053</td>
<td>0.070</td>
<td>0.084</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

• We provide a methodology to obtain the rational price bounds which is useful for MLS valuation.
• The price bounds allow model misspecification such as preference, mortality processes and risk adjustment method.
• The price bound is applicable to most existing MLS valuation models; it can be regarded as an add-on analysis tool for MLS pricing.
• The price bound can be derived by the utility-based or the no-arbitrage method, and the MLS transaction data are not necessary.
The Limitations

- We assume a frictionless market, where the model builders can buy and sell an arbitrary amount of the basis assets.

- The price bound still suffers some model risk; if the benchmark model is biased, the ceiling value must be increased to loosen the price bounds to contain such risk.

- We do not consider the counter-party risk and loading fees, which may result in the undervaluation of the MLS.
Thank you for listening!
Mortality projection models

• Many stochastic mortality models have been proposed to capture mortality process pattern.
  – Dowd et al. (2010), Lin and Cox (2008), Chen and Cox (2009)

• Swiss Re mortality catastrophe bond
  – Mortality process with jumps is a significant phenomenon and should be captured in the stochastic mortality model.
  – We take the mortality catastrophe bond as an valuation example
An Example 1

- A two-period economy, three basis assets and four possible states.
- Three basis assets: risk-free bonds, annuities and life insurances; all the prices equal to 1.
- The four possible states have equal (subjective) probabilities: $\pi_j = 1/4$, for $j = 1, \ldots, 4$.
- The payoffs of basis assets at $t=1$ are:
  
  $x^{b1} = (1, 1, 1, 1)$ for the risk-free bond,
  
  $x^{b2} = (0, 1, 1, 2)$ for the annuity,
  
  and $x^{b3} = (3, 2, 0, 0)$ for the life insurance.

- A new MLS with a terminal payoff $x = (0, 0, 0, 3)$ is introduced into the market with price $P$. What is the reasonable bound for $P$?
An Example 2

• Utility-based pricing method assigns a pricing kernels in each state

\[ m^* = (m_1^*, m_1^*, m_3^*, m_4^*) \]

where \( m^* \) is the benchmark pricing kernel representing the marginal rate of substitution across different states of the benchmark investor.

• Given \( m^* \), MLS can be priced as

\[ E[m^* x] = P^* \]

where \( E[ ] \) is the expectation operator under the subjective probability measure. We call the \( P^* \) benchmark price.

• For example, a risk-neutral investor with pricing kernel \( m^* = (1, 1, 1, 1) \) will price the MLS by \( P^* = 1/4 \times 3 = 3/4 \).
An Example 3

- The no-arbitrage pricing rule requires \( m_1, m_2, m_3, m_4 > 0 \), and pricing kernels correctly price the basis assets:

\[
\frac{1}{4}(m_1) + \frac{1}{4}(m_2) + \frac{1}{4}(m_3) + \frac{1}{4}(m_4) = 1 \quad \text{for the risk-free bond},
\]
\[
\frac{1}{4}(m_2) + \frac{1}{4}(m_3) + \frac{1}{4}(m_4 \times 2) = 1 \quad \text{for the annuity}
\]
\[
\frac{1}{4}(m_1 \times 3) + \frac{1}{4}(m_2 \times 2) + 0 + 0 = 1 \quad \text{for the life insurance}.
\]

- The admissible pricing kernel sets are

\[
m_2 = 2 - 3/2 \ m_1, \ m_3 = 2 - 1/2 \ m_1, \ m_4 = m_1 \quad \text{and} \quad 0 < m_1 < 4/3
\]

- This set yields the no-arbitrage price bound for the MLS is

\[
0 < P < 1.
\]
An Example 4

• Narrow the price bound by the gain-loss ratio constraint.
• For example, the risk-neutral investor with regards the excess payoffs of the MLS not to be greater than some value $L$,

\[
\sup_j (m_1, 2 - \frac{3}{2} m_1, 2 - \frac{1}{2} m_1) \leq L \quad \text{and} \quad m_1 > 0.
\]

Then, the bounds on $m$:

\[
\frac{4}{1 + 2L} < m_1 < \frac{(4L - 4)}{(3L - 1)}, \quad m_2 = 2 - \frac{3}{2} m_1,
\]

\[
m_3 = 2 - \frac{1}{2} m_1 \quad \text{and} \quad m_4 = m_1.
\]

• If we set $L=3$, (the investor regards the gain on the investment not to be better than 3 times of the loss), yield $\frac{4}{7} < m_1 < 1$, or $\frac{1}{7} < P < \frac{1}{4}$. 