Optimal Stops and Algorithmic Trading

Giuseppe Di Graziano

giuseppe.di-graziano@db.com

November 20, 2013
Overview

1. Motivation and Set Up
2. Optimal Stops with Constant P&L Drift
3. Optimal Stops with Unknown P&L Drift
4. Optimal Stops with Stochastic Drift
5. Numerical Examples
6. Calibration
Stop loss and target profit thresholds are commonly used by practitioners.

Intuitively they must be related to the risk preferences of the individual trader/desk.

They must also depend on the track record of a given trader/strategy.

We propose an approach which allows to link the risk preferences of the trader to the characteristics of a given strategy in an algorithmic context.
Set Up

- Underlying: single asset, time spreads, inter asset spreads, basket of assets
- When a position is entered a fixed number of contracts $N$ is bought/sold
- When a position is exited all the outstanding $N$ contracts are sold/bought back
Optimal Stops with Constant P&L Drift

- [Imkeller and Rogers] model the P&L of a position as a Brownian motion with constant drift

\[ X_t = \sigma W_t + \mu t \] (1)

and assume that the cost of exiting the position at the random (stopping) time \( T \) is equal to \( c \)

- Consider the simple stopping strategy

\[ T = \inf \{ t : X_t = -a \quad \text{or} \quad X_t = b \} \] (2)

- One approach to the exit problem is to maximise the expected utility of the P&L, i.e.

\[ \phi = E[e^{-\rho T}U(X_T - c)] \] (3)

for some increasing and concave utility function \( U(x) \)
If \( f(x) \) is in \( C^2 \) and the following ODE is satisfied

\[
\frac{1}{2} \sigma^2 f''(x) + \mu f'(x) - \rho f(x) = 0 \tag{4}
\]

then the process \( M_t = e^{-\rho t} f(X_t) \) is a (Local) Martingale and

\[
f(x) = E^x[e^{-\rho T} U(X_T - c)] \tag{5}
\]

It is thus sufficient to solve the ODE above with the appropriate boundary conditions in the interval \([-a, b] \) to obtain an explicit function for \( \phi = f(0; a, b) \)

The optimal stop loss and target profit thresholds are obtained by maximising \( f(0; a, b) \) as a function of the parameters \( a \) and \( b \).
Example: CARA Utility

- If we choose the utility function to be

\[ U(x) = 1 - \exp(-\gamma x) \]  

we can find an explicit solution for the objective function \( \phi \)

- The objective function can be written as

\[
\phi \equiv f(0; a, b) = E[e^{-\rho T}] - e^{-\gamma c} E[e^{-\rho T - \gamma X_T}]
= L(\rho, 0) - e^{\gamma c} L(\rho, \gamma)
\]

where

\[ L(\rho, \gamma) = E[e^{-\rho T - \gamma X_T}] \]

- Solving the ODE, we obtain

\[
L(\rho, \gamma) = \frac{e^{\gamma a} (e^{\beta b} - e^{\alpha b}) + e^{-\gamma b} (e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}
\]
Example: CARA Utility

... where $\alpha$ and $\beta$ are equal to

$$\alpha = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} - 2\rho}$$  \hspace{1cm} (11)$$

and

$$\beta = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} - 2\rho}$$  \hspace{1cm} (12)$$
However, when the drift is deterministic and positive, it is not optimal to place stop losses. If the drift is deterministic and negative, it does not make sense to trade in the first place.

[Imkeller and Rogers] suggest to let $\mu$ be a random variable with known distribution.

The optimisation problem thus becomes

$$
\phi(\mu; a, b) = \int E(\mu)[e^{-\rho T} U(X_T - c)]\psi(\mu)d\mu
$$

(13)

As before we can solve for the stops $a$ and $b$ which maximise the function above.
Assume now that the drift of the P&L changes over time in a stochastic fashion.

For example, the drift may be high and positive when the trade is entered and weaken over time (or even become negative) as other market participants spot the same opportunity or other exogenous factors start affecting the price of the asset.

In order to capture such a behaviour, we can model the P&L as a Markov-modulated diffusion (MMD)

$$dX_t = \mu(y_t)dt + \sigma(y_t)dW_t$$ (14)

where $y_t$ is a continuous time Markov chain with infinitesimal generator $Q$, independent from $W_t$

For example $y_t \in \{1, 2\}$ and $\mu(1) = \bar{\mu} > 0$ and $\mu(2) = 0$, i.e. the P&L has an initial positive drift which dies out at the random time when the chain changes state. In this case $y_t = 2$ is an absorbing state.
In order to solve the optimisation problem

\[ \phi = E^x[e^{-\rho T}U(X_T - c)] \quad (15) \]

when \( X_t \) is a MMD, consider the function \( f(x, y) \in C^{2,0} \).

Applying Ito’s formula to the function \( \tilde{f} \equiv e^{-\rho t}f(X_t, y_t) \) we obtain

\[
d(e^{-\rho t} f(X_t, y_t)) = e^{-\rho t}(\mu(y_t)f_x(X_t, y_t) + \frac{1}{2}\sigma^2(y_t)f_{xx}(X_t, y_t) + (Qf)(X_t, y_t) - \rho f(X_t, y_t))dt + dM^f_t\]

where \( M^f_t \) is a local Martingale.
Since \( y_t \) can only take a finite number of values, with a slight abuse of notation we can think of \( f(X_t) \) as a vector valued function with element \( i \) equal to \( f_i(X_t) \equiv f(X_t, i) \)

For \( \tilde{f} \) to be a local Martingale we require that

\[
\frac{1}{2} \sum f''(x) + Mf'(x) + (Q - R)f(x) = 0
\]

Here \( \Sigma, M \) and \( R \) are diagonal matrices whose \( i^{th} \) diagonal entry is equal to \( \sigma(i), \mu(i) \) and \( \rho(i) \) respectively. \( Q \) is the infinitesimal generator of the chain.
Since $f(x)$ is bounded in the interval $[-a, b]$, it follows from the optional stopping theorem that

$$f_i(x) = E^{x,i}[e^{-\rho T} U(X_T - c)]$$

where $y_0 = i$ is the initial state of the chain and we have used the boundary conditions

$$\begin{cases} f_i(-a) = U(-a - c) & i \in \{1, \ldots, n\} \\ f_i(b) = U(b - c) & i \in \{1, \ldots, n\} \end{cases}$$
The system of ODEs above admits solutions of the form

\[ f(x) = ve^{-\lambda x} \]  

where \( v \) is a \( n \) dimensional vector

Substituting (16) into the system (12) and re-arranging we obtain

\[ \lambda^2 v - 2\lambda \Sigma^{-1} M v + 2\Sigma^{-1} (Q - R)v = 0 \]

The quadratic eigenvalue problem (14) can be reduced to a canonical eigenvalue problem

\[
\begin{pmatrix}
2\Sigma^{-1} M & -2\Sigma^{-1} (Q - R) \\
I & 0
\end{pmatrix}
\begin{pmatrix}
h \\
v
\end{pmatrix} = \lambda
\begin{pmatrix}
h \\
v
\end{pmatrix}
\]

where

\[ h = \lambda v \]
This is a standard eigenvalue problem which admits \( n \) solutions in the positive half plane and \( n \) in the negative half plane.

The solution to our ODE system will thus take the form

\[
f(x) = \sum_{i=1}^{2n} w_i v_i e^{-\lambda_i x}
\]

The \( 2n \) coefficients \( w_i \) can be derived by solving the system

\[
\begin{align*}
\sum_{i=1}^{2n} w_i v_i e^{-\lambda_i b} &= \bar{U}(b - c) \\
\sum_{i=1}^{2n} w_i v_i e^{-\lambda_i a} &= \bar{U}(-a - c)
\end{align*}
\]

Here \( \bar{U}(z) \) is an \( n \) dimensional vector with \( i^{th} \) entry equal to \( U(z) \).
Numerical Examples

Constant drift - $\mu = 0.15$, $\sigma = 0.25$, $c = 0.01$
Signal with slow decay - $\mu(1) = 0.15$, $\mu(2) = 0$, $q = 0.5$, $c = 0.01$
Numerical Examples

Signal with fast decay - $\mu(1) = 0.15$, $\mu(2) = 0$, $\sigma = 0.25$, $q = 2$, $c = 0.01$
Numerical Examples

Illiquid Security - $\mu(1) = 0.15, \mu(2) = 0, \sigma = 0.25, q = 0.5, c = 0.15$
Numerical Examples

**Low risk aversion** - $\mu(1) = 0.15$, $\mu(2) = 0$, $\sigma = 0.25$, $q = 0.5$, $c = 0.15$, $\gamma = 0.05$
### Table: Optimal stops for different parameters

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\mu(1)$</th>
<th>$\sigma$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.025</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.06</td>
<td>0.14</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.05</td>
<td>0.05</td>
<td>0.21</td>
</tr>
<tr>
<td>10</td>
<td>0.025</td>
<td>0.05</td>
<td>0.0325</td>
<td>0.04</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.05</td>
<td>0.0275</td>
<td>0.1</td>
</tr>
</tbody>
</table>
In the simple model of the example above, calibration can be carried out analytically.

For the calibration to be reliable a relatively high number of backtests are necessary, which is possible in the high frequency domain.

The volatility $\sigma$ can be calculated using standard methods.

The cost $c$ can be easily estimated using bid-ask data.
Expected P&L: $E[X(t)]$
Parameters $\mu(1)$ and $q$ can be calibrated using integral transforms

$$A_1 \equiv \int_0^\infty E[X_t] \lambda e^{-\lambda t} dt$$  \hspace{1cm} (17)

$$A_2 \equiv \int_0^\infty tE[X_t] e^{-\lambda t} dt$$  \hspace{1cm} (18)

for some appropriately chosen parameter $\lambda$
In order to calculate $A_1$ and $A_2$ we need an explicit expression for $E[X_t]$

The expected P&L can be easily calculated using the fact that the jump times of the chain are exponentially distributed.

In particular

$$E[\mu(y_s)] = \mu(1)P(T > s) = \mu(1)e^{-qs}$$

which in turn implies

$$E[X_t] = \int_0^t E[\mu(y_s)]ds$$

$$= \int_0^t \mu(1)e^{-qs}ds = \frac{\mu(1)}{q}(1 - e^{-qt})$$
Calibration: Two State Model

- It follows that

\[
A_1 = \int_0^\infty \frac{\mu(1)}{q} (1 - e^{-qt}) \lambda e^{-\lambda t} \, dt
= \frac{\mu(1)}{q + \lambda}
\]  

(21)

(22)

- Similarly

\[
A_2 = \int_0^\infty \frac{\mu(1)}{q} (1 - e^{-qt}) te^{-\lambda t} \, dt
= \frac{\mu(1)}{\lambda^2} \frac{q + 2\lambda}{q + \lambda}
\]  

(23)

(24)
Calibration: Two State Model

- We can estimate $A_1$ and $A_2$ empirically from the backtested sample path

\[
A_1 \approx \int_0^{\bar{t}} \frac{1}{n} \sum_{j=1}^{n} \tilde{X}_t^j \lambda e^{-\lambda t} \quad (25)
\]

\[
A_2 \approx \int_0^{\bar{t}} \frac{1}{n} \sum_{j=1}^{n} \tilde{X}_t^j t e^{-\lambda t} \quad (26)
\]

where $\bar{t}$ is the cut off time for each individual backtesting
Using equations (22) and (24) in conjunction with the estimates above and solving for \( \mu(1) \) and \( q \), we obtain

\[
q = \frac{\lambda (2A_1 - A_2 \lambda^2)}{A_2 \lambda^2 - A_1} \tag{28}
\]

\[
\mu(1) = \frac{\lambda A_1^2}{A_2 \lambda^2 - A_1} \tag{29}
\]
In order to test the accuracy of the calibration algorithm, we tested the approach on simulated data. The table below shows the results.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\mu(1)$</th>
<th>$\sigma$</th>
<th>$\tilde{q}$</th>
<th>$\tilde{\mu}(1)$</th>
<th>$\tilde{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.015</td>
<td>2.0841</td>
<td>0.0502</td>
<td>0.0152</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>1.9526</td>
<td>0.0505</td>
<td>0.0505</td>
</tr>
<tr>
<td>10</td>
<td>0.025</td>
<td>0.015</td>
<td>9.625</td>
<td>0.0261</td>
<td>0.0150</td>
</tr>
<tr>
<td>10</td>
<td>0.025</td>
<td>0.05</td>
<td>11.87</td>
<td>0.0260</td>
<td>0.0493</td>
</tr>
</tbody>
</table>
Calibration: $n$-States Model

- Consider now an $n$-states Markov chain with generator $Q$ and initial state distribution $\pi$.
- One possible calibration approach is minimising the following function:

$$F \equiv \int_0^{\bar{t}} \left( E[X_t] - \hat{X}_t \right)^2 dt \approx m(\bar{t}) \sum_{j=1}^{m(\bar{t})} \left( E[X_{t_j}] - \hat{X}_{t_j} \right)^2 \Delta j$$

where

$$\hat{X}_t \equiv \frac{1}{n} \sum_{j=1}^{n} \tilde{X}_j$$

and $n$ is the number of backtesting.
The expected P&L conditional of the initial state of the chain, can be calculated analytically:

\[
E[X_t \mid y_0 = i] = E^i \left[ \int_0^t \mu(y_s) ds \right]
\]

\[
= \int_0^t (e^{Qs} \mu)_i ds
\]

\[
= \left( Q^{-1} \left( e^{Qt} - I \right) \mu \right)_i
\]

The unconditional expected P&L is given by:

\[
E[X_t] = \sum_{i=1}^{n} \left( Q^{-1} \left( e^{Qt} - I \right) \mu \right)_i \pi_i
\]
References

Imkeller and Rogers (2011)
Trading to Stop. Working Paper, University of Cambridge

Di Graziano and Rogers (2005)
Barrier option pricing for assets with Markov-modulated dividends. Journal of Computational Finance, 9, 75-87