Measuring the impact of jumps in multivariate price processes using bipower covariation

Ole E. Barndorff-Nielsen
Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark
oebn@imf.au.dk

Neil Shephard
Nuffield College, University of Oxford, Oxford OX1 1NF, UK
neil.shephard@nuf.ox.ac.uk

Abstract

Realised bipower variation consistently estimates the quadratic variation of the continuous component of prices. In this paper we generalise this concept to realised bipower covariation, study its properties, illustrate its use, derive its asymptotic distribution and use it to test for jumps in multivariate price processes. We also introduce a new concept called cojumping, where individual assets all have jumps, but it is possible to construct portfolios which have continous sample paths.

NOTE: This paper is not finished.

Keywords: Bipower variation; Cojumping; Covariation; Jump process; Quadratic covariation; Realised variance; Semimartingales; Stochastic volatility.

1 Introduction

In the theory of financial economics the variation of asset prices is measured by looking at sums of outer products of returns calculated over very small time periods. The mathematics of this is based on the quadratic variation (QV) process (e.g. Protter (2004)). Asset pricing theory links the dynamics of increments of the QV process to the increments of the risk premium process (e.g. Chamberlain (1988) and Back (1991)). There has been considerable recent econometric work on this topic, estimating QV using equally spaced discrete returns. Such an estimator is called the realised QV process, while the increments of this process are called realised covariations in the multivariate case and realised variances or volatilities in the univariate case. A time series of realised covariations was studied in the context of the methodology of volatility forecasting by Andersen, Bollerslev, Diebold, and Labys (2001) and Andersen, Bollerslev, Diebold, and Labys (2003), while central limit theories for the realised QV process and realised covariations were developed by Jacod (1994), Barndorff-Nielsen and Shephard (2002), Barndorff-Nielsen and Shephard (2004a) and Mykland and Zhang (2005). See Andersen, Bollerslev, and Diebold (2004) for an incisive survey of this area and references to related work.

In this paper we will measure the contribution of the continuous component of prices to their covariation and form robust tests for the presence of jumps on individual days in financial markets. This will be based on what we call realised bipower covariation, which we will outline in a moment. Being able to distinquish between jumps and continuous sample path price movements is important as it has implications for risk management and asset allocation. Further, one of the properties of price processes which survive the change to equivilent martingale measures is the presence or absense of jumps. This means that we can test for jumps either using option data or using time series of underlying assets. A stream of recent papers in financial econometrics has addressed this issue using low frequency return data (e.g. the parametric models of Eraker, Johannes, and Polson (2003), Andersen, Benzoni, and Lund (2002), Chernov, Gallant, Ghysels, and Tauchen (2003) and the Markovian, non-parametric analysis of Aït-Sahalia (2002), Johannes (2004) and Bandi and Nguyen (2003)) and options data (e.g. Bates (1996), Carr and Wu (2004) and the review by Garcia, Ghysels, and Renault (2004)). Our approach will be non-parametric and exploit high frequency multivariate data.

In two recent papers Barndorff-Nielsen and Shephard (2004b) and Barndorff-Nielsen and Shephard (2003) introduced a partial generalisation of the univariate QV process called the bipower variation (BPV) process. They showed that in some cases relevant to financial economics the BPV equals the QV of the continuous component of a single price process. This means we can, in theory, split up the individual components of the univariate QV into that due to the continuous part of prices and that due to jumps. In turn the bipower variation process can be consistently estimated using an equally spaced discretisation of financial data. This estimator is called the realised BPV process. Following the introduction of bipower variation, Andersen, Bollerslev, and Diebold (2003) have used daily increments to realised BPV as an input into new reduced form forecasting devices for modelling future values of daily realised variances (which in turn proxies the variability of future prices). This follows the influential line of thinking of Andersen, Bollerslev, Diebold, and Labys (2003) who modelled realised variances in terms of lags of previous realised variances. Huang and Tauchen (2003) have reported extensive Monte Carlo experiments on the performance of the central limit theory for realised BPV developed by Barndorff-Nielsen and Shephard (2003). Their results suggest the theory performs well when carried out over short periods of time. Theoretical guidance for using the asymptotic theory over longer time intervals is provided by Corradi and Distaso (2004).

In this paper we introduce a multivariate definition of the BPV process. This equals the QV process of the continuous component of a vector of prices, which means we can use it to split up the multivariate QV into that due to the continuous and jump components of prices. We use these ideas to define a new concept of dependence called cojumping. We illustrate the use of the multivariate BPV process in financial economics. We then derive the asymptotic distribution of the estimator of this, the multivariate realised BPV process. We then use this distribution theory to test for jumps in multivariate exchange rate data.

Finally, we should note that in this paper we will ignore the effect of market microstructure effects. Potentially this is an important omission and it would be good to overcome this weakness. In the case of realised QV quite a lot of interesting work has been carried out in this area. Zhang, Mykland, and Aït-Sahalia (2003) address the noise problem and propose a subsampling procedure for estimating the integrated volatility of the log price process. Such subsampling ideas may be able to be applied to reduce the impact of noise on BPV, but some new theoretical tools will have to be developed before we can do this from a theoretical viewpoint and we are not in a position to discuss them as yet. Hansen and Lunde (2004) have initiated a study of how the realised quadratic variation may be bias corrected to alleviate the noise effect. See also the work of Bandi and Russell (2003). The latter line of investigation is continued in joint ongoing work by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004). That work considers a general class of kernel estimators of the QV and relates it to subsampling. The main thrust of the Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004) work consists in determining, from optimality criteria, another type of kernel estimator that has turned out to yield very accurate estimates for almost all lags. Zhang (2004) has shown that subsampling can be generalised, yielding an estimator with the same rate of convergence as the modified kernel of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004).

The outline of the paper is as follows. In Section 2 we introduce the notation of a semi-martingale, and recall the definitions of the QV process. We also introduce the multivariate BPV process and the idea of cojumping. In Section 3 we study consistent estimators of the QV and BPV processes, which we call the realised QV and realised BPV processes, respectively. In Section 4 we discuss the daily discretisation of the realised QV and BPCV processes. We informally illustrate these concepts in the context of some high frequency exchange rate data in Section 5, comparing the traditional measure of codependence using realised covariation, with those developed out of the new realised BPV. We see that on tranquil days there is very little difference, but when there are jumps the statistics provide very different sets of information.

In Section 6 we derive a joint asymptotic distribution theory for the realised QV and BPCV processes. We show this can be implemented under rather weak assumptions. In Section 7 we show how to use this distribution theory to develop multivariate tests for jumps in the vector of prices. In Section 8 we illustrate the use of our asymptotic theory on our exchange rate data.

In Section 9 we draw out the conclusions from our paper. We also have a lengthy Appendix. All but one of its sections contains proofs of the main results in the paper. The last section of the paper gives a multivariate generalisation of the Barndorff-Nielsen and Shephard (2002) approach to preprocessing the data to remove the effects of breaks in the datafeed.

2 Definitions

2.1 Semimartingale notation

Let the prices of a p-dimensional vector of assets be written as

$$Y_t = \left(Y_{(1)t}, Y_{(2)t}, ..., Y_{(p)t}\right)', \quad \text{for} \quad t \geq 0.$$

Here t represents continuous time. We assume Y lives on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$. We assume Y is a semimartingale (written $Y \in \mathcal{SM}$), which means it can be decomposed as $Y = A + M$, where A is a vector process with elements of finite variation (written $A \in \mathcal{FV}$) paths and M is vector of local martingales (written $M \in \mathcal{M}_{loc}$). For an accessible discussion of probabilistic aspects of this see Protter (2004), while its attraction from an economic viewpoint is discussed by Back (1991). We will often restrict various classes of processes to those with continuous or purely discontinuous sample paths. We generically denote this with superscripts c and d respectively, e.g. $\mathcal{M}_{loc}^c$ stands for the class of continuous local martingales, while M^c denotes the continuous component of M.

Our analysis will revolve around the Brownian semimartingale (written $Y \in \mathcal{BSM}$)

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u,$$

where the a vector process is predictable and has locally bounded sample paths, the σ matrix process is càdlàg, while W is a vector of independent, standard Brownian motions. In the semimartingale notation, $A_t = \int_0^t a_u \mathrm{d}u$, while $M_t = \int_0^t \sigma_u \mathrm{d}W_u$. Clearly M is a multivariate stochastic volatility process (see the reviews in Ghysels, Harvey, and Renault (1996) and Shephard (2005, Ch. 1)). We also define the spot covariance as

$$\Sigma_t = \sigma_t \sigma_t',$$

and assume that (for all $t < \infty$)

$$\int_0^t \Sigma_{(ll)u} \mathrm{d}u < \infty, \quad l = 1, 2, ..., p,$$

where $\Sigma_{(kl)}$ is the k,l-th element of the Σ matrix. The latter is needed to ensure that $M \in \mathcal{M}_{loc}$. The structure of $Y \in \mathcal{BSM}$ means $A \in \mathcal{FV}^c$, which means that $\mathcal{BSM} \subset \mathcal{SM}^c$. The form of A

follows from an assumption of lack of arbitrage once M is assumed to be a SV process, while the form of M follows from the martingale representation theorem of Doob so long as quadratic variation of Y is absolutely continuous. This is discussed at some length in Barndorff-Nielsen and Shephard (2004a). Thus the $\mathcal{BSM}$ is close to being the general class of vector continuous sample path processes.

When we add jumps to the price process, we assume they are of finite activity. The resulting process is called a Brownian semimartingale plus finite activity jump process (written $Y \in \mathcal{BSMJ}_{FA}$). This has

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + \sum_{j=1}^{N_t} C_j. \qquad (1)$$

The simple counting process N has (for all $t < \infty$) $N_t < \infty$ and we assume that (for all $t < \infty$)

$$\sum_{j=1}^{N_t} C_{(l)j}^2 < \infty, \quad l = 1, 2, ..., p.$$

We do not assume the c process has zero mean, so the jumps can potentially contribute both to A and M in the semimartingale decomposition. Clearly $\mathcal{BSMJ}_{FA} \subseteq \mathcal{SM}$. The $\mathcal{BSMJ}_{FA}$ has the attractive feature that it is closed under stochastic integration which, in particular, means that for any comformable matrix of constants B the rotated process $BY \in \mathcal{BSMJ}_{FA}$.

2.2 Quadratic variation

The quadratic variation process plays a leading role in this paper.

Definition 1 *(e.g. Jacod and Shiryaev (1987, p. 55)) So long as it exists, the quadratic variation $p \times p$ matrix process is defined as*

$$[Y] = \text{p-}\lim_{n\to\infty} \sum_{j=1}^{n} \left(Y_{t_j} - Y_{t_j-1}\right)\left(Y_{t_j} - Y_{t_j-1}\right)', \qquad (2)$$

for any sequence of non-stochastic partitions $t_0 = 0 < t_1 < ... < t_n = t$ with $\sup_j\{t_j - t_{j-1}\} \to 0$ for $n \to \infty$. Here p-lim denotes convergence is locally uniform in time and in probability.

It is well known that the QV always exists if $Y \in \mathcal{SM}$.

It is convenient to refer to the l,k-th element of QV as the l,k-th quadratic covariation

$$\begin{aligned} [Y]_{(l,k)} &= [Y_{(l)}, Y_{(k)}], \quad l,k = 1,2,...,p \\ &= \text{p-}\lim_{n\to\infty} \sum_{j=1}^{n} \left(Y_{(l)t_j} - Y_{(l)t_j-1}\right)\left(Y_{(k)t_j} - Y_{(k)t_j-1}\right). \end{aligned}$$

The conventional notation is to write $[Y_{(l)}, Y_{(l)}] = [Y_{(l)}]$, the QV of $Y_{(l)}$. Thus

$$[Y] = \left(\begin{array}{cccc} [Y_{(1)}] & [Y_{(1)}, Y_{(2)}] & \cdots & [Y_{(1)}, Y_{(p)}] \\ [Y_{(2)}, Y_{(1)}] & [Y_{(2)}] & \cdots & [Y_{(2)}, Y_{(p)}] \\ \vdots & \vdots & \ddots & \vdots \\ [Y_{(p)}, Y_{(1)}] & [Y_{(p)}, Y_{(2)}] & \cdots & [Y_{(p)}] \end{array} \right).$$

The quadratic covariation $[Y_{(l)}, Y_{(k)}]$ can be calculate solely from the QV of a sequence of univariate processes. This follows from the fact that

$$[Y_{(l)} + Y_{(k)}] = [Y_{(l)}] + [Y_{(k)}] + 2[Y_{(l)}, Y_{(k)}]. \tag{3}$$

Rearranging produces the so-called *polarisation* results (e.g. Revuz and Yor (1999, p. 125)) that

$$[Y_{(l)}, Y_{(k)}] = \frac{1}{2}\left([Y_{(l)} + Y_{(k)}] - [Y_{(l)}] - [Y_{(k)}]\right) \tag{4}$$

or

$$[Y_{(l)}, Y_{(k)}] = \frac{1}{4}\left([Y_{(l)} + Y_{(k)}] - [Y_{(l)} - Y_{(k)}]\right). \tag{5}$$

It will be (5) which will prove to be central in this paper, rather than the more obvious (4). The reasons for this will become clear in a moment.

It is well known that

$$\begin{aligned} [Y] &= [M^c] + \sum_{0 \le s \le t} \Delta Y_s \Delta Y_s' \\ &= [M^c] + [Y^d], \end{aligned} \tag{6}$$

where $\Delta Y_t = Y_t - Y_{t-}$ are the jumps in the process. This means that the QV of Y aggregates the QV of M^c and the QV of Y^d. In the special case where $Y \in \mathcal{BSMJ}_{FA}$ then

$$[Y]_t = \int_0^t \Sigma_u \mathrm{d}u + \sum_{j=1}^{N_t} C_j C_j'.$$

2.3 Bipower variation

The definition of BPV is given in terms of equally spaced time intervals of length $\delta > 0$ and the corresponding vector of returns

$$y_j = Y_{j\delta} - Y_{(j-1)\delta}, \quad j = 1, 2, ..., \lfloor t/\delta \rfloor = n,$$

where $\lfloor x \rfloor$ is the integer part of x. The definition is the multivariate generalisation of the univariate case due to Barndorff-Nielsen and Shephard (2003).

Definition 2 *So long as it exists, the q-th lag bipower variation (BPV) $p \times p$ matrix process is*

$$\{Y;q\} = \begin{pmatrix} \{Y_{(1)};q\} & \{Y_{(1)},Y_{(2)};q\} & \cdots & \{Y_{(1)},Y_{(p)};q\} \\ \{Y_{(2)},Y_{(1)};q\} & \{Y_{(2)};q\} & \cdots & \{Y_{(2)},Y_{(p)};q\} \\ \vdots & \vdots & \ddots & \vdots \\ \{Y_{(p)},Y_{(1)};q\} & \{Y_{(p)},Y_{(2)};q\} & \cdots & \{Y_{(p)};q\} \end{pmatrix}.$$

The l,l-th element of $\{Y;q\}$ is

$$\{Y_{(l)};q\} = \text{p}-\lim_{n\to\infty} \sum_{j=q+1}^{n} \left|y_{(l)j-q}\right| \left|y_{(l)j}\right|,$$

while the l,k-th bipower covariance process, is

$$\{Y_{(l)},Y_{(k)};q\} = \frac{1}{4}\left(\{Y_{(l)}+Y_{(k)};q\} - \{Y_{(l)}-Y_{(k)};q\}\right). \tag{7}$$

Again, p-lim denotes convergence is locally uniform in time and in probability.

The limit in Definition 2 can be given explicitly under broad conditions.

Theorem 1 *Assume $Y \in \mathcal{BSMJ}_{FA}$. Then*

$$\{Y;q\}_t = \mu_1^2 \int_0^t \Sigma_u \mathrm{d}u, \quad l = 1,2,...,p, \tag{8}$$

where

$$\mu_1 = \mathrm{E}\left|u\right| = \sqrt{2}/\Gamma\left(\frac{1}{2}\right) = \sqrt{2}/\sqrt{\pi} \simeq 0.79788$$

and $u \sim N(0,1)$.

Proof. Our proof is carried out element by element on the matrix $\{Y;q\}$. We use the polarisation result

$$[Y_{(l)},Y_{(k)}] = \frac{1}{4}\left([Y_{(l)}+Y_{(k)}] - [Y_{(l)}-Y_{(k)}]\right). \tag{9}$$

Thus the required result would follow if we can show that

$$\begin{aligned} \{Y_{(l)}+Y_{(k)};q\} &= \mu_1^2[Y_{(l)}^c + Y_{(k)}^c] && (10) \\ &= \mu_1^2 \int_0^t \Sigma_{(ll)u}\mathrm{d}u + \mu_1^2 \int_0^t \Sigma_{(kk)u}\mathrm{d}u + 2\mu_1^2 \int_0^t \Sigma_{(lk)u}\mathrm{d}u. && (11) \end{aligned}$$

If there are no jumps then $Y_{(l)} + Y_{(k)} \in \mathcal{BSM}$ which implies (10) holds immediately by the convergence in probability result of Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004) for general Brownian semimartingales and generalised bipower measures of variation. The robustness to added finite activity jumps follows immediately by the argument in Barndorff-Nielsen and Shephard (2003).

□

Remark 1 *(i) For simplicity of exposition it is sometimes convenient to write* $\{Y\} = \{Y; 1\}$.

(ii) The bipower covariance process could have been been defined as

$$\frac{1}{2}\left(\left\{Y_{(l)} + Y_{(k)}; q\right\} - \left\{Y_{(l)}; q\right\} - \left\{Y_{(k)}; q\right\}\right),$$

but it will be seen shortly that this has less virtue than using (7).

(iii) $\{Y; q\}$ *must be symmetric and positive semi-definite.*

(iv) Clearly

$$[Y] - \mu_1^{-2}\{Y; q\} = [Y^d] = \sum_{j=1}^{N} C_j C_j'. \tag{12}$$

This shows that it is theoretically possible to seperately identify the continuous and discontinuous components of the multivariate QV.

2.4 Cojumping

Suppose again that

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + \sum_{j=1}^{N_t} C_j$$

and each element of Y exhibits jumps in the time interval $[0, t]$. Then in general we can define a new process

$$\begin{aligned} X_t &= \int_0^t D_{u-} \mathrm{d}Y_u \\ &= \int_0^t D_u a_u \mathrm{d}u + \int_0^t D_u \sigma_u \mathrm{d}W_u + \sum_{j=1}^{N_t} D_{\tau_j -} C_j, \end{aligned}$$

where D is a non-zero $k \times p$ matrix process whose elements are càdlàg and adapted to the filtration generated by Y and τ_1, τ_2,...,τ_{N_t} are the arrival times of the counting process N. This type of process is important for it could represent, for example, the value of a portfolio, where D are the investment weights.

It is clearly possible that there may exist a D process such that some of the elements of X have continuous sample paths in the time interval $[0, t]$ even though each of the elements of Y have discontinuities. When this is the case we say that the Y process *cojumps*. Cojumping has some similiarities to the idea of cobreaking introduced by Hendry (1995), but in some sense the concept is more straightforward here as the notion of a continuous sample path is unambigious.

Remark 2 *The assumption that* D *is adapted to the filtration generated by* Y *implies* D_{t-} *has to be determined before we see* $\Delta Y_t = Y_t - Y_{t-}$. *This rules out the trivial construction*

$$D_{\tau_j -} = I_p - C_j (C_j' C_j)^{-1} C_j',$$

which would allow us to see the jumps coming and then adjust D *accordingly.*

Clearly

$$\begin{aligned}
[X]_t &= [X^c]_t + [X^d]_t \\
&= \left[\int_0^t D_u \mathrm{d}Y_u^c\right]_t + \left[\int_0^t D_{u-} \mathrm{d}Y_u^d\right]_t \\
&= \int_0^t D_u \sigma_u \sigma_u' D_u' \mathrm{d}u + \sum_{j=1}^{N_t} D_{\tau_j -} C_j C_j' D_{\tau_j -}'.
\end{aligned}$$

Thus we can see that cojumping occurs in the time interval $[0, t]$ with respect to an adapted D if some of the diagonal elements of

$$\sum_{j=1}^{N_t} D_{\tau_j -} C_j C_j' D_{\tau_j -}'$$

are exactly zero. If we constrain our attention to time-invariant D matrices, then the search for cojumping amounts to asking if

$$[Y^d]_t = \sum_{j=1}^{N_t} C_j C_j'$$

is of reduced rank with probability one.

Example 1 *Suppose*

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + \Pi f_t,$$

where Π *is a* $p \times k$ *non-stochastic matrix,* f *is a* $k \times 1$ *dimensional process with elements which have purely discontinuous sample paths*

$$f_t = \sum_{j=1}^{N_t} g_j,$$

and $p > k$*. Then* $C_j = \Pi g_j$*. Clearly we can, if* $\Pi'\Pi$ *is full rank, construct*

$$\Pi_\perp = I_p - \Pi(\Pi'\Pi)^{-1}\Pi',$$

which has the property that $\Pi_\perp \Pi = 0$*. Thus*

$$\Pi_\perp Y_t = \int_0^t \Pi_\perp a_u \mathrm{d}u + \int_0^t \Pi_\perp \sigma_u \mathrm{d}W_u,$$

which has a continuous sample path.

3 Realised variation

3.1 Realised QV process

The *realised* QV *process* is defined as

$$[Y_\delta] = \sum_{j=1}^{n} y_j y_j' \xrightarrow{p} [Y],$$

as $\delta \downarrow 0$ so long as $Y \in \mathcal{SM}$. It is studied at some length in Andersen, Bollerslev, Diebold, and Labys (2003) and Barndorff-Nielsen and Shephard (2004a) from the viewpoint of volatility forecasting and asymptotic distribution theory, respectively.

Remark 3 *For any conformable matrix of constants B then $[BY_\delta] = B[Y_\delta]B'$. Thus $[Y_\delta]$ implies the QV matrix of BY_δ. This is convenient as BY_δ can be thought of as the price process for q static portfolios based on the p asset prices Y_δ. This result does not generalise to the addition of $A^+ \in \mathcal{FV}^c$ for $[A^+ + BY_\delta] \neq B[Y_\delta]B'$, although $[A^+ + BY_\delta] \xrightarrow{p} B[Y]B'$.*

3.2 Realised BPV process

The BPV process can be estimated using the realised BPV process.

Definition 3 *The q-th lag realised bipower variation (BPV) $p \times p$ matrix process is*

$$\{Y_\delta; q\} = \begin{pmatrix} \{Y_{(1)\delta}; q\} & \{Y_{(1)\delta}, Y_{(2)\delta}; q\} & \cdots & \{Y_{(1)\delta}, Y_{(p)\delta}; q\} \\ \{Y_{(2)\delta}, Y_{(1)\delta}; q\} & \{Y_{(2)\delta}; q\} & \cdots & \{Y_{(2)\delta}, Y_{(p)\delta}; q\} \\ \vdots & \vdots & \ddots & \vdots \\ \{Y_{(p)\delta}, Y_{(1)\delta}; q\} & \{Y_{(p)\delta}, Y_{(2)\delta}; q\} & \cdots & \{Y_{(p)\delta}; q\} \end{pmatrix}.$$

The l,l-th element of $\{Y_\delta; q\}$ is

$$\{Y_{(l)}; q\} = \gamma_{q,\delta} \sum_{j=q+1}^{n} \left| y_{(l)j-q} \right| \left| y_{(l)j} \right|, \quad \gamma_{q,\delta} = \frac{n}{n-q},$$

while the l,k-th bipower covariance process, is

$$\{Y_{(l)}, Y_{(k)}; q\} = \frac{\gamma_{q,\delta}}{4} \left(\{Y_{(l)} + Y_{(k)}; q\} - \{Y_{(l)} - Y_{(k)}; q\} \right). \tag{13}$$

The constant $\gamma_{q,\delta} \downarrow 1$ as $\delta \downarrow 0$ and so plays no asymptotic role, but it does improve the finite sample performance of the process. Clearly $\{Y_{(l)\delta}, Y_{(l)\delta}; q\}$ equals $\{Y_{(l)\delta}; q\}$.

If $Y \in \mathcal{BSMJ}_{FA}$ then

$$\{Y_\delta; q\} \xrightarrow{p} \mu_1^2 [Y^c].$$

This implies

$$\mu_1^{-2} \{Y_\delta; q\} \xrightarrow{p} [Y^c]$$

and

$$[Y_\delta] - \mu_1^{-2} \{Y_\delta; q\} \xrightarrow{p} [Y^d].$$

It is unfortunately not the case that $\{BY_\delta\}$ equals $B\{Y_\delta\}B'$. This follows immediately from the fact that for a general real b and $\delta > 0$ so $\{bY_{(l)\delta}, Y_{(k)\delta}\}$ does not equal $b\{Y_{(l)\delta}, Y_{(k)\delta}\}$. However, $\{Y_{(l)\delta}, Y_{(k)\delta}\}$ does possess some elegant properties.

Theorem 2 *Suppose the realised BPV between assets l and k exists. Then*

$$\left\{Y_{(l)\delta}, Y_{(k)\delta}\right\} = \left\{Y_{(k)\delta}, Y_{(l)\delta}\right\}, \tag{14}$$

$$\left\{Y_{(l)\delta}, -Y_{(k)\delta}\right\} = \left\{-Y_{(l)\delta}, Y_{(k)\delta}\right\} = -\left\{Y_{(l)\delta}, Y_{(k)\delta}\right\}. \tag{15}$$

Further, suppose b is any real numbers, then the following holds

$$\left\{bY_{(l)\delta}, bY_{(k)\delta}\right\} = b^2\left\{Y_{(l)\delta}, Y_{(k)\delta}\right\}, \tag{16}$$

$$\left\{Y_{(l)\delta}, bY_{(l)\delta}\right\} = b\left\{Y_{(l)\delta}\right\}. \tag{17}$$

Proof. Given in the Appendix.

Result (14) implies $\{Y_\delta\}$ is symmetric, but for finite δ it is not necessarily positive semi-definite. This latter point is, in our view, the main drawback of the use of this measure. In practice, one possible remedy for the lack of positive semi-definiteness would be to determine the spectral decomposition $B_\delta\Lambda_\delta B_\delta'$ of $\{Y_\delta\}$, define $\widetilde{\Lambda}_\delta$ as the diagonal matrix obtained from Λ_δ by changing all negative eigenvalues to 0, and, finally modify $\{Y_\delta\}$ to $B_\delta\widetilde{\Lambda}_\delta B_\delta'$. Clearly, $\{Y_\delta\}$ is converging to $\mu_1^2[Y^c]$ which is positive semi-definite, which implies all the corresponding eigenvalues are becoming non-negative in the limit so $B_\delta\widetilde{\Lambda}_\delta B_\delta' \xrightarrow{p} \mu_1^2[Y^c]$. When we carry out empirical work we will continually use this modified version of the realised BPV process. The same goes for $[Y_\delta] - \mu_1^{-2}\{Y_\delta; q\}$ which estimates $[Y^d]$. This is also symmetric, but is not necessarily positive semi-definite.

3.3 Alternative bipower variation estimator

The same type of argument could have been based on the following result. Define

$$\xi\left(Y_{(l)\delta}, Y_{(k)\delta}\right) = \frac{1}{2}\left(\left\{Y_{(l)\delta} + Y_{(k)\delta}\right\} - \left\{Y_{(l)\delta}\right\} - \left\{Y_{(k)\delta}\right\}\right),$$

then if $Y \in \mathcal{BSMJ}_{FA}$ so $\xi\left(Y_{(l)\delta}, Y_{(k)\delta}\right)$ converges in probability to $\mu_1^2[Y_{(l)}^c, Y_{(k)}^c]$. This alternative estimator has a number of advantages, but lacks the attractive property (15) exhibited by realised BPV. Further, work not reported here suggest that this estimator is less efficient than $\{Y_\delta; q\}$. For these reasons we will favour $\left\{Y_{(l)\delta}, Y_{(k)\delta}; q\right\}$ from now on.

This type of result is reminisicent of the use of triangular arbitrage constraints to define a covariance between exchange rates, e.g. Brandt and Diebold (2004) who then estimate covariances solely through the ranges on daily exchange rates. Write the log of Yen/DM rate by D, the log of the Yen/Dollar rate as $Y_{(l)}$ and the log of the DM/Dollar rate as $Y_{(k)}$. Assume $D = Y_{(l)} - Y_{(k)}$ holds exactly. Then $[Y_{(l)}, Y_{(k)}]$ equals $\left([Y_{(l)}] + [Y_{(k)}] - [D]\right)/2$, which is consistently estimate by $\left([Y_{(l)\delta}] + [Y_{(k)\delta}] - [D_\delta]\right)/2$. Likewise $[Y_{(l)}^c, Y_{(k)}^c]$ can be estimated by $\left(\left\{Y_{(l)\delta}\right\} + \left\{Y_{(k)\delta}\right\} - \left\{D_\delta\right\}\right)/2$.

4 Time series of realised quantities

We remarked in the Introduction that considerable attention has recently been given to discretisations of the realised QV process. We can define, for a fixed time interval $\hbar > 0$, which we will refer to as a day for concreteness, a sequence of *daily realised covariations*

$$V(Y_\delta)_i = [Y_\delta]_{\hbar i} - [Y_\delta]_{\hbar(i-1)}, \quad i = 1, 2, ..., T.$$

This can be written explicitly as

$$V(Y_\delta)_i = \sum_{j=1}^{\lfloor \hbar/\delta \rfloor} y_{j,i} y_{j,i}',$$

where

$$y_{j,i} = Y_{\delta j + \hbar(i-1)} - Y_{\delta(j-1)+\hbar(i-1)},$$

the j-th high frequency return on the i-th day. Clearly

$$V(Y_\delta)_i \xrightarrow{p} V(Y)_i = [Y]_{\hbar i} - [Y]_{\hbar(i-1)}.$$

This was the starting point of Barndorff-Nielsen and Shephard (2004a) who derived the asymptotic distribution of $\delta^{-1/2}(V(Y_\delta)_i - V(Y)_i)$ and Andersen, Bollerslev, Diebold, and Labys (2003) who studied forecasting future values of $V(Y)_i$ based on the history of $V(Y_\delta)_i$. Here we briefly discuss the corresponding definition and basic convergence results for the realised BPCV process. These follow straightforwardly from our previous theoretical results.

We can define a sequence of T daily *q-th lag daily realised bipower covariations* as

$$\widetilde{V}(Y_\delta; q)_i = \{Y_\delta; q\}_{\hbar i} - \{Y_\delta; q\}_{\hbar(i-1)} \xrightarrow{p} \mu_1^2 V(Y^c)_i.$$

The l-th diagonal elements of $\widetilde{V}(Y_\delta; q)_i$ will be

$$\widetilde{v}(y_{(l)})_i = \widetilde{\gamma}_{q,\delta} \sum_{j=q+1}^{\lfloor \hbar/\delta \rfloor} \left| y_{(l)j-q,i} y_{(l)j,i} \right|, \quad \widetilde{\gamma}_{q,\delta} = \frac{\lfloor \hbar/\delta \rfloor}{\lfloor \hbar/\delta \rfloor - q},$$

while the l,k-th element will be

$$\widetilde{V}(Y_\delta; q)_{(lk)i} = \frac{1}{4}\left(\widetilde{v}(y_{(l)} + y_{(k)})_i - \widetilde{v}(y_{(l)} - y_{(k)})_i \right).$$

5 Empirical illustration

5.1 A tranquil day

To illustrate some of the empirical features of realised covariation and realised BPCV we have used a subset of the return data employed by Andersen, Bollerslev, Diebold, and Labys (2001), although we have made slightly different adjustments to deal with some missing data. These adjustments are described in detail in the Appendix and represent a multivariate generalisation of the stochastic adjustment schemes introduced by Barndorff-Nielsen and Shephard (2002). The bivariate series in question records the United States Dollar/ German Deutsche Mark and Dollar/ Japanese Yen series. It covers the ten year period from 1st December 1986 until 30th November 1996. The original dataset records every 5 minutes the most recent mid-quote to appear on the Reuters screen. It has been kindly supplied to us by Olsen and Associates in Zurich, who document their pathbreaking work in this area in Dacorogna, Gencay, Müller, Olsen, and Pictet (2001).

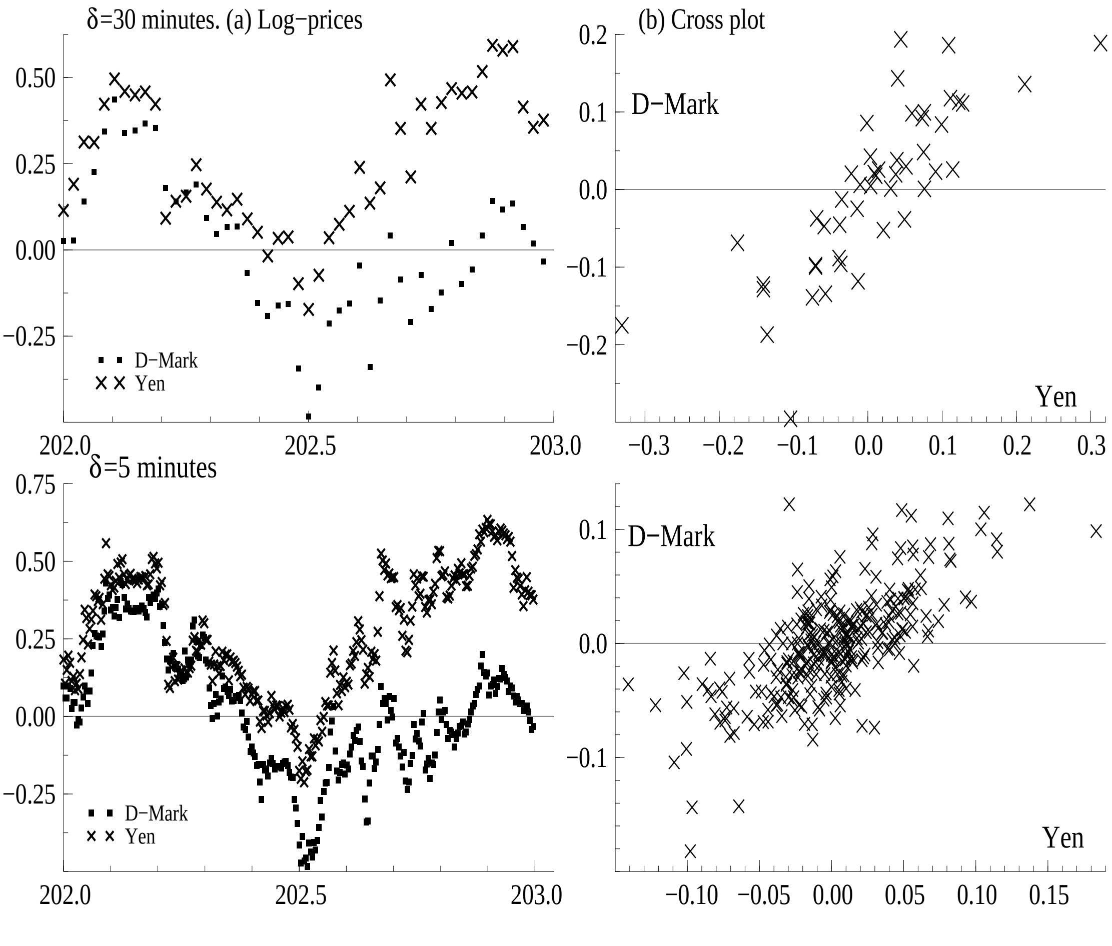

Figure 1: *Data on 26th October 1988 for the D-Mark and Yen against the Dollar. Top line: based on 30 minute data, bottom line based on 5 minute. Left hand side: renormalised log-price. Right hand side: plot of returns of D-Mark against Yen. Code: bpcv.ox.*

The top and bottom parts of the left hand side of Figure 1 shows the sample path of the bivariate y_δ based on δ representing 30 and 5 minutes respectively for a single, randomly choosen day. This day was 26th October 1988 and was a relatively tranquil day on the foreign exchange markets. On the right hand side of the Figure we plot the returns for the D-Mark against the

Yen, computed using 30 and 5 minute data. Here we can see a broadly positive relationship between the two sets of returns. The realised variation summary statistics are given in Table 1. The realised BPCV is computed in this Section as the average version (**??**) using $Q = 6$. The Table shows that the two estimates are rather similiar and that the statistics do not change very much as δ changes.

202nd day	D-Mark, $y_{(1)\delta}$		Yen, $y_{(2)\delta}$		Joint, $y_{(1)\delta}$, $y_{(2)\delta}$	
Minutes, δ	30	5	30	5	30	5
Realised QV	.528	.590	.550	.571	.413	.418
Realised BPV	.590	.605	.484	.578	.440	.427

Table 1: Estimates of measures of covariation. Based on δ being 30 minutes and 5 minutes, respectively, for 26th October 1988, which is the 202nd day in the sample. Code: bpcv.ox

Figure 2 gives a slightly broader view on some of these issues, showing corresponding results for the first 100 days in the sample. On the top row of the Figure we plot a time series of the daily realised variances, realised covariances and their bipower versions. These are computed using 5 minute return data. The realised variance for the D-Mark is shown using crosses and it is quite variable, being quite high in the first 40 days, before falling substantially. The daily realised bipower variations are similiar, although they seem to be quite a lot lower on high volatility days. This provides informal evidence of jumps on those days, an issue we will return to formally later on in this paper. A similiar result holds for the Yen, although there is a second volatility spike in this series towards the last third of this period. The daily realised covariance of the two rates has a similiar pattern to the Yen series. The bottom row of Figure 2 returns to the results for 26th October 1988. This repeats the results in Table 1, but now shows how these variables change with δ, ranging from 5 minutes to 30 minutes. We can see the movements are modest for this day.

5.2 An extreme day: a balance of payments shock

On 15th January 1988 at 8.30 Eastern Standard Time (EST) in the United States of America a significant shock hit the foreign exchange markets. The Financial Times reported on its front page the next day

> "The dollar and share prices soared in hectic trading on world financial markets yesterday after the release of official figures showing that the US trade deficit had fallen to \$13.22 bn in November from October's record level of \$17.63 bn. The US currency surged 4 pfennigs and 4 yen within 10 minutes of the release of the figures and maintained the day's highest levels in late New York business ... ."

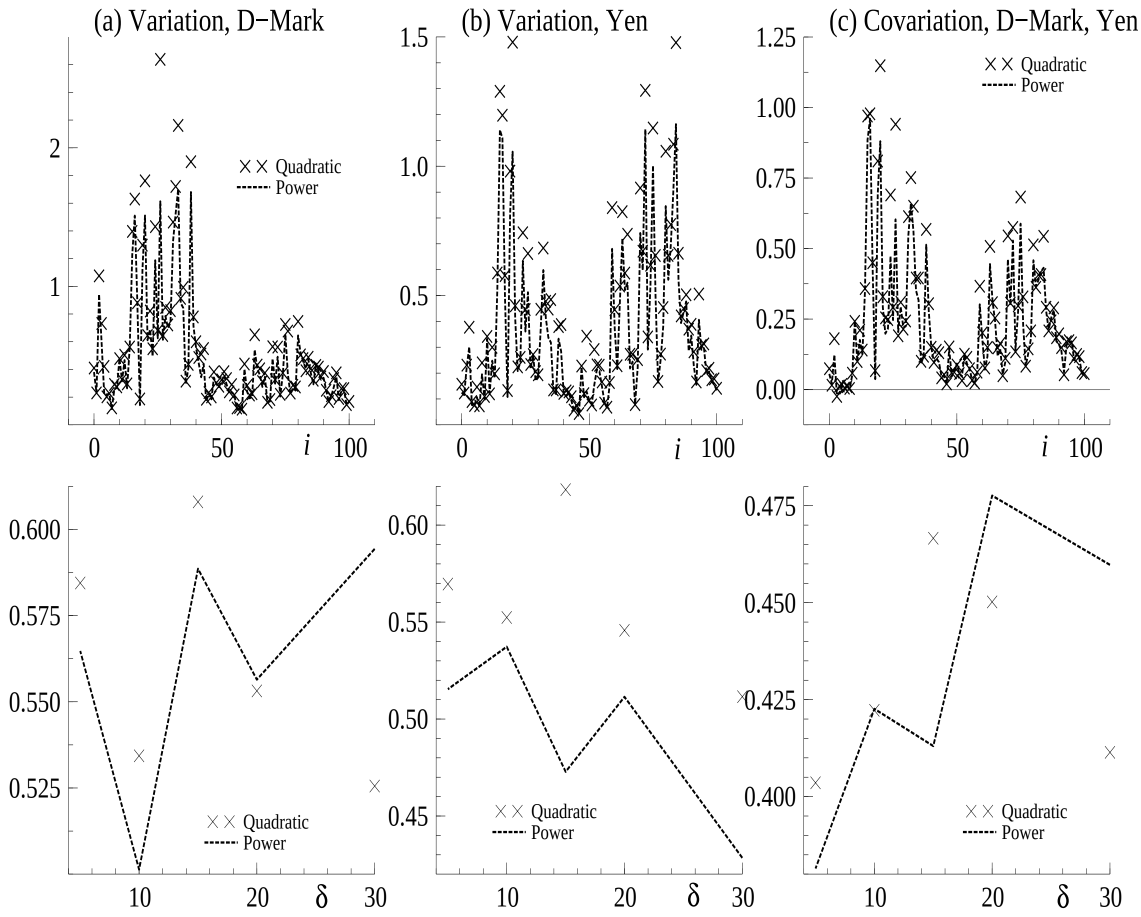

Figure 2: *Top line of graphs gives typical realised variances, covariances and bipower versions for the first 100 days from 1st December 1986. (a) Realised variance and realised bipower variation for D-Mark against U.S. Dollar. (b) Realised variance and realised bipower variation for Yen against U.S. Dollar. (c) Realised covariance and realised BPCV. Based on five minute return data. Bottom line: graphs of realised covariations and realised BPCV against δ for a randomly selected day in the sample. We have chosen the 26th October 1988. Code: bpcv.ox.*

The left hand panels of Figure 3 show the movements in D-Mark and Yen against the Dollar, with the rate imputted by the Olsen group every 30 minutes at the top of the graph and every 20 minutes at the bottom. There appears to be a important jump in the series around 8.30 EST. The right hand panels of the graph shows the cross plot of the bivariate return series. When δ is 30 minutes we see that the two jumps appear at the same time and really dominate the movements in the prices. This is the expected pattern, with the balance of payment shock strengthening the U.S. Dollar both against the German and Japanese currency. However, when we move δ down to 20 minutes something unexpected happens. We see at the bottom right of the Figure that the jumps in the D-Mark occurs earlier than in the Yen. Why did this happen?

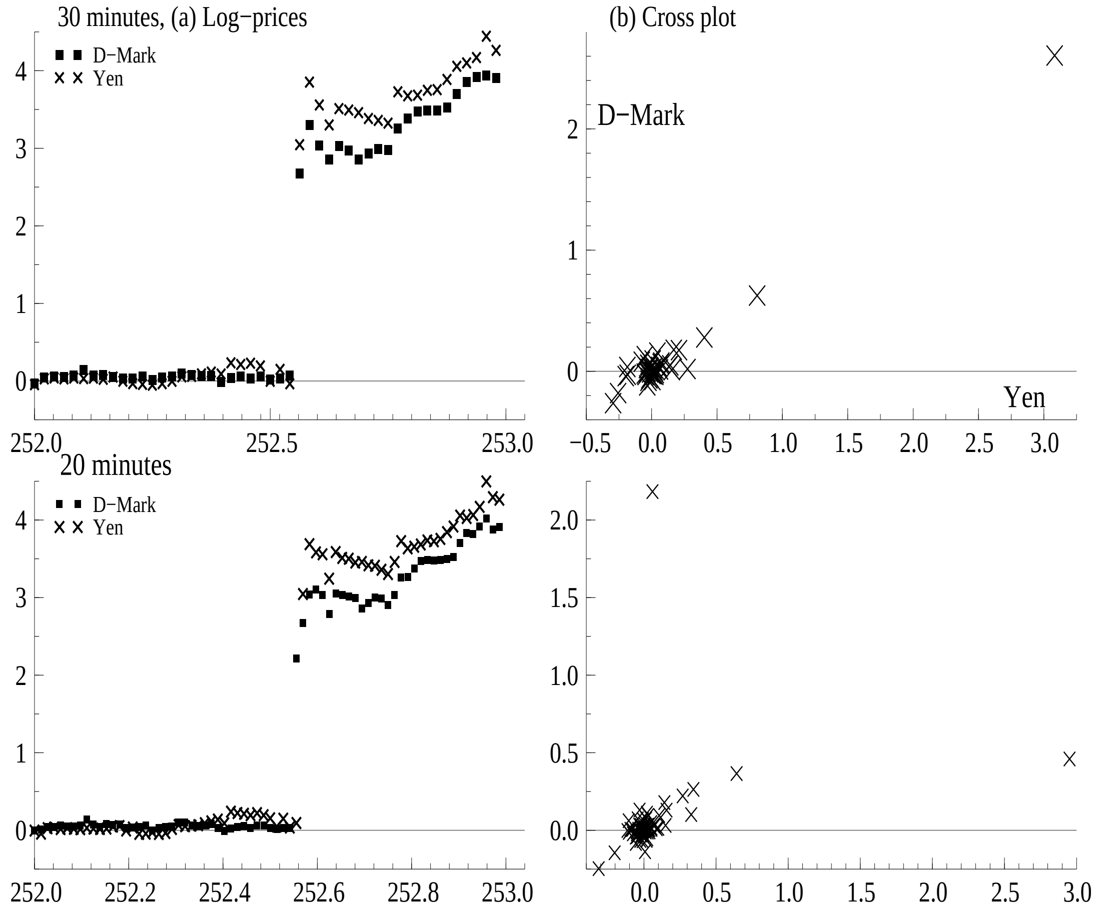

Figure 3: *DM and Yen against the Dollar, based on the Olsen dataset. Data is 15th January 1988. (a) recording of log-prices every 30 minutes. (b) Cross-plot of the 30-minute returns. Y-axis records D-Mark, x-axis records Yen returns, all against the US Dollar. Bottom graphs repeat this based on 20 minute records.*

Table 2 shows the Olsen's groups imputed returns for the D-Mark and Yen during the period around the balance of payments announcement. Recall the Olsen's groups data is computed off quotes from the Reuter's screen. We can see that the quotes for the D-Mark moved quickly, but it took a little time for the news to impact the Reuter's quotes on the Yen. It is clear that fully informed traders would not be actually trading at these out-dated Yen rates.

These market microstructure effects are important, not so much because they impact the univariate volatility measures, but because our measures of codependence can be completely messed up by this timing issue. In particular in the calculation of realised covariance we multiply high frequency returns, to measure codependence. Clearly if we use 30 minute returns we have no problem, but if we exploit 20, 10 or 5 minute returns then the dependence patterns will be destroyed. This is shown in Figure 4, which shows the cross plot of 30 and 20 minute returns. These plots indicate remarkably different levels of correlation. Rather curiously 15

Time (EST)	Time	D-Mark	Yen
8.25-8.30	162	0.0811	-0.0321
8.30-8.35	163	2.0278	0.0792
8.35-8.40	164	0.1178	0.0514
8.40-8.45	165	0.0899	2.1707

Table 2: Olsen's imputed returns based on quote data every 5 minutes for the D-Mark and Yen against U.S. Dollar. Code: bpcv.ox

minute returns, in this example, will be robust to the measurement errors.

The top panel in Figure 4 shows the realised variance and realised BPV for the D-Mark drawn against δ. This shows relatively stable statistics, with the realised variance being much higher than the corresponding realised BPV. This was noted in Barndorff-Nielsen and Shephard (2003) who show that there is statistically significant evidence that there is a jump in the price process. The mid-panel shows a similar result for the Yen, again suggesting a jump. The most important aspect of Figure 4 is the bottom panel which shows the realised covariance and realised BPCV. When δ is around 30 minutes the realised covariance is around 10, when δ is 20 the covariance is around 2.5. This is a dramatic change. The realised BPCV hardy changes with δ in this case, giving a value which is quite small compared with some of the recorded realised covariances. We will come back to the meaning of the stable realised BPCV in a later section.

6 Asymptotic theory for realised BPCV

6.1 General asymptotic theory

The asymptotic distribution of realised BPCV can be found by applying the general methods developed by Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004) in the context of the multivariate Brownian semimartingale

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u.$$

In this Section we will assume the following on the σ matrix process, although it is possible to relax these assumptions to allow for jumps in the volatility process. We refer the interested reader to Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004).

Definition 4 ***Assumption (H0):*** *We have*

$$\sigma_t = \sigma_0 + \int_0^t a_u^* \mathrm{d}u + \int_0^t v_{u-}^* \mathrm{d}W_u^* + \int_0^t \sigma_{u-}^* \mathrm{d}W_u, \tag{18}$$

where W^* is a vector of Brownian motion independent from W and the processes a^*, v^* and σ^* are adapted càdlàg arrays, with a^* also being predictable and locally bounded.

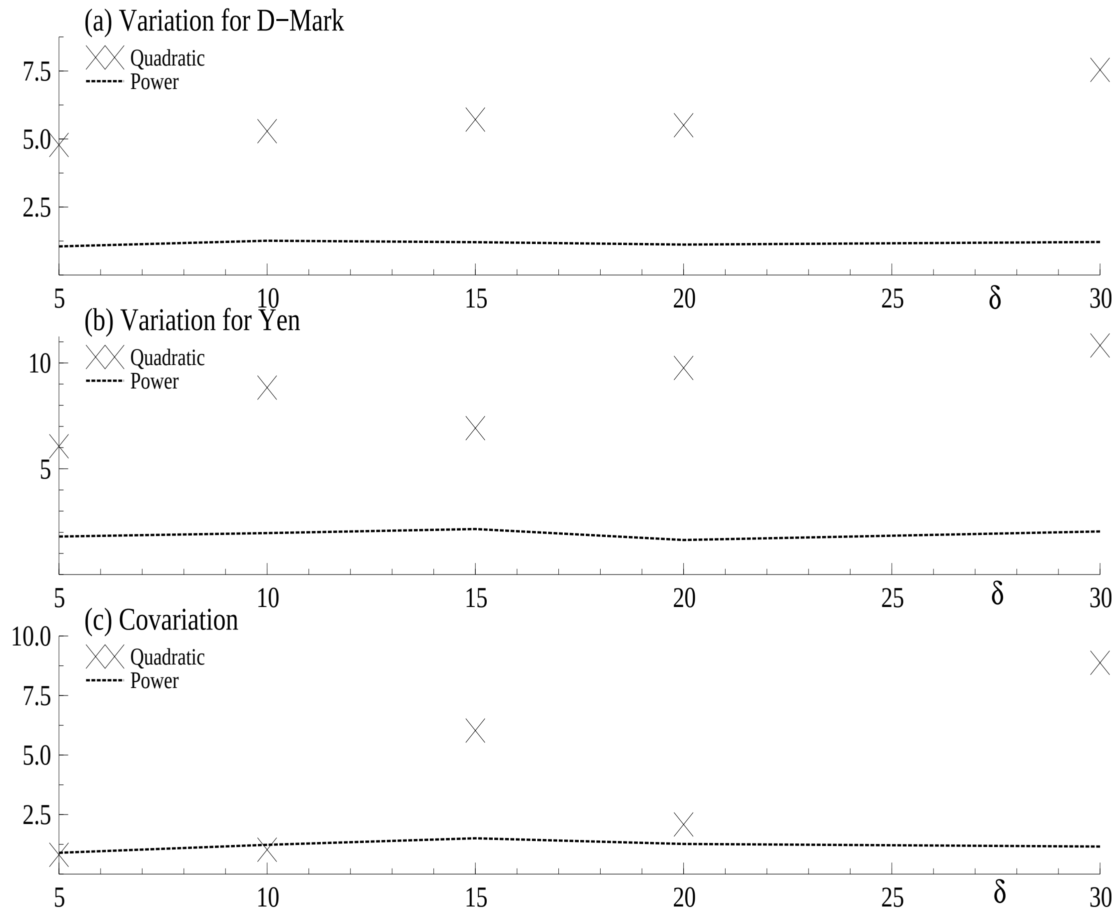

Figure 4: Data based on 15th January 1988. *(a) For the D-Mark series. Records realised variance and average realised bipower variation. Both are graphed against* δ. (b) *For the Yen series, we repeat (a). (c) Plot of the realised covariance and the average realised bipower covariation, both plotted against* δ. Code: bpcv.ox.

For simplicity of exposition we will put $p = 2$. We introduce the notation

$$\Sigma_t = \begin{pmatrix} \left(\sigma_1^2\right)_t & \left(\sigma_{1,2}\right)_t \\ \left(\sigma_{2,1}\right)_t & \left(\sigma_2^2\right)_t \end{pmatrix},$$

and then write

$$\left(\sigma_{1,2}^{+}\right)^2 = \sigma_1^2 + \sigma_2^2 + 2\sigma_{1,2}, \quad \left(\sigma_{1,2}^{-}\right)^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2},$$

$$\rho^{+,-} = \frac{\left(\sigma_1^2 - \sigma_2^2\right)}{\sigma_{1,2}^{+}\sigma_{1,2}^{-}} \in [-1, 1],$$

$$\vartheta(\rho) = S^2(\rho) + 2S(\rho) - 3.$$

Here

$$\begin{aligned} S(\rho) &= \sqrt{(1-\rho^2)} + \rho \arcsin \rho \\ &= \mu_1^{-2} \mathrm{E}\left\{|xy|\right\}, \end{aligned}$$

where x and y be two standard normal random variables with correlation coefficient ρ. Note $S(\rho) = S(-\rho)$. Suppose $\rho \geq 0$, then

$$\frac{\partial S(\rho)}{\partial \rho} = \arcsin \rho + \left(\rho + \frac{1}{2}\right)\left(1 - \rho^2\right)^{-1/2} > 0.$$

So $S(0) = 1$ and $S(\rho)$ monotocially increases on $[0, 1]$ to obtain its maximum $S(1) = \pi/2$. This implies that

$$0 = \vartheta(0) \leq \vartheta(\rho) \leq \vartheta(1) = \vartheta.$$

Note that

$$\begin{aligned} \vartheta &= \mu_1^{-4} + 2\mu_1^{-2} - 3 \\ &= \left(\pi^2/4\right) + \pi - 3 \\ &\simeq 2.609 \end{aligned}$$

Theorem 3 *Assume $Y \in \mathcal{BSM}$ and additionally condition (H0) holds. Then as $\delta \downarrow 0$ so the process*

$$\sqrt{n}\ \left[\left\{Y_{(1)\delta}, Y_{(2)\delta}\right\}_t - 4\mu_1^2 \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u\right],$$

converges stably in law towards a limiting process $U(g, h)$ having the form

$$U(g, h)_t = \int_0^t \sqrt{A(\sigma_u)}\ \mathrm{d}B_u, \tag{19}$$

where

$$A(\sigma_t) = \mu_1^4 \left[\left\{\left(\sigma_{1,2}^+\right)_t^4 + \left(\sigma_{1,2}^-\right)_t^4\right\} \vartheta(1) - 2\left\{\left(\sigma_{1,2}^+\right)_t \left(\sigma_{1,2}^-\right)_t\right\}^2 \vartheta(\rho_t^{+,-})\right].$$

In particular, for a single point in time t,

$$\sqrt{n}\ \left(Y^n(g, h)_t - 4\mu_1^2 \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u\right) \xrightarrow{L} MN\left(0, \int_0^t A(\sigma_u, g, h)\mathrm{d}u\right),$$

where MN denotes a mixed Gaussian distribution. Finally, $\int_0^t A(\sigma_u, g, h)\mathrm{d}u$ can be consistently estimated under the Brownian semimartigale assumption by constructing

$$s_j = \left|y_{(1)j-1} + y_{(2)j-1}\right| \left|y_{(1)j} + y_{(2)j}\right| - \left|y_{(1)j-1} - y_{(2)j-1}\right| \left|y_{(1)j} - y_{(2)j}\right|,$$

$$S_k = n \sum_{j=k+1}^{n} s_{j-k} s_j.$$

Then

$$S_0 + 2S_1 - 2S_2 - S_3 \xrightarrow{p} \int_0^t A(\sigma_u, g, h)\ \mathrm{d}u.$$

We should note that the estimator of $\int_0^t A(\sigma_u, g, h)\, \mathrm{d}u$ is not contained in Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004) and is novel. It is given for general problems in the Appendix, before being specialised to the case we are interested in for the proof of this theorem.

Remark 4 *Obviously* $\left(\sigma_{1,2}^{+}\right)_t^2 = \sigma_1^2 + \sigma_2^2 + 2\sigma_{1,2}$ *and* $\left(\sigma_{1,2}^{-}\right)_t^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}$*, which implies*

$$\left\{ \left(\sigma_{1,2}^{+}\right)_t^4 + \left(\sigma_{1,2}^{-}\right)_t^4 \right\} = 2\left[\left(\sigma_1^2 + \sigma_2^2\right)^2 + 4\sigma_{1,2}^2 \right],$$

while

$$\begin{aligned} \left(\sigma_{1,2}^{+}\right)_t^2 \left(\sigma_{1,2}^{-}\right)_t^2 &= \left(\sigma_1^2 + \sigma_2^2 + 2\sigma_{1,2}\right)\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}\right) \\ &= \left(\sigma_1^2 + \sigma_2^2\right)^2 - 4\sigma_{1,2}^2. \end{aligned}$$

Remark 5 *The corresponding realised quadratic variation result was first given in Barndorff-Nielsen and Shephard (2004a). It has*

$$\sqrt{n}\, \left[\left[Y_{(1)\delta}, Y_{(2)\delta}\right]_t - \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u \right] \xrightarrow{L} MN\left(0, \int_0^t \left(\sigma_{1,u}^2 \sigma_{2,u}^2 + \left(\sigma_{1,2}\right)_u^2\right) \mathrm{d}u\right).$$

If $\sigma_1^2 = \sigma_2^2$ *then* $\rho^{+,-} = 0$*, which implies*

$$\begin{aligned} A(\sigma_t) &= \mu_1^4 \vartheta(1) \left\{ \left(\sigma_{1,2}^{+}\right)_t^4 + \left(\sigma_{1,2}^{-}\right)_t^4 \right\} \\ &= 8\mu_1^4 \vartheta(1) \left\{ \left(\sigma_{1,t}^2\right)^2 + \left(\sigma_{1,2}\right)_t^2 \right\}, \end{aligned}$$

as $\vartheta(0) = 0$*. Thus*

$$\sqrt{n}\, \left[\frac{1}{4\mu_1^2} \left\{Y_{(1)\delta}, Y_{(2)\delta}\right\}_t - \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u \right] \xrightarrow{L} MN\left(0, \frac{1}{2}\vartheta(1) \int_0^t \left(\sigma_{1,u}^4 + \left(\sigma_{1,2}\right)_u^2\right) \mathrm{d}u\right).$$

This implies that when the variances of the two processes are roughly equal then realised bipower variation is around 30% less efficient at estimating the integrated covariance than the realised covariation measure when there are no jumps.

Remark 6 *As the realised quadratic variation is, asymptotically, a fully efficient estimator of* $\int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u$ *when the process is a Brownian semimartingale, it follows from a Hausman (1978) type argument that*

$$\begin{aligned} &\sqrt{n} \begin{bmatrix} \left[Y_{(1)\delta}, Y_{(2)\delta}\right]_t - \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u \\ \frac{1}{4\mu_1^2} \left\{Y_{(1)\delta}, Y_{(2)\delta}\right\}_t - \int_0^t \left(\sigma_{1,2}\right)_u \mathrm{d}u \end{bmatrix} \\ &\xrightarrow{L} MN\left(0, \begin{array}{cc} \int_0^t \left(\sigma_{1,u}^2 \sigma_{2,u}^2 + \left(\sigma_{1,2}\right)_u^2\right) \mathrm{d}u & \int_0^t \left(\sigma_{1,u}^2 \sigma_{2,u}^2 + \left(\sigma_{1,2}\right)_u^2\right) \mathrm{d}u \\ \int_0^t \left(\sigma_{1,u}^2 \sigma_{2,u}^2 + \left(\sigma_{1,2}\right)_u^2\right) \mathrm{d}u & \frac{1}{16\mu_1^4} \int_0^t A(\sigma_u, g, h) \mathrm{d}u \end{array}\right). \end{aligned}$$

This implies that we can use a two sided Hausman-type test for jumps in the covariation process based upon

$$\sqrt{n}\left[\left[Y_{(1)\delta},Y_{(2)\delta}\right]_t-\frac{1}{4\mu_1^2}\left\{Y_{(1)\delta},Y_{(2)\delta}\right\}_t\right]$$
$$\xrightarrow{L} MN\left(0,\frac{1}{16\mu_1^4}\int_0^t A(\sigma_u,g,h)\mathrm{d}u-\int_0^t\left(\sigma_{1,u}^2\sigma_{2,u}^2+(\sigma_{1,2})_u^2\right)\mathrm{d}u\right).$$

Example 2 *To see how the asymptotic theory performs in finite samples, and to study the relative efficiency of the realised BPCV and realised covariance, we report some results from a simple Monte Carlo experiment. Throughout we use the bias adjusted realised BPCV. We look at three cases where, each where $\sigma_1^2\sigma_2^2=1$. We vary σ_1^2, ρ and n, reporting the asymptotic results using the case where $n=\infty$. Our focus is on the variability of*

$$\left[X_{(1)\delta},X_{(2)\delta}\right]_1-\sigma_{1,2}\quad \text{and}\quad \frac{1}{4\mu_1^2}\left\{X_{(1)\delta},X_{(2)\delta}\right\}_1-\sigma_{1,2},$$

for both estimators have no discernable bias. Throughout we report n times the sampling variance. The results given in Table 3 show the expected result that the realised covariance is more efficient that the realised BPCV. Importantly the degree of inefficiency of realised BPCV increases as σ_1^2 and σ_2^2 become more different, while the inefficiency does not vary much with ρ.

Remark 7 *Suppose X,Y is a bivariate Brownian semimartingale and we define, using an obvious notation,*

$$C(X,Y)_t=\mu_1^4\int_0^t\vartheta\left(\rho_{(x,y)}\right)_u\left(\sigma_x^2\right)_u\left(\sigma_y^2\right)_u\mathrm{d}u.$$

Also define

$$c(x_j,y_j)=a_jb_j+2a_jb_{j-1}-3a_jb_{j-2}$$
$$a_j=|x_j|\,|x_{j-1}|\,,\quad b_j=|y_j|\,|y_{j-1}|\,,$$

then

$$\widehat{C}(X,Y)_t=n\sum_{j=1}^n c(x_j,y_j)\xrightarrow{p}C(X,Y)_t.$$

ISSUE TO BE CLARRIFIED. It is not clear if $\widehat{C}$ is robust to common jumps as terms of the type a_jb_j and a_jb_{j-1} have problems.

7 Discussion and extensions

7.1 Building reduced form models

Following Barndorff-Nielsen and Shephard (2004b), Andersen, Bollerslev, and Diebold (2003) have used bipower variation as an input into new reduced form forecasting devices for modelling

n	$\rho = 0.9$		$\rho = 0.5$		$\rho = 0$	
	RCV	RBPCV	RCV	RBPCV	RCV	RBPCV
1,1						
5	1.79	2.66	1.21	1.78	0.98	1.42
10	1.81	2.49	1.23	1.70	0.99	1.35
25	1.82	2.44	1.24	1.65	0.99	1.32
100	1.81	2.39	1.25	1.63	1.00	1.31
∞	1.81	2.36	1.25	1.62	1	1.30
3,1/3						
5	1.79	2.76	1.23	2.07	.989	1.73
10	1.81	2.58	1.25	1.96	1.00	1.65
25	1.82	2.53	1.25	1.92	.996	1.60
100	1.81	2.48	1.26	1.88	1.01	1.58
∞	1.81	2.44	1.25	1.85	1	1.56
5,1/5						
5	1.79	2.78	1.23	2.20	.989	1.92
10	1.81	2.60	1.25	2.08	1.00	1.82
25	1.82	2.55	1.25	2.03	.996	1.76
100	1.81	2.50	1.26	1.99	1.01	1.73
∞	1.81	2.47	1.25	1.96	1.00	1.71

Table 3: Finite sample behaviour of the realised covariance and the realised bipower covariation based on bivariate Brownian motion with common variances. Based on $50,000$ simulations. Results are n times sample variance of the realised quantities. Code: bpcv.ox

future values of realised variances (which in turn proxy the variability of future prices). This follows the influential line of thinking of Andersen, Bollerslev, Diebold, and Labys (2003) who modelled realised variances in terms of lags of previous realised variances. Following initial versions of the work reported in this paper, they used the test for a jump given in result given in Barndorff-Nielsen and Shephard (2004b) to truncate the BPV based estimator of the QV of the jump component if the jumps are not significant. This shrinkage style estimator seems remarkably successful in empirical work, yielding fresh insights and improved forecast accuracy.

8 Empirical illustration: the distribution theory

9 Conclusions

10 Acknowledgments

Ole E. Barndorff-Nielsen's work is supported by CAF (`www.caf.dk`), which is funded by the Danish Social Science Research Council, and by MaPhySto (`www.maphysto.dk`), which is funded by the Danish National Research Foundation. Neil Shephard's research is supported by the UK's ESRC through the grant "High frequency financial econometrics based upon power variation."

All the calculations made in this paper are based on software written by the authors using the Ox language of Doornik (2001). We thank Tim Bollerslev and Xin Huang for discussions on this topic.

A Appendix

A.1 Proof of Theorem 2

Result (14) is trivial. We start with (15). Clearly

$$\begin{aligned}
\frac{4}{\gamma_{1,\delta}}\left\{Y_{(l)\delta}, -Y_{(k)\delta}\right\} &= \sum_{j=2}^{\lfloor t/\delta \rfloor}\left(\left|y_{(l)j-1} - y_{(k)j-1}\right|\left|y_{(l)j} - y_{(k)j}\right| - \left|y_{(l)j-1} + y_{(k)j-1}\right|\left|y_{(l)j} + y_{(k)j}\right|\right) \\
&= -\sum_{j=2}^{\lfloor t/\delta \rfloor}\left(\left|y_{(l)j-1} + y_{(k)j-1}\right|\left|y_{(l)j} + y_{(k)j}\right| - \left|y_{(l)j-1} - y_{(k)j-1}\right|\left|y_{(l)j} - y_{(k)j}\right|\right) \\
&= -\frac{4}{\gamma_{1,\delta}}\left\{Y_{(l)\delta}, Y_{(k)\delta}\right\},
\end{aligned}$$

which delivers that result. Next (16)

$$\begin{aligned}
\frac{4}{\gamma_{1,\delta}}\left\{aY_{(l)\delta}, aY_{(k)\delta}\right\} &= \sum_{j=2}^{\lfloor t/\delta \rfloor}\left(\left|ay_{(l)j-1} + ay_{(k)j-1}\right|\left|ay_{(l)j} + ay_{(k)j}\right| - \left|ay_{(l)j-1} - ay_{(k)j-1}\right|\left|ay_{(l)j} - ay_{(k)j}\right|\right) \\
&= \frac{4}{\gamma_{1,\delta}}a^2\left\{Y_{(l)\delta}, Y_{(k)\delta}\right\}.
\end{aligned}$$

Finally, (17) follows as

$$\begin{aligned}
\frac{4}{\gamma_{1,\delta}}\left\{Y_{(l)\delta}, aY_{(l)\delta}\right\} &= \sum_{j=2}^{\lfloor t/\delta \rfloor}\left(\left|y_{(l)j-1} + ay_{(l)j-1}\right|\left|y_{(l)j} + ay_{(l)j}\right| - \left|y_{(l)j-1} - ay_{(l)j-1}\right|\left|y_{(l)j} - ay_{(l)j}\right|\right) \\
&= \sum_{j=2}^{\lfloor t/\delta \rfloor}\left|y_{(l)j-1}y_{(l)j}\right|\left\{|1+a|\,|1+a| - |1-a|\,|1-a|\right\} \\
&= \frac{4}{\gamma_{1,\delta}}a\left\{Y_{(l)\delta}, Y_{(l)\delta}\right\}.
\end{aligned}$$

A.2 Proof of Theorem 3

We first recall a special case of the result of Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004). Let us write returns in the form

$$y_i = Y_{i/n} - Y_{(i-1)/n} \tag{20}$$

where n and i are positive integers. Then they studied the behaviour of processes of the form

$$Y^n(g,h)_t = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} g(\sqrt{n}\ y_{i-1})'h(\sqrt{n}\ y_i), \tag{21}$$

as $n \to \infty$. Here we simplify their results and assume g and h are $d \times 1$ functions. They used the following notation

$$\rho_\sigma(g) = \mathrm{E}_{X|\sigma}\left\{g(X)\right\}, \quad \text{where} \quad X|\sigma \sim N(0, \sigma\sigma'),$$

$$\rho_\sigma(gh) = \mathrm{E}_{X|\sigma}\left\{g(X)h(X)\right\},$$

and

$$Y(g,h)_t = \int_0^t \rho_{\sigma_u}(g)\rho_{\sigma_u}(h)\mathrm{d}u. \tag{22}$$

Theorem 4 *(Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004)) Assume $Y \in \mathcal{BSM}$ and additionally condition (H0) holds. Then as $\delta \downarrow 0$ so the process*

$$\sqrt{n}\ \left(Y^n(g,h)_t - Y(g,h)_t\right)$$

converges stably in law towards a limiting process $U(g,h)$ having the form

$$U(g,h)_t = \int_0^t \sqrt{A(\sigma_u, g, h)}\ \mathrm{d}B_u, \tag{23}$$

where

$$A(\sigma, g, h) = \sum_{i=1}^{d_2}\sum_{j=1}^{d_2}\left\{\begin{array}{l} \rho_\sigma\left(g^i g^j\right)\rho_\sigma\left(h^i h^j\right) + \rho_\sigma\left(g^i\right)\rho_\sigma\left(h^j\right)\rho_\sigma\left(g^j h^i\right) \\ +\rho_\sigma\left(g^j\right)\rho_\sigma\left(h^i\right)\rho_\sigma\left(g^i h^j\right) \\ -3\rho_\sigma\left(g^i\right)\rho_\sigma\left(g^j\right)\rho_\sigma\left(h^i\right)\rho_\sigma\left(h^j\right). \end{array}\right\}$$

Our result follows from the application of this Theorem. In particular, write

$$\begin{aligned} Y^n(g,h)_t &= \frac{1}{n}\sum_{i=1}^n \left(\left|y_{i-1}^{(1)} + y_{i-1}^{(2)}\right|, -\left|y_{i-1}^{(1)} - y_{i-1}^{(2)}\right|\right)\left(\left|y_i^{(1)} + y_i^{(2)}\right|, \left|y_i^{(1)} - y_i^{(2)}\right|\right)' \\ &= \frac{1}{n}\sum_{i=1}^n \left(g_{i-1}^{(1)}, g_{i-1}^{(2)}\right)\left(h_i^{(1)}, h_i^{(2)}\right)' \\ &= \frac{1}{n}\sum_{i=1}^n \left(k_{i-1}^{(1)}, -k_{i-1}^{(2)}\right)\left(k_i^{(1)}, k_i^{(2)}\right)', \end{aligned}$$

where

$$k^{(1)} = \left|y^{(1)} + y^{(2)}\right|, \quad k^{(2)} = \left|y^{(1)} - y^{(2)}\right|$$

$$g^{(1)} = h^{(1)} = k^{(1)}, \quad h^{(2)} = -g^{(2)} = k^{(2)}.$$

Let x and y be two standard normal random variables with correlation coefficient ρ. Then it is known from Proposition 1, which is given in the next subsection, that

$$S(\rho) = \mu_1^{-2}\mathrm{E}\left\{|xy|\right\} = \sqrt{(1-\rho^2)} + \rho \arcsin\rho. \tag{24}$$

Note $S(\rho) = S(-\rho)$, so the asymptotic standard deviations in Theorem 3 are symmetric in ρ around zero. Suppose $\rho \geq 0$, then

$$\frac{\partial S(\rho)}{\partial \rho} = \arcsin \rho + \left(\rho + \frac{1}{2}\right)\left(1-\rho^2\right)^{-1/2} > 0.$$

So $S(0) = 1$ and $S(\rho)$ monotocially increases on $[0,1]$ to obtain its maximum $S(1) = \pi/2$. Note also that

$$0 = \vartheta(0) \leq \vartheta(\rho) \leq \vartheta(1).$$

Let us define $\lambda = \left(\sigma_1^2 - \sigma_2^2\right) / \left(\sigma_{1,2}^{+}\sigma_{1,2}^{-}\right)$. Then

$$\rho_\sigma(k^{(1)}) = \mu_1 \sigma_{1,2}^{+}, \quad \rho_\sigma(k^{(2)}) = \mu_1 \sigma_{1,2}^{-},$$

$$\rho_\sigma(k^{(1)}k^{(1)}) = \sigma_{1,2}^{+2}, \quad \rho_\sigma(k^{(2)}k^{(2)}) = \left(\sigma_{1,2}^{-}\right)^2, \quad \rho_\sigma(k^{(1)}k^{(2)}) = \mu_1^2 S(\lambda)\sigma_{1,2}^{+}\sigma_{1,2}^{-}.$$

Then

$$\begin{aligned}
A(\sigma, g, h) &= \sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j+2}\left\{\rho_\sigma^2(k^ik^j) + 2\rho_\sigma(k^i)\rho_\sigma(k^j)\rho_\sigma(k^jk^i) - 3\rho_\sigma^2(k^i)\rho_\sigma^2(k^j)\right\} \\
&= \left\{\rho_\sigma^2(k^1k^1) + 2\rho_\sigma^2(k^1)\rho_\sigma(k^1k^1) - 3\rho_\sigma^4(k^1)\right\} \\
&\quad -2\left\{\rho_\sigma^2(k^1k^2) + 2\rho_\sigma(k^1)\rho_\sigma(k^2)\rho_\sigma(k^1k^2) - 3\rho_\sigma^2(k^1)\rho_\sigma^2(k^2)\right\} \\
&\quad +\left\{\rho_\sigma^2(k^2k^2) + 2\rho_\sigma^2(k^2)\rho_\sigma(k^2k^2) - 3\rho_\sigma^4(k^2)\right\} \\
&= \mu_1^4\left[\left\{\left(\sigma_{1,2}^{+}\right)^4 + \left(\sigma_{1,2}^{-}\right)^4\right\}\vartheta(1) - 2\left(\sigma_{1,2}^{+}\sigma_{1,2}^{-}\right)^2\vartheta(\lambda)\right].
\end{aligned}$$

This is the required result.

□

The remaining part of the proof requires us to estimate $\int_0^t A(\sigma_u, g, h)\ \mathrm{d}u$. This has not been previously discussed in the literature. Here we give a general solution to this problem, which may be helpful outside the scope of this paper.

Theorem 5 *Suppose Y is a Brownian semimartingale and*

$$A(\sigma, g, h) = \sum_{i=1}^{d_2}\sum_{j=1}^{d_2}\left\{\begin{array}{l}\rho_\sigma\left(g^ig^j\right)\rho_\sigma\left(h^ih^j\right) + 2\rho_\sigma\left(g^i\right)\rho_\sigma\left(h^j\right)\rho_\sigma\left(g^jh^i\right) \\ -3\rho_\sigma\left(g^i\right)\rho_\sigma\left(g^j\right)\rho_\sigma\left(h^i\right)\rho_\sigma\left(h^j\right).\end{array}\right\}$$

Define

$$S_k = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\left(g_{i-k-1}'h_{i-k}\right)\left(g_{i-1}'h_i\right).$$

Then

$$S_0 + 2S_1 - 2S_2 - S_3 \xrightarrow{p} \int_0^t A(\sigma_u, g, h)\ \mathrm{d}u.$$

Proof. Clearly

$$S_0 = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} (g_{i-1}h_i)^2 \xrightarrow{p} \sum_{i=1}^{d_2}\sum_{j=1}^{d_2}\int_0^t \rho_{\sigma_u}\left(g^i g^j\right)\rho_{\sigma_u}\left(h^i h^j\right) \mathrm{d}u,$$

by applying the convergence in probability result of Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004), that for generic g and h statistics (21) converge to (22), to the components of this statistics of the form. In particular, for example,

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\left(g_{i-1}^{(1)}h_i^{(1)}\right)^2 \xrightarrow{p} \int_0^t \rho_{\sigma_u}\left(g^1 g^1\right)\rho_{\sigma_u}\left(h^1 h^1\right) \mathrm{d}u.$$

The higher lagged versions, used to construct the statistics S_1 and S_2, can be handled in the same way using the results in Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2004).

□

We now just need to specialise this result to our case, which is straightforward. This completes the Proof of Theorem 3.

A.3 Proposition 1

In our work on realised power variation we have explored taking sums of powers of absolute values of increments. In order to develop a multivariate version of these objects we need to understand the properties of absolute values of products of normal variables. Some results on this are given in the following Proposition.

Proposition 1 *Let x and y be two standard normal random variables with correlation coefficient ρ. Define s and q by*

$$q = |xy| \quad \text{and} \quad s = \sqrt{x^2 + y^2},$$

then the joint density of q and s is

$$p(q, s; \rho) = \frac{2}{\pi\sqrt{(1-\rho^2)}} \cosh\left(\frac{1}{1-\rho^2}q\right) \exp\left(-\frac{1}{2}\frac{1}{1-\rho^2}s^2\right) s(s^4 - 4q^2)^{-1/2}, \tag{25}$$

for $s \geq \sqrt{2q}$ and 0 otherwise; and the marginal law of q has density

$$p(q; \rho) = \frac{2}{\pi\sqrt{(1-\rho^2)}} \cosh\left(\frac{\rho}{1-\rho^2}q\right) K_0\left(\frac{1}{1-\rho^2}q\right), \tag{26}$$

where K_0 is a modified Bessel function of the third kind. Finally

$$R(q) = \mathrm{E}\left\{|xy|\right\} = \frac{2}{\pi}\left\{\sqrt{(1-\rho^2)} + \rho \arcsin \rho\right\}. \tag{27}$$

Proof The mapping from (x, y) to $(q = |xy|, u = (|x| - |y|)^2)$ is eight-to-one with, for $y, x > 0$

$$\left| \begin{array}{cc} \frac{\partial q}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial u}{\partial y} \end{array} \right| = \left| \begin{array}{cc} y & 2(x-y) \\ x & -2(x-y) \end{array} \right| = 2\left(y^2 - x^2\right).$$

The same holds for all 8 quadrants. Thus the Jacobian of the transformation is

$$\frac{1}{2\left|y^2 - x^2\right|} = \frac{1}{2\sqrt{u(u+4q)}}.$$

As

$$p(x, y; \rho) = \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}(x^2 + y^2 - 2\rho xy)\right\}$$

and $u = x^2 + y^2 - 2q$ we obtain

$$\begin{aligned} p(q, u; \rho) &= \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}(u + 2q - 2\rho q)\right\} \frac{4}{2\sqrt{u(u+4q)}} \\ &\quad + \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}(u + 2q + 2\rho q)\right\} \frac{4}{2\sqrt{u(u+4q)}} \\ &= \frac{1}{\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}(u + 2q)\right\} \frac{1}{\sqrt{u(u+4q)}} \\ &\quad \times \left\{\exp\left(\frac{\rho}{1-\rho^2}q\right) + \exp\left(-\frac{\rho}{1-\rho^2}q\right)\right\} \\ &= \frac{1}{\pi\sqrt{(1-\rho^2)}} \exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}(u + 2q)\right\} \frac{1}{\sqrt{u(u+4q)}} \cosh\left(\frac{\rho}{1-\rho^2}q\right). \end{aligned} \tag{28}$$

This is the required result for $(q, u) \in \mathbb{R}_+^2$.

A well-known Bessel function identity states that

$$\int_0^\infty \{x(x+a)\}^{-1/2} e^{-\theta x} \mathrm{d}x = e^{a\theta/2} K_0\left(\frac{a\theta}{2}\right)$$

(cf. Gradshteyn and Ryzhik (1965, 3.364.3). The expression (26) now follows by integration of (28) with respect to u. □

The mean value of q^r, for positive real r, enters our subsequent considerations. An explicit expression for $\mathrm{E}\{q^r\}$ does not seem to exist in general, but for integer r simple formulae are available. Below we derive the formula for $r = 1$ from (26). For r integer and greater than 1 a similar approach works.[1] However, our interest is mainly in the values of $r \in \left[\frac{1}{2}, 2\right)$, particularly $r = 1/2$ and $r = 1$.

The relation saying that (26) integrates to 1 is a special case of the formula (Gradshteyn and Ryzhik (1965, 6.661.2))

$$\int_0^\infty \cosh(ax) K_0(bx) \mathrm{d}x = \frac{\pi}{2}(b^2 - a^2)^{-1/2} \tag{29}$$

[1] The resulting expressions have in fact be derived a long time ago, in the Thirties, by Kamat and Nabeya by a different approach that more generally yields expressions for $\mathrm{E}\{|x|^r|y|^s\}$ for positive integer r and s in terms of the hypergeometric functions, see Johnson and Kotz (1972, pp. 91-92).

which holds for $|a| < b$. With the same restriction on a and b we have (Gradshteyn and Ryzhik (1965, 6.661.1))

$$\int_0^\infty \sinh(ax)K_0(bx)\mathrm{d}x = \arcsin(a/b)(b^2 - a^2)^{-1/2}. \tag{30}$$

From the identity $\sin(\arcsin y) = y$ and the fact that $\cos(\arcsin y) = (1 - y^2)^{1/2}$ we have $\arcsin' y = (1 - y^2)^{-1/2}$. It follows, by differentiation of (30), that

$$\begin{aligned}\int_0^\infty x\cosh(ax)K_0(bx)\mathrm{d}x &= b^{-1}(1 - a^2/b^2)^{-1/2}(b^2 - a^2)^{-1/2} + a(b^2 - a^2)^{-3/2}\arcsin(a/b) \\ &= (b^2 - a^2)^{-1} + a\arcsin(a/b)(b^2 - a^2)^{-3/2}. \qquad (31)\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{E}\{q\} &= \frac{2}{\pi\sqrt{(1-\rho^2)}}\left\{1 - \rho^2 + \frac{\rho}{1-\rho^2}(1-\rho^2)^{3/2}\arcsin\rho\right\} \\ &= \frac{2}{\pi}\left\{\sqrt{(1-\rho^2)} + \rho\arcsin\rho\right\}. \qquad (32)\end{aligned}$$

We shall use the shorthand $R(\rho)$ for the resulting expression, i.e.

$$R(\rho) = \mathrm{E}\{|xy|\} = \frac{2}{\pi}\left\{\sqrt{(1-\rho^2)} + \rho\arcsin\rho\right\}.$$

A.4 Multivariate interpolation

A.4.1 The algorithm

The Olsen group have kindly made available to us an exchange rate dataset which records every five minutes the most recent quote to appear on the Reuters screen from 1st December 1986 until 30th November 1996. When prices are missing they have interpolated them. Details of this processing are given in Dacorogna, Gencay, Müller, Olsen, and Pictet (2001). The same dataset was analysed by Andersen, Bollerslev, Diebold, and Labys (2001). We follow the extensive work of Torben Andersen and Tim Bollerslev on this dataset, who remove much of the times when the market is basically closed. This includes almost all of the weekend, while they have taken out most US holidays. The result is what we will regard as a single time series of length 705,313 observations. Although many of the breaks in the series have been removed, sometimes there are sequences of very small price changes caused by, for example, unmodelled non-US holidays or data feed breakdowns. This is problematic for high frequency volatility modelling, for it will result in periods with artifically extremely low aggregate volatility measures. Barndorff-Nielsen and Shephard (2002) have addressed this issue by replacing missing data in each price process by samples from independent Brownian bridges, with rather modest variances. This produces sensible volatility measures, but the inherent univariate nature of the analysis means that the

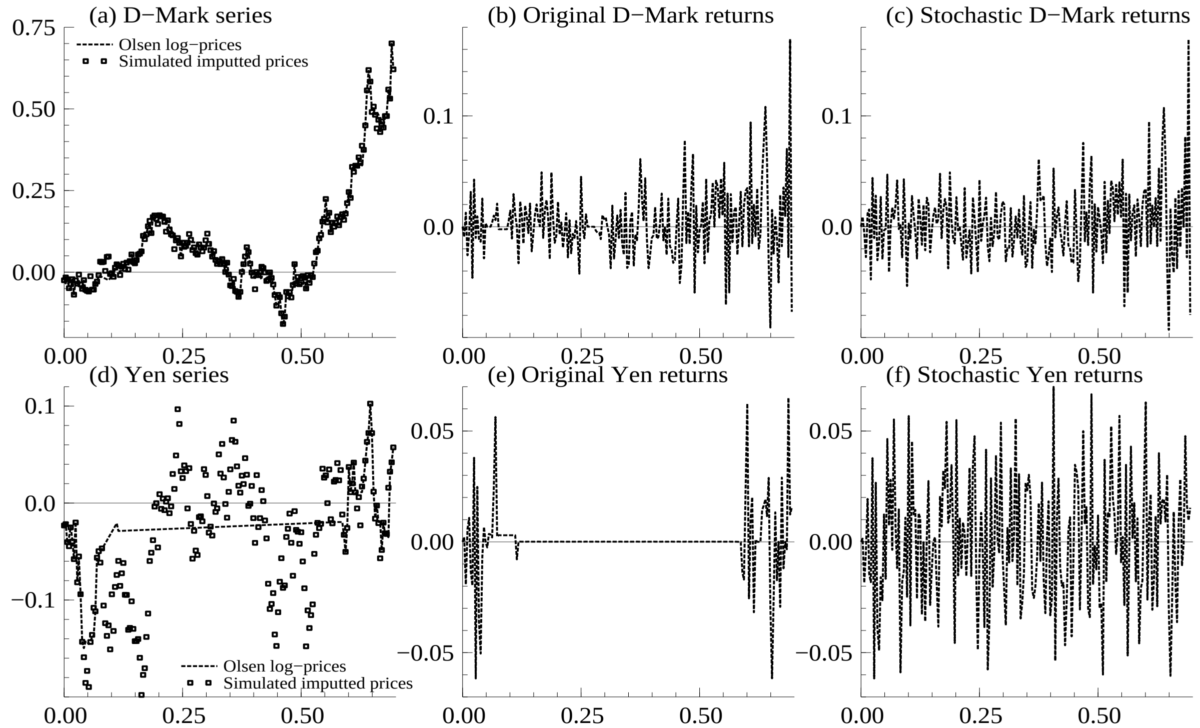

Figure 5: *Top line of graphs are the raw and interpolated data using a Brownian bridge interpolator. Bottom line of graphs is the corresponding returns. The x-axes are marked off in days.*

imputation ignores dependence between asset prices. To deal with this we need a more flexible approach.

We deal with this by using a calibrated statistical model

$$y_i = \alpha_i + D_i \varepsilon_i, \quad \alpha_{i+1} = \alpha_i + \eta_i, \quad \varepsilon_i \overset{i.i.d.}{\sim} N(0, \infty I), \quad \eta_i \overset{i.i.d.}{\sim} N(0, c\Upsilon), \quad \varepsilon_i \perp\!\!\!\perp \eta_j,$$

where y_i is the vector of Olsen prices recorded every 5 minutes at time i. We regard α_i as the "true" rate if the data was not missing. Here D_i is diagonal, with binary elements. Value 1 occurs if the rate is missing, 0 if not. Hence we can have situations with 0, 1, 2, 3, ... up to p rates being missing. If the l-th rate is observed at time l then trivially $y_{(l)i} = \alpha_{(l)i}$. Throughout we will calibrate this model by taking

$$\Upsilon = \frac{1}{\lfloor t/\delta \rfloor} \sum_{i=1}^{\lfloor t/\delta \rfloor} y_i y_i',$$

the long-run 5 minute covariance between the rates. We preselected $c = 0.1$, thus the extraction devices always produce modestly valued missing returns. Interpolation can, in principle, be carried out deterministically or stochastically. We can report either $\mathrm{E}(\alpha|y)$ or simulate from $\alpha|y$. We favour the latter as it gives more realistic price processes from the viewpoint of volatility modelling. This sampling can be carried out using the simulation smoother of de Jong and Shephard (1995), which is available in the software Ox using the package SsfPack, documented in Koopman, Shephard, and Doornik (1999). Carrying out the simulation efficiently is important in our context, as our sample size is over a half of a million. Having said that, the stochastic interpolation algorithm we use takes only a handful of seconds on a Pc.

In practice we have taken prices as being missing if the corresponding absolute return is below 0.01%. By using this simulation procedure we are not affecting the long run trajectory of prices, while the impact on realised covariation is usually very small indeed unless the vast majority of observations on a specific day is missing. The procedure is illustrated in Figure 5, which shows the first 200 observations in the Dollar/DM and Dollar/Yen series we have used in this paper. Please note that this is a very extreme case of many missing data points. Later stretches of the data have much fewer breaks in them, but this graph illustrates the effects of our intervention. Clearly our approach is ad hoc. However, a proper statistical modelling of these breaks is very complicated due to their many causes and the fact that our dataset is enormous.

A.4.2 Summary statistics

In this subsection we give average values of the different codependence measures. We start with the empirical covariance of daily returns over the entire sample period, which is

$$\begin{pmatrix} .503 & \mathit{.658} \\ .328 & .495 \end{pmatrix},$$

where the number in italics is the corresponding correlation. We can compute corresponding realised quantities by calculating them each day and then averaging the results over the entire sample. The corresponding 30 minute returns based realised covariance and average realised bipower covariation (based on $Q = 6$) are, respectively,

$$\begin{pmatrix} .470 & \mathit{.616} \\ .288 & .465 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} .383 & \mathit{.621} \\ .236 & .380 \end{pmatrix}.$$

We can see the realised covariation numbers are broadly higher than the corresponding bipower versions, although the correlation between assets is estimated to be similiar. The corresponding results based on 5 minute returns yield

$$\begin{pmatrix} .505 & \mathit{.573} \\ .291 & .510 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} .459 & \mathit{.592} \\ .272 & .463 \end{pmatrix}.$$

These have higher average levels of volatility, but somewhat lower levels of dependence.

We can compare the results with those generated when we do not use the stochastic adjustment scheme. In the case of 5 minute returns, the results would be

$$\begin{pmatrix} .528 & \mathit{.455} \\ .243 & .538 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} .446 & \mathit{.464} \\ .208 & .449 \end{pmatrix}.$$

This shows the interpolation method has a big impact on the measurement of codependence, although much less on measuring volatility. Finally, we should note that when we use 30 minute returns the impact of using stochastic interpolation is much more modest.

References

Aït-Sahalia, Y. (2002). Telling from discrete data whether the underlying continuous-time model is a diffusion. *Journal of Finance 57*, 2075–2112.

Andersen, T. G., L. Benzoni, and J. Lund (2002). An empirical investigation of continuous-time equity return models. *Journal of Finance 57*, 1239–1284.

Andersen, T. G., T. Bollerslev, and F. X. Diebold (2003). Some like it smooth, and some like it rough: untangling continuous and jump components in measuring, modeling and forecasting asset return volatility. Unpublished paper: Economics Dept, Duke University.

Andersen, T. G., T. Bollerslev, and F. X. Diebold (2004). Parametric and nonparametric measurement of volatility. In Y. Ait-Sahalia and L. P. Hansen (Eds.), *Handbook of Financial Econometrics*. Amsterdam: North Holland. Forthcoming.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2001). The distribution of exchange rate volatility. *Journal of the American Statistical Association 96*, 42–55. Correction published in 2003, volume 98, page 501.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2003). Modeling and forecasting realized volatility. *Econometrica 71*, 579–625.

Back, K. (1991). Asset pricing for general processes. *Journal of Mathematical Economics 20*, 371–395.

Bandi, F. M. and T. H. Nguyen (2003). On the functional estimation of jump-diffusion models. *Journal of Econometrics 116*, 293–328.

Bandi, F. M. and J. R. Russell (2003). Microstructure noise, realized volatility, and optimal sampling. Unpublished paper.

Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard (2004). A central limit theorem for realised power and bipower variations of continuous semimartingales. Economics working paper 2004-W29, Nuffield College, Oxford.

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2004). Regular and modified kernel-based estimators of integrated variance: the case with independent noise. Unpublished paper: Nuffield College, Oxford.

Barndorff-Nielsen, O. E. and N. Shephard (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B 64*, 253–280.

Barndorff-Nielsen, O. E. and N. Shephard (2003). Econometrics of testing for jumps in financial economics using bipower variation. Unpublished discussion paper: Nuffield College, Oxford.

Barndorff-Nielsen, O. E. and N. Shephard (2004a). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica 72*, 885–925.

Barndorff-Nielsen, O. E. and N. Shephard (2004b). Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics 2*, 1–48.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies 9*, 69–107.

Brandt, M. W. and F. X. Diebold (2004). A no-arbitrage approach to range-based estimation of return covariances and correlations. *Journal of Business*. Forthcoming.

Carr, P. and L. Wu (2004). Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 113–141.

Chamberlain, G. (1988). Asset pricing in multiperiod securities markets. *Econometrica 56*, 1283–1300.

Chernov, M., A. R. Gallant, E. Ghysels, and G. Tauchen (2003). Alternative models of stock price dynamics. *Journal of Econometrics 116*, 225–257.

Corradi, V. and W. Distaso (2004). Specification tests for daily integrated volatility, in the presence of possible jumps. Unpublished paper: Queen Mary College, London.

Dacorogna, M. M., R. Gencay, U. A. Müller, R. B. Olsen, and O. V. Pictet (2001). *An Introduction to High-Frequency Finance*. San Diego: Academic Press.

de Jong, P. and N. Shephard (1995). The simulation smoother for time series models. *Biometrika 82*, 339–350.

Doornik, J. A. (2001). *Ox: Object Oriented Matrix Programming, 3.0*. London: Timberlake Consultants Press.

Eraker, B., M. Johannes, and N. G. Polson (2003). The impact of jumps in returns and volatility. *Journal of Finance 53*, 1269–1300.

Garcia, R., E. Ghysels, and E. Renault (2004). The econometrics of option pricing. In Y. Ait-Sahalia and L. P. Hansen (Eds.), *Handbook of Financial Econometrics*. North Holland. Forthcoming.

Ghysels, E., A. C. Harvey, and E. Renault (1996). Stochastic volatility. In C. R. Rao and G. S. Maddala (Eds.), *Statistical Methods in Finance*, pp. 119–191. Amsterdam: North-Holland.

Gradshteyn, I. and I. W. Ryzhik (1965). *Table of Integrals. Series and Products*. New York: Academic Press.

Hansen, P. R. and A. Lunde (2004). An unbiased measure of realized variance. Unpublished paper: Aarhus University.

Hausman, J. A. (1978). Specification tests in econometrics. *Econometrica 46*, 1251–1271.

Hendry, D. F. (1995). A theory of co-breaking. Unpublished paper: Nuffield College, Oxford.

Huang, X. and G. Tauchen (2003). The relative contribution of jumps to total price variation. Unpublished paper: Department of Economics, Duke University.

Jacod, J. (1994). Limit of random measures associated with the increments of a Brownian semimartingale. Preprint number 120, Laboratorie de Probabilities, Universite P and M Curie, Paris.

Jacod, J. and A. N. Shiryaev (1987). *Limit Theorems for Stochastic Processes*. Springer-Verlag: Berlin.

Johannes, M. (2004). The statistical and economic role of jumps in continuous-time interest rate models. *Journal of Finance 59*, 227–260.

Johnson, N. and S. Kotz (1972). *Distributions in Statistics: Continuous Multivariate Distributions*. New York: Wiley.

Koopman, S. J., N. Shephard, and J. A. Doornik (1999). Statistical algorithms for models in state space using SsfPack 2.2. *The Econometrics Journal 2*, 107–166.

Mykland, P. and L. Zhang (2005). ANOVA for diffusions. *Annals of Statistics 33*. Forthcoming.

Protter, P. (2004). *Stochastic Integration and Differential Equations*. New York: Springer-Verlag.

Revuz, D. and M. Yor (1999). *Continuous Martingales and Brownian motion* (3 ed.). Heidelberg: Springer-Verlag.

Shephard, N. (2005). *Stochastic Volatility: Selected Readings.* Oxford: Oxford University Press. Forthcoming.

Zhang, L. (2004). Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach. Unpublished paper: Department of Statistics, Carnegie Mellon University.

Zhang, L., P. Mykland, and Y. Aït-Sahalia (2003). A tale of two time scales: determining integrated volatility with noisy high-frequency data. Unpublished paper.