Local-Stochastic Volatility for Vanilla Modelling: a Tractable and Arbitrage Free Approach

Dominique Bang

Bank of America Merrill Lynch, London

Cass University, 30th of January 2019

Outline

- Smile Modeling with Focus on Vanilla
  - Requirements
  - Local-Stochastic Volatility Models
  - Other examples
- LSV: breaking down the Model
  - Lamperti transform
  - Modified dynamics
  - Option pricing
- Application: SABR LV
  - Normal SABR: a revisit.
  - Combining with a tailor-made LV.
  - Model calibration to Swaptions and CMS
- Conclusion

Smile Modeling: Requirements

- Flexibility:
  - Need to reflect the support of the underlying (e.g. compatible with negative IR)
  - Calibration to (typically) 5 european options
  - Potentially incorporate market value of convex products $\rightarrow$ control smile in the wings
- Intuitive parameterization:
  - Transparent mapping of parameters to ATM level, skew and convexity
  - Control over the dynamics of the model ('backbone')
- Tractability:
  - Robustness and performance critical $\rightarrow$ closed form solutions for european options

Local Stochastic Volatility Models

$$dF_t = \alpha_t \cdot \sigma(F_t) \cdot dW_t$$

- $\sigma()$ deterministic function
  - Usually determines the support of the distribution (e.g. shifted CEV)
  - Determines (partially) the skew
  - Controls the backbone

- α_t stochastic volatility process
  - Controls convexity via its volatility ('VolVol' parameter)
  - Controls skew via its correlation to the driving Brownian Motion W_t

LSV: Examples

- Pure Local Vol ($\alpha_t = 1$)
  - Case $\sigma(F) = F^\beta$ well known. $\sigma(F) = a + bF + cF^2$ completely solved (see Andersen [2]).
  - Piecewise constant and piecewise linear studied in Lipton and Sepp[9] and Itkin and Lipton[5].
  - Most other cases rely on numerical methods.

- Pure Stochastic Vol ($\sigma(F) = 1$)
  - Usually supported in $]-\infty, +\infty[$
  - (Normal) Heston: $dz_t = \kappa(1 - z_t)dt + \nu\sqrt{z_t}dW_t^z$, $\alpha_t = \sqrt{z_t}$.
  - (Normal) SABR: α_t logNormal $d\alpha_t = \nu\alpha_t dW_t^\alpha$

- Local Stochastic Vol
  - Heston (CIR+linear), Tremor (CIR + quadratic), Blacher[6] (LNMR + cubic), Lipton[8] ('universal models') and Jaeckel and Kahl[6] (Hyp + Hyp).
  - SABR (LN + CEV), see Hagan[2]

LSV generically not tractable: numerical methods required to compute European Options prices.

Other examples

- Levy Processes:
  - Finite activity: Poisson process.
  - Infinite activity: NIG, see Barndorff-Nielsen [5]. Applied to Inflation, see Ticot[13]
- Path Dependent
  - Path Dependent Volatility. See Guyon[10] and Shelton[12].
- Mixture of Models
  - Density weighted average of other model densities. See e.g. Antonov et al[1].
  - Moment matching techniques
  - Sometimes over parameterized.
- Implied Vol Parameterization
  - SVI see Gatheral[9]

Disentangling Local from Stoch Vol

$$dF_t = \alpha_t \cdot \sigma(F_t) \cdot dW_t$$

- Lamperti Transform
  - $G(F) = \int_{F_0}^{F} \frac{du}{\sigma(u)}$
  - Ito Lemma: $G(F_t) = M_T - \frac{1}{2} \int_0^T \alpha_t^2 \sigma'(F_t) dt$ with $M_T \triangleq \int_0^T \alpha_t dW_t$ (pure SV).
- Drift Reduction: replace stochastic drift with a function of maturity μ_T.
  - μ_T impacts primarily $\mathbb{E}(F_T) \to$ designed to ensure correct forward F_0.
  - Mild impact on the volatility skew.
  - Similar approach applied for a pure LV process (piecewise linear), see Schlenkrich[11] for a full discussion.

We adopt the definition:

$$F_T \triangleq G^{-1}\left(M_T - \mu_T\right)$$

Local Vol encapsulated in functional G.

Option Pricing

- Notation: $C^Y(K) \triangleq \mathbb{E}(Y_T - K)$. we have:

$$C^F(K) = \sigma(K)C^M(\mu_T + G(K)) + \int_K^\infty \sigma'(k)C^M(\mu_T + G(k))\,dk$$

- proof (sketch):
  - Pay-off decomposition (carr-Madan, see e.g. Rouah[10]) applied to $F_T = G^{-1}(M_T - \mu_T)$
  - Change of variables using $\left(G^{-1}\right)' = \sigma \circ G^{-1}$ and $\left(G^{-1}\right)'' = [\sigma\sigma'] \circ G^{-1}$.
- Similar decomposition for any function of F_T (e.g. puts)
- General case: efficient formula required for C^M.

Next: focus on SABR models family

Normal SABR

Set M_t as a normal SABR process:

$$dM_t = \alpha_t dW_t,$$
$$d\alpha_t = \nu \alpha_t dW_t^\alpha$$

where dW_t and dW_t^α are $\rho-$correlated BM under pricing measure $\mathbb{Q}$.

- Closed form solutions available as 2-d integral of elementary fonctions (see Henry-Labordere[3], Islah[4] and Korn & Tang[7]) or as a 1-d integral of special functions (see Antonov, Konikov & Spector[1])...
- .. but computationally too expensive when embedded into our method (already involves one integral for generic LV).
- Need for efficient arbitrage-free and accurate approximations.

Normal SABR: LV projection

- Gyongy's Lemma (see Gyongy[1]): $M_t \stackrel{d}{=} L_t$ where:

$$dL_t = \sqrt{V(L_t)}dW_t \tag{1}$$

$$V(x) = \mathbb{E}\left(\alpha_t^2 | M_t = x\right) \tag{2}$$

- Remarkable result: $V(x)$ is quadratic (elegant proof in the line of Balland and Tran[3]).

$$dL_t = \alpha_0 \sqrt{1 + 2\rho\nu \frac{L_t - M_0}{\alpha_0} + \nu^2 \left(\frac{L_t - M_0}{\alpha_0}\right)^2} dW_t$$

- Coordinate Transform

$$L_t = M_0 + \frac{\alpha_0}{\nu} z_t$$

$$dz_t = \nu \sqrt{1 + 2\rho z_t + z_t^2} dW_t, z_0 = 0.$$

Normal SABR: Jamshidian's Trick

- Reconstruction Formula:

$$L_t - M_0 = \frac{\alpha_0}{\nu}\left[-\rho + \frac{1}{2}\left((1+\rho)\xi_T - \frac{(1-\rho)}{\xi_T}\right)\right]$$

$$\xi_T \triangleq exp\left(\int_0^{z_T} dz \left(1 + 2\rho z + z^2\right)^{-\frac{1}{2}}\right)$$

Monotonic relationship between L_t and ξ_T

- Jamshidian's trick
  Let ξ_K defined via $K - M_0 = \frac{\alpha_0}{\nu}\left[-\rho + \frac{1}{2}\left((1+\rho)\xi_K - \frac{(1-\rho)}{\xi_K}\right)\right]$. Thus:

$$L_t - K = \frac{\alpha_0}{2\nu}\left((1+\rho)(\xi_T - \xi_K) + (1-\rho)(\frac{1}{\xi_K} - \frac{1}{\xi_T})\right)$$

  and

$$(M_t - K)^+ \overset{d}{=} \frac{\alpha_0}{2\nu}\left((1+\rho)(\xi_T - \xi_K)^+ + (1-\rho)(\frac{1}{\xi_K} - \frac{1}{\xi_T})^+\right)$$

Normal SABR: a novel representation of option price

- Exact representation of the call price

$$C^M(K) = \frac{\alpha_0}{2\nu}\left((1+\rho)C^{\xi}(\xi_K) + (1-\rho)P^{\frac{1}{\xi}}(\frac{1}{\xi_K})\right)$$

$$\frac{d\xi_t}{\xi_t} = \nu dW_t + \frac{(1-\rho)\nu^2}{(1+\rho)\xi_t^2 + (1-\rho)}dt, \ \xi_0 = 1$$

- LN approximation for subordinate process ξ_T
  - Assuming $\xi_T = \tilde{\Gamma}\exp(\tilde{\nu}W_T)$, Γ and $\tilde{\nu}$ computed via moment matching.
  - Pricing formula for Normal SABR requires one BS call and one BS Put
  - Model arbitrage free. Works well.

- Represent more closely dynamics of ξ_T for even better accuracy and small computational overhead.

Normal SABR: measure change.

- Define h_t via $\xi_t = e^{\nu h_t}$. We have:

$$dh_t = dW_t - \frac{\nu}{2} \tanh\left(\nu(h_t + \bar{h})\right) dt$$
$$\bar{h} \triangleq \frac{1}{2\nu} \ln\left(\frac{1+\rho}{1-\rho}\right)$$

- Change of measure using the martingale $\theta_t^\star$

$$\frac{d\theta_t^\star}{\theta_t^\star} = \frac{\nu}{2} \tanh(\nu(h_t + \bar{h})) dW_t,$$

- Associated measure $\mathbb{Q}^\star$ defined via $\frac{d\mathbb{Q}^\star}{d\mathbb{Q}} = \theta_T^\star$
- Per construction, h_T standard (driftless) Brownian under $\mathbb{Q}^\star$.

Normal SABR: local projection and density

- Local projection of Radon-Nikodym derivative
  - solve for f such that $d\left[\frac{f(h_t)}{\theta_t^\star}\right] = O(dt) \to f(h) = \sqrt{\cosh\left(\nu(h+\bar{h})\right)}$
  - Tractable projection

$$\mathbb{E}^\star\left[\frac{1}{\theta_T^\star}|h_T\right] = \frac{1}{f(h_T)}\mathbb{E}^\star\left[\frac{f(h_T)}{\theta_T^\star}|h_T\right],$$

$$\approx \frac{1}{\gamma}e^{-\frac{\nu}{2}|h_T+\bar{h}|}\frac{3-e^{-\nu|h_T+\bar{h}|}}{2} \triangleq \frac{1}{\theta^\dagger(h_T)},$$

- Density

$$\mathbb{E}\left(\delta_h(h_T)\right) = \mathbb{E}^\star\left(\frac{\delta_h(h_T)}{\theta_T^\star}\right) = \mathbb{E}^\star\left(\mathbb{E}^\star\left[\frac{1}{\theta_T^\star}|h_T\right]\delta_h(h_T)\right) \quad (3)$$

$$\approx \mathbb{E}^\star\left(\frac{\delta_h(h_T)}{\theta^\dagger(h_T)}\right) = \frac{e^{-\frac{h^2}{2T}}}{\sqrt{2\pi T}\theta^\dagger(h_T)} \quad (4)$$

Normal SABR: Summary

- Normal SABR option price

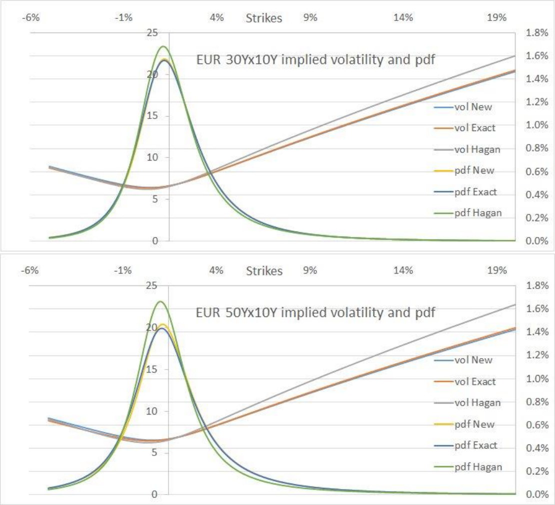
$$C^M(K) = \frac{\alpha_0}{2\nu}\left((1+\rho)C^{\xi}(\xi_K) + (1-\rho)P^{\frac{1}{\xi}}(\frac{1}{\xi_K})\right)$$

- $\xi_T = e^{\nu h_T} \approx \Gamma e^{\nu h_T^{\dagger}}$
  - Γ enforce re-pricing of the Forward (closed form).
  - Density of $h_T^{\dagger}$ is $\varrho(h) \triangleq \frac{e^{-\frac{\nu}{2}|h+\bar{h}|}\left(\frac{3-2e^{-\nu|h+\bar{h}|}}{2}\right)}{\gamma} \frac{e^{-\frac{h^2}{2T}}}{\sqrt{2\pi T}}$
  - $C^{\xi}()$ and $P^{\frac{1}{\xi}}()$ closed form (few calls to the normal CDF)
- Arbitrage free by construction for any configuration.
- Formula accurate for expiries as long as 50Y.

Normal SABR: Comparison Exact, Hagan and New

- Normal SABR: swaption tenor is 10 years, maturities 10 to 50 years. Forward $F_0 = 2\%$, ATM normal Vol $\hat{\sigma} = 0.5\%$, Vol-of-vol $\nu = 20\%$ and correlation $\rho = 50\%$. Comparison of implied PDF (left vertical axis) and normal implied volatility (right vertical axis) for our new approach ("New"), the exact approach in Antonov et al[1] ("Exact"), and the asymptotic approximation in Hagan et al[2] ("Hagan") for the Normal SABR model.

Normal SABR: Exact, Hagan and New



Local Volatility specification

- Standard Local Vol $\sigma(F) = (\max(F+m, 0))^{\beta}$
  - Allows for negative IR, but shift m somehow arbitrary. Approach doesn't support stress tests scenarios.
  - Little control over the dynamics.
  - No control over high strikes.
- Our approach provides total control over the LV. Possible spec:
  - $\max(\cdot, 0)$ replaced by a regularized version, e.g $\eta(\epsilon, f) = \epsilon \ln(1 + e^{\frac{f}{\epsilon}})$.
  - β can be made spot dependent by introducing two levels $(\beta_l,\ \beta_h)$.
  - Add a term for high strikes, e.g. $\Psi \max(F - F_h, 0)$ (or regularized version).

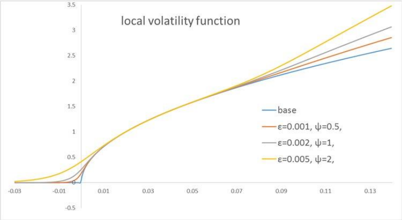


Figure: Control over the wings of the LV (normalized at the forward $F_0 = 0.02$) varying ϵ and Ψ. Base case (square-root LV): $\beta_l = \beta_h = \frac{1}{2}$, $\epsilon = 0$, $F_h = F_0 + 8\%$ and $\Psi = 0$.

SABR LV: Calibration

Assuming a LV σ has been chosen:

- Pricing Equations
  - $C^F(K) = \sigma(K)C^M(\mu_T + G(K)) + \int_K^\infty \sigma'(k)C^M(\mu_T + G(k))\,dk$
  - $P^F(K) = \sigma(K)P^M(G(K) + \mu) - \int_{-\infty}^K \sigma'(k)P^M(G(k) + \mu)\,dk$
  - $C^M(K) = \frac{\alpha_0}{2\nu}\left((1+\rho)C^\xi(\xi_K) + (1-\rho)P^{\frac{1}{\xi}}(\frac{1}{\xi_K})\right)$
- Model Calibration
  - Forward: $C^F(F_0) - P^F(F_0) = 0$, mostly governed by μ_T
  - ATM Straddle: $C^F(F_0) + P^F(F_0) = \hat{\sigma}\sqrt{\frac{2T}{\pi}}$, mostly controled by α_0.
  - Good initial guess for (μ_T, α_0) available.
  - Standard 2-d solver performs well.

LSV: example EUR 10Yx30Y

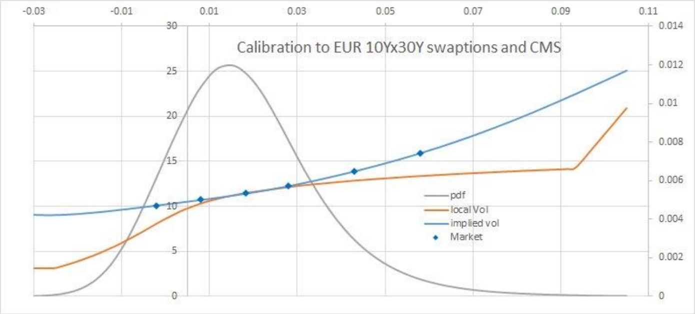


Figure: implied PDF (left vertical axis), LV and implied Normal volatility (right vertical axis) for a model calibrated to swaptions implied volatilities (Market) and the CMS convexity adjustments (61 basis points). Forward $F_0 = 1.833\%$, ATM Vol $\hat{\sigma} = 0.533\%$, Vol of vol $\nu = 16.5\%$, correlation $\rho = 7\%$, $\beta_l = 26.5\%$, $\beta_h = 0\%$, $\epsilon = 0.0032$ and $\psi = 47.7$

Conclusion

- New mechanism to combine arbitrary LV and SV.
- Tractable when efficient option pricing under SV available
- Applied to SABR LV
  - New arbitrage-free, accurate and efficient proxy for pricing under Normal SABR.
  - Example of practical LV. Calibration to Swaptions and CMS.
  - Resulting distribution smooth and well behaved.
- Method generic and can be applied to other LSV models.

References

- Antonov, A., Konikov, M. and Spector, M. (2015) Mixing SABR Models for Negative Rates, SSRN 2015.
- Andersen L. (2011) Option Pricing with Quadratic Volatility: a Revisit, F&S, Springer.
- Balland, P. (2013) SABR Goes Normal, Risk.
- Bang, D. (2018) Local-Stochastic Volatility Models for Vanilla Modeling: a Tractable and Arbitrage Free Approach to Option Pricing, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3171877.
- Barndorff-Nielsen, O. (1998) *Processes of normal inverse Gaussian type*, F&S, Springer.
- Blacher, G. (2001) A New Approach for Designing and Calibrating Stochastic Volatility Models for Optimal Delta Vega Hedging of Exotic Options. Global Derivatives
- Breeden D. and Litzenberger, R. (1978) Prices of State-Contingent Claims Implicit in Option Prices, The Journal of Business.
- Carr, P., M. Tari, and T. Zariphopoulou (1999). Closed Form Option Valuation with Smiles. Tech. rep. www.math.nyu.edu/research/carrp/papers/pdf/closed4. pdf.
- Gatheral, J. and Jacquier, A. (2013) Arbitrage-Free SVI Volatility Surfaces, QF.
- Guyon, J.(2014) Path-dependent volatility, Risk Magazine, October 2014.

- Gyongy, I. (1986) Mimicking the One-Dimensional Marginal Distributions of Processes Having an Ito Differential, Probability Theory and Related Fields.
- Hagan, P. (2002) Managing Smile Risk, Wilmott.
- Henry-Labordere P. (2008) Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing, Chapman & Hall.
- Islah O.(2009) Solving SABR in exact form and unifying it with LIBOR market model, SSRN paper.
- Itkin, A., Lipton, A. (2017) Filling the gaps smoothly. Journal of Computational Sciences.
- Jaeckel, P. and Kahl, C. (2010). Hyp Hyp Hooray.
- Korn R. and Tang S. (2013) Exact Analytical Solution for the Normal SABR Model, Wilmott, v. 13, pp. 64-69.
- Lipton, A. (2002) The Vol Smile Problem, Risk.
- Lipton A. and Sepp A.(2011) Filling the gaps
- Rouah, F.(1978) Payoff Function Decomposition, *http : //www.frouah.com/finance%20notes/Payoff%20function%20decomposition.pdf*
- Schlenkrich, Sebastian (2018) Approximate Local Volatility Model for Vanilla Rates Options. *http : //dx.doi.org/*10.2139/*ssrn*.3150689.
- Shelton, D. (2016) Path Dependent Local Vol Surface Interpolation, BAML Internal Doc.
- Ticot, Y. (2011) Pricing Inflation Vanillas and Exotics, BAML Internal Doc.