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# Optimal Funding and Investment Strategies in Defined Contribution Pension Plans under Epstein-Zin Utility

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## Abstract

A defined contribution pension plan allows consumption to be redistributed from the plan member's working life to retirement in a manner that is consistent with the member's personal preferences. The plan's optimal funding and investment strategies therefore depend on the desired pattern of consumption over the lifetime of the member.

We investigate these strategies under the assumption that the member has an Epstein-Zin utility function, which allows a separation between risk aversion and the elasticity of intertemporal substitution, and we also take into account the member's human capital.

We show that a stochastic lifestyling approach, with an initial high weight in equity-type investments and a gradual switch into bond-type investments as the retirement date approaches is an optimal investment strategy. In addition, the optimal contribution rate each year is not constant over the life of the plan but reflects trade-offs between the desire for current consumption, bequest and retirement savings motives at different stages in the life cycle, changes in human capital over the life cycle, and attitude to risk.

Key words: defined contribution pension plan, funding strategy, investment strategy, Epstein-Zin utility

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# **Optimal Funding and Investment Strategies in Defined Contribution Pension Plans under Epstein-Zin Utility**

## **1 Introduction**

### **1.1 The role of the pension plan in allocating consumption across the life cycle**

A typical individual's life cycle consists of a period of work followed by a period of retirement. Individuals therefore need to reallocate consumption from their working life – when the lifetime's income is earned – to retirement – when there might be no other resources available, except possibly a subsistence level of support from the state. A defined contribution (DC) pension plan can achieve this reallocation in a way that is consistent with the preferences of the individual plan member. There are three key preferences to take into account.

The first relates to the desire to smooth consumption across different states of the nature in any given time period. The second relates to the desire to smooth consumption across different time periods. Saving for retirement involves the sacrifice of certain consumption today in exchange for, generally, uncertain consumption in the future. This uncertainty arises because both future labour income and the returns on the assets in which retirement savings are invested are uncertain. The plan member therefore needs to form a view on both the trade-off between consumption in different states of nature in the same time period and the trade-off between consumption in different time periods. Attitudes to these trade-offs will influence the optimal funding and investment strategies of the pension plan.

In a DC pension plan, the member allocates part of the labour income earned each year to the pension plan in the form of a contribution and, thus, builds up a pension fund prior to retirement. Then, on retirement, the member uses a proportion of the accumulated pension fund to purchase a life annuity. The decisions regarding the contribution rate

each year before retirement (i.e., the funding strategy) and the annuitisation ratio (i.e., the proportion of the fund at retirement that is used to purchase a life annuity) are both driven by the member's preference between current and future consumption, as well as the desire to leave a bequest, the third key preference that we need to take into account. Should the member die before retirement, the entire accumulated pension fund will be available to bequest; after retirement, only that part of the residual pension fund that has not been either annuitised or spent can be bequested.

The investment strategy (i.e., the decision about how to invest the accumulated fund across the major asset categories, such as equities and bonds) will influence the volatility of the pension fund and, hence, consumption in different time periods, and so will be influenced by the member's attitude to that volatility.

In this paper, we investigate the optimal funding and investment strategies in a DC pension plan.<sup>1</sup> To do this, we use a model that differs radically from existing studies in this field in three key respects.

The first key feature of the model is the assumption of Epstein-Zin recursive preferences by the plan member. This enables us to separate relative risk aversion (RRA) and the elasticity of intertemporal substitution (EIS). Risk aversion is related to the desire to stabilise consumption across different states of nature in a given time period (e.g., an individual with a high degree of risk aversion wishes to avoid consumption uncertainty in that period, and, in particular, a reduction in consumption in an unfavourable state of nature) and the EIS measures the desire to smooth consumption over time (e.g., an individual with a low EIS wishes to avoid consumption volatility over time, and, in particular, a reduction in consumption relative to the previous time period).<sup>2</sup> Thus, risk

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<sup>1</sup> This research focuses on the investment and funding strategies for a DC plan during the accumulation stage and the only form of saving we allow is pension saving. Non-pension saving, housing-related investments and post-retirement investment strategies are beyond the scope of this study.

<sup>2</sup> The EIS is defined as

aversion and EIS are conceptually distinct and, ideally, should be parameterised separately. In this paper, we consider four different types of member according to different RRA and EIS combinations, as shown in Table 1.1.

<b>Table 1.1 Pension plan member types</b>		
	<b>High RRA (risk averse)</b>	<b>Low RRA (risk tolerant)</b>
<b>Low EIS (likes consumption smoothing)</b>	<ul style="list-style-type: none"> <li>● risk-averse member who dislikes consumption volatility over time</li> <li>● low equity allocation, particularly as retirement approaches</li> <li>● e.g., low income member with dependants</li> </ul>	<ul style="list-style-type: none"> <li>● risk-tolerant member who dislikes consumption volatility over time</li> <li>● high equity allocation at all ages</li> <li>● e.g., low income member without dependants</li> </ul>
<b>High EIS (accepts consumption volatility)</b>	<ul style="list-style-type: none"> <li>● risk-averse member who does not mind consumption volatility over time</li> <li>● low equity allocation, particularly as retirement approaches</li> <li>● e.g., high income member with dependants</li> </ul>	<ul style="list-style-type: none"> <li>● risk-tolerant member who does not mind consumption volatility over time</li> <li>● high equity allocation at all ages</li> <li>● e.g., high income member without dependants</li> </ul>

$$\varphi = -\frac{\frac{d(c_{t1}/c_{t2})}{c_{t1}/c_{t2}}}{\frac{d[U'(c_{t1})/U'(c_{t2})]}{U'(c_{t1})/U'(c_{t2})}} = -\frac{d[\ln(c_{t1}/c_{t2})]}{d[\ln(U'(c_{t1})/U'(c_{t2}))]},$$

where  $c_{t1}$  is consumption in period  $t1$  and  $U'(c_{t1})$  is the marginal utility of  $c_{t1}$ , etc. The sign and size of the EIS reflects the relationship between the substitution effect and income effect of a shock to a state variable, such as an increase in the risk-free interest rate. The substitution effect is always negative, since current consumption decreases when the risk-free rate increases because future consumption becomes relatively cheap and this encourages an increase in savings. The income effect will be positive if an increase in the risk-free rate (which induces an increase in wealth) leads to an increase in current consumption; it will be negative otherwise. If the income effect dominates, the EIS will be negative and an increase in the risk-free rate leads to an increase in current consumption. If the substitution effect dominates (which is the usual assumption), the EIS will be positive and an increase in the risk-free rate leads to a decrease in current consumption. If the income and substitution effects are of equal and opposite sign, the EIS will be zero and current consumption will not change in response to an increase in the risk-free rate: in other words, consumption will be smooth over time in response to interest rate volatility.

Within the commonly used power utility framework, the coefficient of relative risk aversion (RRA) is the reciprocal of EIS (see, for example, Campbell and Viceira (2002)). This restriction has been criticised because it does not reflect empirical observations. For example, based on the consumption capital asset pricing model,<sup>3</sup> Schwartz and Torous (1999) disentangle these two concepts using the term structure of asset returns. Using US data, their best estimate for RRA is 5.65 (with a standard error of 0.22) and their best estimate of the IES is 0.226 (with a standard error of 0.008). Thus, a high RRA is associated with a low level of EIS, but the estimated parameter values do not have the reciprocal relationship assumed by power utility. Blackburn (2006) also rejects the reciprocal relationship on the basis of a time series of RRA and EIS parameters estimated from observed S&P 500 option prices for a range of different expiry dates between 1996 and 2003.<sup>4</sup>

The second key feature of the model is the recognition that the optimal investment strategy will depend not just on the properties of the available financial assets, but also on the plan member's human capital. A commonly used investment strategy in DC pension plans is "deterministic lifestyling". With this strategy, the pension fund is invested entirely in high risk assets, such as equities, when the member is young. Then, at some arbitrary date prior to retirement (e.g., 10 years), the assets are switched gradually (and usually linearly) into lower risk assets such as bonds and cash. However, there has been no strong empirical evidence to date demonstrating that this is an optimal strategy.

If equity returns are assumed to be mean reverting over time, then the lifestyle strategy of holding the entire fund in equities for an extended period prior to retirement may be justified, as the volatility of equity returns can be expected to decay over time (as a result of the effect of "time diversification"). However, there is mixed empirical evidence about whether equity returns are genuinely mean reverting: Blake (1996), Lo and Mackinley (1988) and Poterba and Summers (1988) find supporting evidence for the UK and US,

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<sup>3</sup> Breeden's 1979 extension of the traditional CAPM which estimates future asset prices based on aggregate consumption rather than the return on the market portfolio.

<sup>4</sup> In particular, Blackburn (2006) found that, over the period 1996 to 2003, the level of risk aversion changed dramatically whilst the level of elasticity of intertemporal substitution stayed reasonably constant.

while Howie and Davies (2002) and Kim et al (1991) find little support for the proposition in the same countries. We would therefore not wish an optimal investment strategy to rely on the assumption of mean reversion holding true in practice.

A more appropriate justification for a lifestyle investment strategy comes from recognising the importance of human capital in individual financial planning. Human capital (i.e., the net present value of an individual's future labour income) can be interpreted as a bond-like asset in which future labour income is the "dividend" on the individual's implicit holding of human capital. Young pension plan members therefore implicitly have a significant holding of bond-like assets and, thus, should weight the financial element of their overall portfolio towards equity-type assets.<sup>5</sup> But to date, there has been no quantitative research exploring the human capital dimension in a DC pension framework.

This paper presents an intertemporal model to solve the life-cycle asset allocation problem for a DC pension plan member. The model assumes two assets (a risky equity fund and a risk-free cash fund), a constant investment opportunity set (i.e., a constant return on the risk-free asset, and a constant expected return and volatility on the risky asset) and stochastic labour income. We consider two aspects of labour income risk: the volatility of labour income and the correlation between labour income and equity returns which determines the extent to which labour income affects portfolio choice (e.g., a positive correlation reduces the optimal asset allocation to equities).

The third key feature of the model concerns the annuitisation decision at retirement. A member with a strong "bequest" savings motive will not wish to annuitise all the accumulated pension wealth. In our model, the member chooses to annuitise a proportion of the accumulated pension fund at retirement by buying a life annuity which will generate a return linked to bonds. We denote this proportion the "annuitisation ratio". This ratio is chosen to maximise the expected utility level at retirement when annuity

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<sup>5</sup> Note this argument might not be appropriate for more entrepreneurial individuals whose pattern of future labour income growth corresponds more to equity than to bonds.

income replaces labour income. The member invests the residual wealth that is not annuitised in higher returning assets in line with the RRA. The member can draw an income from the residual wealth to enhance consumption in retirement, but, unlike the life annuity, the residual wealth can be bequested when the individual dies.

Before considering the model in more detail, we will review Epstein-Zin utility.

## 1.2 Epstein-Zin utility

The classical dynamic asset allocation optimisation model was introduced by Merton (1969, 1971), and shows how to construct and analyse optimal dynamic models under uncertainty. Ignoring labour income, in a single risky asset and constant investment opportunity setting, the optimal portfolio weight in the risky asset for an investor with a power utility function  $U(W) = (W)^{1-\gamma} / (1-\gamma)$  (where  $W$  is wealth and  $\gamma$  is the coefficient of relative risk aversion) is given by:

$$\alpha = \frac{\mu}{\gamma\sigma^2} \quad [1]$$

where  $\mu$  and  $\sigma^2$  are the excess return on the risky asset and the variance of the return on the risky asset, respectively. The investment opportunity set is assumed to be constant.

Equation [1] is appropriate for a single-period myopic investor, rather than a long-term investor such as a pension plan member. Instead of focusing on the level of wealth itself, long-term investors focus on the consumption stream that can be financed by a given level of wealth. As described by Campbell and Viceira (2002, p37), “they consume out of wealth and derive utility from consumption rather than wealth”. Consequently, current saving and investment decisions are driven by preferences between current and future consumption.



To account for this, Epstein and Zin (1989) proposed the following discrete-time recursive utility function,<sup>6</sup> which has become a standard tool in intertemporal investment models, but has not hitherto been applied to pension plans:

$$V_t = \left\{ (1-\beta)C_t^{1-\frac{1}{\varphi}} + \beta \left[ E_t \left( V_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \quad [2]$$

where

- $V_t$  is the utility level at time  $t$ ,
- $\beta$  is the individual's personal discount factor for each year,
- $C_t$  is the consumption level at time  $t$ ,
- $\gamma$  is the coefficient of relative risk aversion (RRA), and
- $\varphi$  is the elasticity of intertemporal substitution (EIS).

The recursive preference structure in [2] is helpful in two ways: first, it allows a multi-period decision problem to be reduced to a series of one-period problems (from time  $t$  to time  $t+1$ ); and second, as mentioned previously, it enables us to separate RRA and EIS.

Ignoring labour income, for an investor with Epstein-Zin utility, there is an analytical solution<sup>7</sup> for the optimal portfolio weight in the risky asset given by:

$$\alpha_t = \frac{\mu_t}{\gamma\sigma_t^2} + \left(1 - \frac{1}{\gamma}\right) \times \frac{\text{cov}_t(R_{t+1}, -V_{t+1})}{\sigma_t^2} \quad [3]$$

This shows that the demand for the risky asset is based on the weighted average of two components. The first component is the short-term demand for the risky asset (or myopic demand, in the sense that the investor is focused on wealth in the next period). The second component is the intertemporal hedging demand, which is determined by the

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<sup>6</sup> Recursive utility preferences focus on the trade-off between current-period utility and the utility to be derived from all future periods. Kreps and Porteus (1978) first developed a generalised iso-elastic utility function which distinguishes attitudes to risk from behaviour toward intertemporal substitution. Following the KP utility function, Epstein and Zin (1989, 1991) proposed a discrete-time recursive utility function that allows the separation of the risk aversion parameter from the EIS parameter. Duffie and Epstein (1992) then extended the Epstein-Zin discrete recursive utility in a continuous-time form called a stochastic differential utility (SDU) function.

<sup>7</sup> For more details, see Merton (1973) and Campbell and Viceira (2002).

covariance of the risky asset return with the investor's utility per unit of wealth over time. Thus, ignoring labour income, the optimal portfolio weights are constant over time, provided that the investment opportunity set remains constant over time (i.e.,  $\mu_t = \mu$  and  $\sigma_t^2 = \sigma^2$  in [3]).

In a realistic life-cycle saving and investment model, however, labour income cannot be ignored. It is risky and cannot be capitalised and traded. But, allowing for labour income volatility in the optimisation process means that an analytical solution for the optimal asset allocation cannot be obtained. To address this, the recent literature has employed a number of numerical methods<sup>8</sup> to approximate the solution of the dynamic portfolio optimisation problem.

In the presence of income risk, the optimal portfolio weight in the risky asset is not constant, but instead follows a lifestyle strategy, as shown by Coco et al. (2005). This can be explained as follows: human capital or wealth can be thought of as the expected net present value (NPV) of future labour income. Thus, an individual's labour income can be seen as the dividend on the individual's implicit holding of human capital. The ratio of human to financial wealth<sup>9</sup> is a crucial determinant of the life-cycle portfolio composition. In early life, as shown in Figure 1.1, this ratio is large since the individual has had little time to accumulate financial wealth and expects to receive labour income for many years. Given that long-term average labour income growth is of a similar order of magnitude as average long-run interest rates in the UK over the last century, as explained in Cairns et al. (2006), labour income can be thought of as an implicit substitute to investing in the

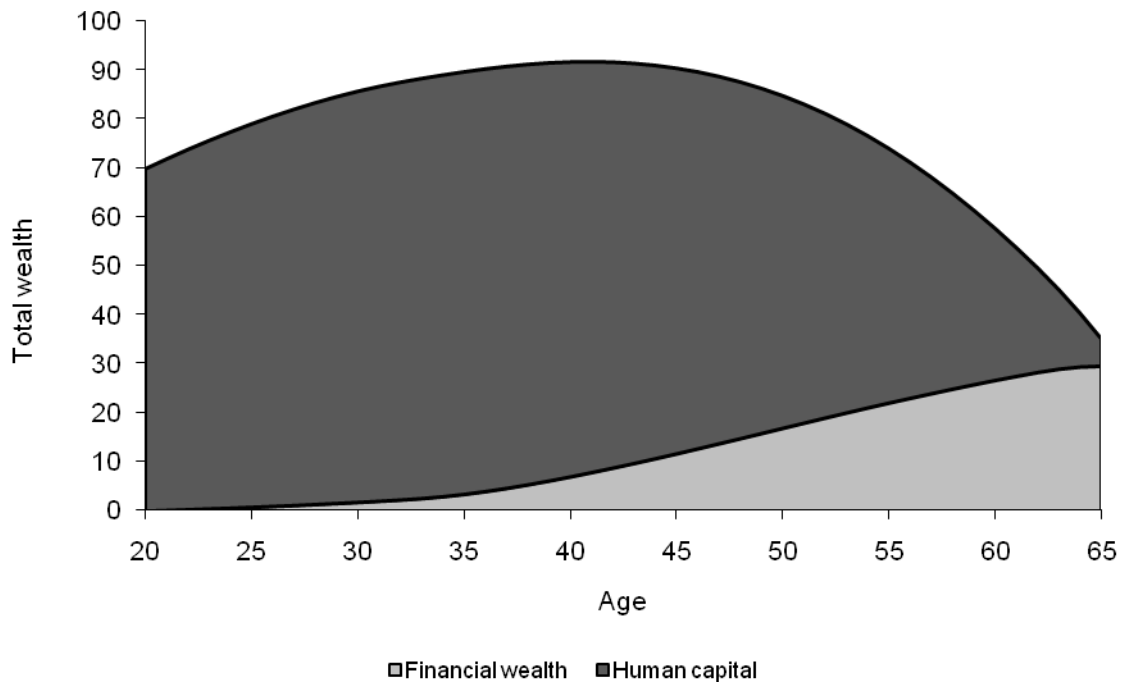
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<sup>8</sup> By far, the most popular approach is value function iteration. Specifically, this involves the discretisation of the state variables by setting up a standard equally-spaced grid and solves the optimisation for each grid point at the next-to-last time period. The expectation term in the resulting Bellman equation is approximated by using quadrature integration and then the dynamic optimisation problem can be solved by backward recursion. It is possible that the accumulated state variable values from the previous time period are not represented by a grid point, in which case, an interpolation method (e.g., bilinear, cubic spline, etc.) must be employed to approximate the value function. However, this approach requires knowledge of the distribution of each of the shocks to the process, so that appropriate quadrature integration (e.g., Gauss quadrature) can be used. Furthermore, this approach cannot handle a large number of state variables. To overcome these limitations, Brandt et al. (2005) proposed a simulation method based on the recursive use of approximated optimal portfolio weights. The idea is to estimate asset return moments using a large number of simulated sample paths, and then to approximate the value function using a Taylor series expansion. If the return is path-dependent, it is necessary to regress the return variable on the simulated state variables from previous time period, before using the Taylor expansion with conditional return moments.

<sup>9</sup> In our model, the only source of financial wealth is pension wealth, and we use these terms interchangeably.

risk-free asset. Thus, younger individuals have a significant holding in this non-tradable risk-free asset and, therefore, should allocate most of their financial wealth to the risky asset to keep the overall portfolio composition constant, as suggested by Equation [3] above. As they grow older, individuals accumulate more financial wealth and draw down human capital.<sup>10</sup> They should therefore rebalance their financial portfolio towards risk-free assets as age increases.

Figure 1.1 – Decomposition of total wealth over the life cycle



More recently, life-cycle asset allocation models with a stochastic labour income process have been extended to include the use of a recursive utility function to allow a separation of RRA and EIS by, for instance, Weil (1989) and Campbell and Viceira (2002). By including a fixed first-time risky-asset entry cost and adopting Epstein-Zin utility, Gomes and Michaelides (2005) present a life-cycle asset allocation model to explain the

<sup>10</sup> In our model, the value of human capital will be zero at the end of age 65. The salary is assumed to be paid at the start of the year, so the human capital at 65 in Figure 1.1 is equal to the individual’s final salary.

empirical observations of low stock market participation and moderate equity holdings for participants<sup>11</sup>.

Turning to DC pension plans, most of the existing literature investigates their optimal dynamic asset allocation strategy by assuming a fixed contribution rate (e.g., 10% of salary per annum prior to retirement) and maximising the utility of the replacement ratio (i.e., pension as a proportion of final salary) at retirement (for example, Cairns et al. (2006)) or by minimising the expected present value of total disutility<sup>12</sup> prior to retirement (for example, Haberman and Vigna (2002)). The EIS is implicitly assumed to be zero and there is no facility for adjusting the contribution rate in response to changes in salary level or in asset performance. However, in practice, most DC plans allow members to make additional voluntary contributions, and often set upper and lower limits on the contribution rate per annum.

Our aim in this study is to investigate the optimal asset allocation strategy for a DC plan member with Epstein-Zin utility, so that an individual member's investment strategy depends on the pattern of preferred consumption levels over the member's entire lifetime. We also derive the optimal profile of contribution rates over the accumulation stage of a DC plan. The rest of the paper is structured as follows. Section 2 outlines the discrete-time model with Epstein-Zin utility, including the parameter calibration process and optimisation method used. In Section 3, we generate simulations of the two key state variables (i.e., wealth and labour income) and derive the optimal funding and investment strategies for the DC pension plan; we also conduct a sensitivity analysis of the results. Section 4 concludes.

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<sup>11</sup> Gomes and Michaelides (2005, page 871) argue that the less risk-averse investors have a weaker incentive to pay the fixed entry cost of equity investment, and therefore stock market participants in aggregate tend to be more risk averse.

<sup>12</sup> The disutility is normally defined using the deviation of actual fund level from interim and final target fund levels.

## 2 The model

### 2.1 The model structure and optimisation problem

We propose a two-asset discrete-time model with a constant investment opportunity set. To ensure conformity with a DC pension plan, a number of constraints need to be specified:

- (i) pension wealth can never be negative,
- (ii) in any year prior to retirement, consumption must be lower than labour income,
- (iii) short selling of assets is not allowed,
- (iv) members are not allowed to borrow from future contributions.<sup>13</sup>

Members are assumed to join the pension plan at age 20 (denoted time  $t = 0$  below) without bringing in any transfer value from a previous plan and the retirement age is fixed at 65. We work in time units of one year and members are assumed to live to a maximum age of  $\omega = 120$ .

#### 2.1.1 Preferences

We assume the plan member possesses the discrete-time recursive utility function proposed by Epstein and Zin (1989):

$$V_{20+t} = \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1-\phi}{\phi}} + \beta \left[ E_{20+t} \left[ p_{20+t} \times V_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \left( \frac{W_{20+t+1}}{b} \right)^{1-\gamma} \right]^{\frac{1-\phi}{1-\gamma}} \right]^{\frac{1-\phi}{1-\gamma}} \right\}^{\frac{1}{1-\phi}} \quad [4]$$

<sup>13</sup> Some studies have assumed that the member can borrow from future contributions (i.e., to incorporate a loan in the portfolio equal to the present value of future contributions). In this way, Boulier et al. (2001) and Cairns et al. (2006) investigate the optimal asset allocation of DC pension plan with guaranteed benefit protection. However, there are arguments against the use of this assumption. In most cases, this would not be allowed in practice. Also, the loan amount depends on assumptions about the level of future contributions and, in practice, there can be a lot of uncertainty about future contributions.

where

- $V_{20+t}$  is the utility level at time  $t$  (or age  $20+t$ ),
- $W_{20+t}$  is the wealth level at time  $t$ ,
- $C_{20+t}$  is the consumption level at time  $t$ ,
- $\gamma$  is the coefficient of relative risk aversion (RRA),
- $\varphi$  is the elasticity of intertemporal substitution (EIS),
- $\beta$  is the discount factor for each year, and
- $p_{20+t}$  is the one-year survival probability at time  $t$  (i.e., the probability that a member who is alive age  $20+t$  survives to age  $20+t+1$ ).

The parameter  $b$  is the “bequest intensity” and determines the strength of the bequest motive. If a member dies during the year of age  $20+t$  to  $20+t+1$ , the deceased member will give the remaining wealth at the end of the year,  $W_{20+t+1}$ , a utility measure of  $b \times (W_{20+t+1}/b)^{1-\gamma} / (1-\gamma)$ . Thus, a higher value of  $b$  implies that the member has a stronger desire to bequest wealth on death.

In the final year of age  $(\omega-1, \omega)$ , where we have  $p_{\omega-1} = 0$ , equation [4] reduces to:

$$V_{\omega-1} = \left\{ C_{\omega-1}^{1-\frac{1}{\varphi}} + \beta \left[ E_{\omega-1} \left[ b \times \frac{\left( \frac{W_{\omega}}{b} \right)^{1-\gamma}}{1-\gamma} \right] \right]^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \quad [5]$$

which provides the terminal condition for the utility function.

### 2.1.2 Financial assets

We assume that there are two underlying assets in which the pension plan can invest:

- (i) a risk-free asset (i.e., a cash fund), and

- (ii) a risky asset (i.e., an equity fund).

The risk-free asset yields a constant rate of interest  $r$ , and the return on the risky asset in year  $t$  is given by:

$$R_{20+t} = r + \mu + \varepsilon_{20+t} \quad [6]$$

where

- $\mu$  is the (constant) risk premium on the risky asset, and
- $\varepsilon_{20+t} = \sigma \times Z_{1,20+t}$ , where  $\sigma$  is the (constant) volatility of the risky asset and  $Z_{1,20+t}$  is an independent and identically distributed (iid) standard Normal random variable

Whilst not necessarily corresponding with the real world, the simplified assumption of iid returns on the risky asset considerably facilitates the numerical method used.

### 2.1.3 Labour and pension income

Before retirement, the member receives an annual salary at the start of each year and contributes a proportion  $\pi_t$  of this into the pension plan at time  $t$ . We adopt the stochastic labour income process used in Cairns et al. (2006) which is illustrated in Figure 2.1. The growth rate in labour income prior to retirement is given by:

$$I_{20+t} = r_t + \frac{S_{20+t+1} - S_{20+t}}{S_{20+t}} + \sigma_1 \times Z_{1,20+t} + \sigma_2 \times Z_{2,20+t} \quad [7]$$

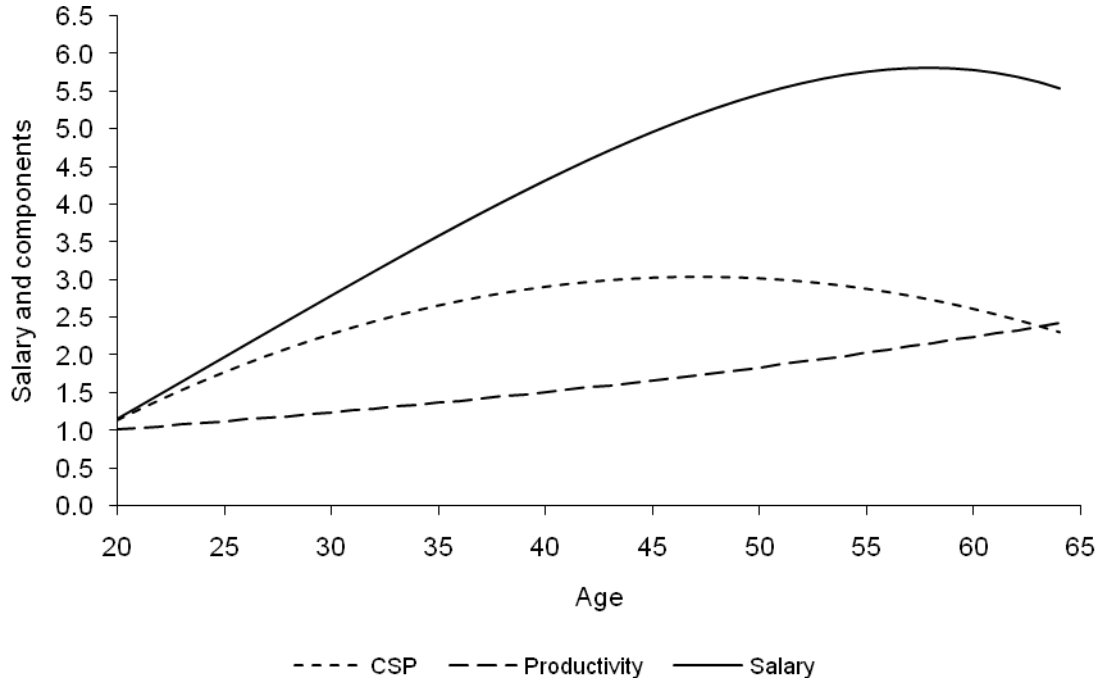
where

- $r_t$  is the long-term average annual real rate of salary growth (reflecting productivity growth in the economy as a whole),
- $S_{20+t}$  is the “career salary profile” (CSP), or salary scale, at time  $t$ , so that the term  $(S_{20+t+1} - S_{20+t})/S_{20+t}$  reflects the promotional salary increase between time  $t$  and time  $t+1$ ,
- $\sigma_1$  represents the volatility of a shock that is correlated with equity returns,
- $\sigma_2$  represents the volatility of the annual rate of salary growth, and

- $Z_{2,20+t}$  is an iid standard Normal random variable.

Equations (6) and (7) are subject to a common stochastic shock,  $Z_{1,20+t}$ , implying that the correlation between the growth rate in labour income and equity returns is given by  $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$ .

Figure 2.1 – Labour income process



Following the work of Blake et al. (2007), we use a quadratic function to model the CSP:

$$S_{20+t} = 1 + h_1 \times \left( -1 + \frac{t}{45} \right) + h_2 \times \left[ -1 + \frac{4t}{45} + 3 \times \left( \frac{t}{45} \right)^2 \right] \quad [8]$$

On retirement at age 65, the member is assumed to annuitise a proportion  $k$  of the accumulated pension fund by buying a life annuity, where  $k$ , the annuitisation ratio, is chosen to maximise the member's utility level at retirement. The amount of annuity income received depends on the accumulated wealth level at retirement, the annuitisation ratio and the price of a life annuity. In this model, the price of a life annuity is calculated



using the risk-free return on the cash fund, so it is fixed over time and no annuity risk is considered. After retirement, the member invests the residual wealth that is not annuitised. Retirement income therefore comes from two sources: the annuity and possible withdrawals from the residual fund until death.

#### 2.1.4 Wealth accumulation

Before retirement, the growth in the member's pension wealth will depend on the investment strategy adopted, the investment returns on both the risk-free asset and the risky asset, and the chosen contribution rate.

The contribution rate at time  $t$  is given by  $\pi_{20+t} = (Y_{20+t} - C_{20+t})/Y_{20+t}$  (for  $0 \leq t \leq 44$ ), where  $Y_{20+t}$  is the labour income level at time  $t$ . We require the contribution rate to be non-negative, so that  $Y_{20+t} \geq C_{20+t}$  before retirement. The contribution rate is allowed to vary over time, so that consumption in any period can adjust to changes in income level and investment performance.

We also need to impose the restriction  $W_{20+t} \geq 0$  (for  $0 \leq t \leq 100$ ), to ensure that the wealth level is always non-negative at each age over the life cycle.

A proportion,  $\alpha_{20+t}$ , of the member's pension account is assumed to be invested in the risky asset at time  $t$ . Then, for  $0 \leq t \leq 43$  (i.e., up to and including the year prior to retirement), we have the following recursive relationship for the wealth process:

$$W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \quad [9]$$

As mentioned above, we assume that that short selling of assets is not allowed and therefore impose the restriction that  $0 \leq \alpha_{20+t} \leq 1$ .

At the start of the year in which the member is aged between 64 and 65, the member receives the final salary payment and makes the final contribution to the pension fund. So, we have:

$$W_{65}^- = (W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})] \quad [10]$$

At the end of this year, the member retires and chooses the annuitisation ratio  $k$ , giving a residual wealth on retirement at exact age 65 of  $W_{65} = (1 - k) \times W_{65}^-$ . The annuitisation ratio  $k$  is chosen to maximise the utility level at retirement. This control variable does not appear in the utility function, but rather in the wealth constraint in the retirement year.

After retirement, the member invests a proportion  $\alpha_{20+t}$  (for  $t \geq 45$ ) of the residual wealth (i.e., that which was not annuitised) in the risky asset and receives annuity payments rather than labour income at the start of each year, provided that the member is still alive. This implies that the recursive relationship for the wealth accumulation process at this stage of the member's life cycle is given by:

$$W_{20+t+1} = \left( W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \quad [11]$$

where  $\ddot{a}_{65}$  is the price of a life annuity at age 65 (and, hence,  $k \times W_{65}^- / \ddot{a}_{65}$  represents the annual annuity income after retirement).

Finally, we must constrain consumption after retirement such that  $C_{20+t} \leq (W_{20+t} + k \times W_{65}^- / \ddot{a}_{65})$ .

### 2.1.5 The optimisation problem and solution method

The model has three control variables: the asset allocation at time  $t$ ,  $\alpha_{20+t}$ , the consumption level at time  $t$ ,  $C_{20+t}$ , and the annuitisation ratio at retirement,  $k$ .

The optimisation problem is then:

$$\max_{\alpha_{20+t}, C_{20+t}, k} E(V_{20+t})$$

subject to the following constraints:

(i) for  $0 \leq t \leq 43$ , we have:

a) a wealth accumulation process satisfying:

$$W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \geq 0,$$

b) an allocation to the risky asset satisfying  $0 \leq \alpha_{20+t} \leq 1$ , and

c) a contribution rate satisfying  $\pi_{20+t} \geq 0$ ;

(ii) for  $t = 44$ , we have:

a) a wealth accumulation process satisfying:

$$W_{65} = (1 - k) \times ((W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})]) \geq 0,$$

b) an allocation to the risky asset satisfying  $0 \leq \alpha_{64} \leq 1$ ,

c) a contribution rate satisfying  $\pi_{64} \geq 0$ , and

d) an annuitisation ratio at age 65 satisfying  $0 \leq k \leq 1$ ;

(iii) and, for  $t \geq 45$ , we have:

a) a wealth accumulation process satisfying:

$$W_{20+t+1} = \left( W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})] \geq 0,$$

b) an allocation to the risky asset satisfying  $0 \leq \alpha_{20+t} \leq 1$ , and

c) consumption satisfying  $C_{20+t} \leq \left( W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} \right)$ .

The Bellman equation at time  $t$  is:

$$\begin{aligned}
& J_{20+t}(W_{20+t}, Y_{20+t}) \\
= & \max_{\alpha_{20+t}, c_{20+t}, k} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{1-\frac{1}{\varphi}} + \beta \left[ E_{20+t} \left( p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \left( \frac{W_{20+t+1}}{b} \right)^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \quad [12]
\end{aligned}$$

An analytical solution to this problem does not exist, because there is no explicit solution for the expectation term in the above expression. Instead, we must use a numerical solution method to maximise the value function and derive the optimal control parameters. We use the terminal utility function at age 120 to compute the corresponding value function for the previous period and iterate this procedure backwards, following a standard dynamic programming strategy.

To avoid choosing a local maximum, we discretise the control variables (i.e., asset allocation, consumption and annuitisation ratio) into equally spaced grids and optimise them using a standard grid search. As an important step in implementing the stochastic dynamic programming strategy, we need to discretise both the state space and shocks in the stochastic processes (i.e., equity return and labour income growth) first. Wealth and labour income level are discretised into 30 and 10 evenly-spaced grid points, respectively, in the computation.<sup>14</sup> Also, the shocks in both the equity return and labour income growth processes are discretised into 9 nodes.<sup>15</sup>

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<sup>14</sup> Clearly, the choice concerning the number of nodes is subjective, but we felt that this choice represents an appropriate trade-off between accuracy and speed of computation.

<sup>15</sup> Again, 9 nodes represents a balance between accuracy and computing time, and is a standard setting in the existing literature.

The expected utility level at time  $t$  is then computed using these nodes and the relevant weights attached to each (i.e., Gauss quadrature weights and interpolation nodes).<sup>16</sup> The advantage of using this set of nodes is that the state variables can be computed more quickly and precisely; however, because we have a fine grid on the control variables and a much coarser grid on the shocks, we may have some state variable values outside of the grid points in the next time period. In this case, cubic spline interpolation is employed to approximate the value function. While this approach does not significantly reduce the accuracy of the results obtained, use of a much finer grid for the shocks in the equity return and labour income growth processes would significantly increase computing time, as mentioned previously.

For each age  $20+t$  prior to the terminal age of 120, we compute the maximum value function and the optimal values for the control variables at each grid point. Substituting these values in the Bellman equation, we obtain the value function of this period, which is then used to solve the maximisation problem for the previous time period. Details of the dynamic programming and integration process are given in Appendix 1. The computations were performed in MATLAB.<sup>17</sup>

## 2.2 Parameter calibration

We begin with a standard set of baseline parameter values (all expressed in real terms) presented in Table 2.1.

The constant net real interest rate,  $r$ , is set at 2% p.a., while, for the equity return process, we consider a mean equity premium,  $\mu$ , of 4% p.a. and a standard deviation,  $\sigma$ , of 20% p.a.. Using an equity risk premium of 4% p.a. (as opposed to the historical average of around 6%) is a common choice in recent literature (e.g., Fama and French (2002), Gomes and Michaelides (2005)). This more cautious assumption reflects the fact that the historical equity risk premium might be higher than can reasonably be expected in future, and thus will reduce the weight given to equities in the optimal portfolios obtained. We

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<sup>16</sup> For more details, see Judd (1998, page 257-266).

<sup>17</sup> <http://www.mathworks.com/products/matlab/>. The code is available on request from the authors.

use the projected PMA92 table<sup>18</sup> as the standard male mortality table, and hence, using a (real) interest rate of 2% p.a., the price of a whole life annuity from age 65 is 15.87.

<b>Table 2.1 Baseline parameter values</b>			
<b>Asset returns</b>		<b>Preference parameters</b>	
risk-free rate, $r$	0.02	RRA, $\gamma$	5
equity premium, $\mu$	0.04	EIS, $\varphi$	0.2
volatility, $\sigma$	0.2	bequest intensity, $b$	1
		discount factor, $\beta$	0.96
<b>Mortality</b>		<b>Labour income process</b>	
mortality table	PMA92	starting salary, $Y_{20}$	1
		average salary growth, $r_l$	0.02
<b>Annuity</b>		volatility of shock correlated with equity returns, $\sigma_1$	0.05
annuity price, $\ddot{a}_{65}$	15.87 (based on PMA92 and a risk-free rate of 0.02)	volatility of annual rate of salary growth, $\sigma_2$	0.02
		$h_1$	-0.276
		$h_2$	0.75835

We start by presenting results for what might be considered as a relatively standard plan member, with RRA of  $\gamma = 5$ , EIS of  $\varphi = 0.2$  and discount factor  $\beta = 0.96$ .<sup>19</sup> As mentioned above, the bequest intensity,  $b$ , plays an important role in life-cycle saving and investment. We set  $b$  equal to unity in the baseline case (which represents a moderate level of bequest saving motive). We later conduct a sensitivity analysis on these parameter values.

The starting salary is normalised on unity. All absolute wealth and income levels are measured in units of the starting salary. In line with post-war UK experience, the annualised real growth rate of national average earnings is assumed to be 2% p.a. with a

<sup>18</sup> PMA92 is a mortality table for male pension annuitants in the UK based on experience between 1991 and 1994; here, we use the projected rates for the calendar year 2010, i.e., the table PMA92(C2010), published by the Continuous Mortality Investigation (CMI) Bureau in February 2004.

<sup>19</sup> This parameter constellation is common in the literature (e.g., Gomes and Michaelides (2004)). The values of RRA and EIS are also consistent with power utility for the baseline case.

standard deviation of 2% p.a. (i.e.,  $r_f = 0.02$  and  $\sigma_2 = 0.02$ ). Following the work of Blake et al. (2007), we estimate the CSP parameters  $h_1$  and  $h_2$  using average male salary data (across all occupations) reported in the 2005 *Annual Survey of Hours and Earnings*. The estimated values are  $h_1 = -0.276$  and  $h_2 = 0.75835$  (see [8]).

## 3 Results

### 3.1 Baseline case

#### 3.1.1 Optimal asset allocation assuming no bequest motive or labour income risk

As suggested by equation [3] above, the optimal portfolio composition should be constant when there is no bequest motive and labour income risk is ignored. Figure 3.1 shows the optimal equity weight for the final time period (i.e., year of age 119 to 120) with no bequest motive. As expected, we can see that when the accumulated wealth level is large (and labour income is small in comparison), the optimal asset allocation is close to the result suggested in equation [3], so that we have:

$$\alpha = \frac{\mu}{\gamma\sigma^2} + \left(1 - \frac{1}{\gamma}\right) \times \frac{\text{cov}(R_{120}, -V_{120})}{\sigma^2} \approx 0.2$$

This result shows that we can approximate an analytical solution numerically reasonably accurately, thereby justifying the use of our grid search numerical method.

#### 3.1.2 Simulation output

The output from the optimisation exercise is a set of optimal control variables (i.e., asset allocation,  $\alpha_{20+t}$ , and consumption level,  $C_{20+t}$ ) for each time period and the optimal annuitisation ratio,  $k$ , at retirement age 65. We generate a series of random variables for both the equity return and labour income shocks, and then generate 10,000 independent simulations of wealth and labour income levels.

Figure 3.1 – Optimal equity allocation for the final time period

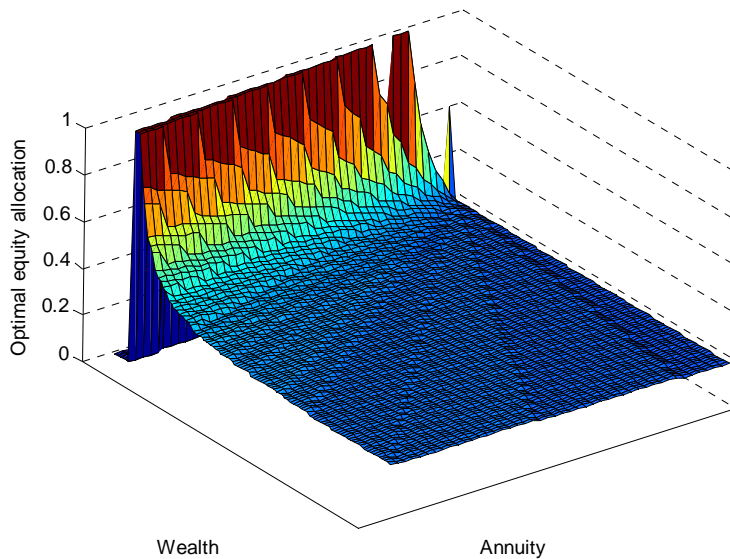


Figure 3.2 shows the simulation means of labour income and optimal wealth and consumption levels for ages 20, 21, ..., 120, and Figure 3.3 shows the consumption profile on a larger scale. We have bequest and retirement saving motives in this model. In the early years of the life cycle (i.e., up to age 45 or so), wealth accumulation is driven by the bequest intensity (i.e., the extent of the desire to protect dependants if the member dies) and by the attitude to risk (i.e., the degree of aversion to cutting consumption in unfavourable states).<sup>20</sup> Consumption increases smoothly during this period. Then, as the member gets older, the retirement motive becomes more important as the member recognises the need to build up the pension fund in order to support consumption after retirement. From age 45 to the retirement age of 65, the retirement savings motive dominates and the pension fund grows significantly. As a result, consumption remains almost constant during this period.

<sup>20</sup> This will become clearer in sections 3.2.1 and 3.2.2.



After retirement, there is a large fall in consumption compared with the period immediately prior to retirement and thereafter consumption remains stable for the remainder of the member's lifetime. Evidence presented in Banks et al (1998) shows that consumption tends to fall after retirement. Part of this is explained by the fact that work-related expenses such as travel and clothes no longer need to be incurred. Part is explained by the fact that precautionary balances need to be maintained to pay for lumpy expenditures such as car repairs or home maintenance, given that working to pay for these expenses is no longer an option. Part is also explained by the fact that retired people tend to stay at home more and so spend less on high-cost items, such as restaurant meals and holidays. We do not attempt to model these complex issues in a formal way. Instead, we try to capture them in the post-retirement consumption constraint  $C_{20+t} \leq (W_{20+t} + (k \times W_{65}^- / \ddot{a}_{65}))$  (i.e., for  $t \geq 45$ ), given in section 2.1.4 above. Consumption after retirement is therefore subject to a cash-in-hand constraint and cannot exceed the sum of pension income and unannuitised wealth. The constraint is tighter the higher the fraction,  $k$ , of pension wealth the member chooses to annuitize at age 65. Given that  $k$  is a control variable, Figure 3.3 shows the optimal reduction in consumption in retirement when  $k$  is chosen optimally (i.e., to maximise  $E(V_{64})$ ).

The size of the fall in consumption at retirement when there is a cash-in-hand constraint will be influenced by the level of non-pension wealth, such as discretionary savings or housing. Since non-pension wealth does not need to be annuitized, it acts as a buffer that can be used to smooth life-cycle consumption. Non-pension wealth is outside the scope of the present paper.

Figure 3.2 – Mean of simulated wealth, consumption and income

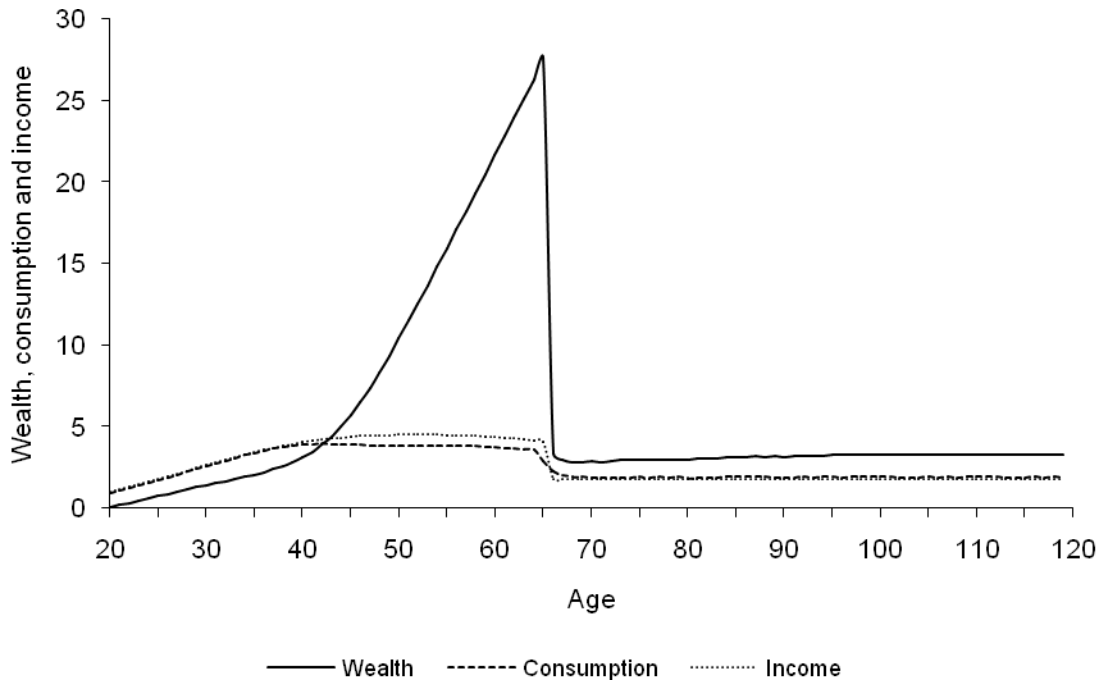


Figure 3.3 – Mean of consumption

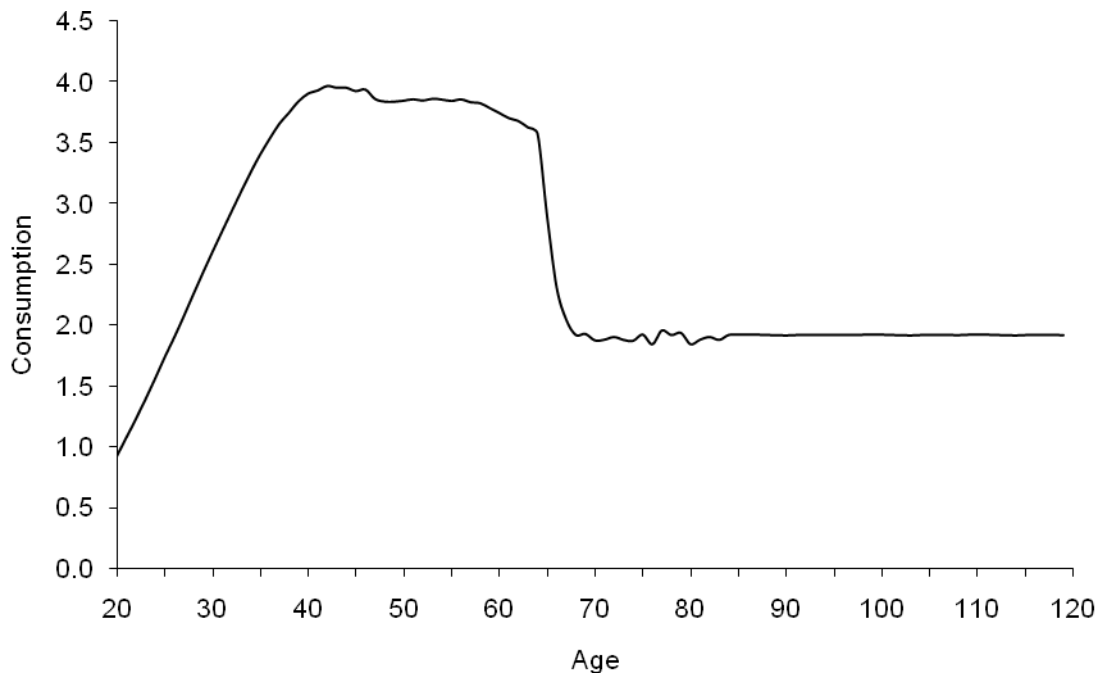


Figure 3.4 shows the expected NPV of total future labour income (i.e., human capital). We can see that human capital increases until about age 35. This is because of the very high rate of salary growth in the early years (relative to the discount rate applied to future labour income).

Figure 3.4 – Human capital

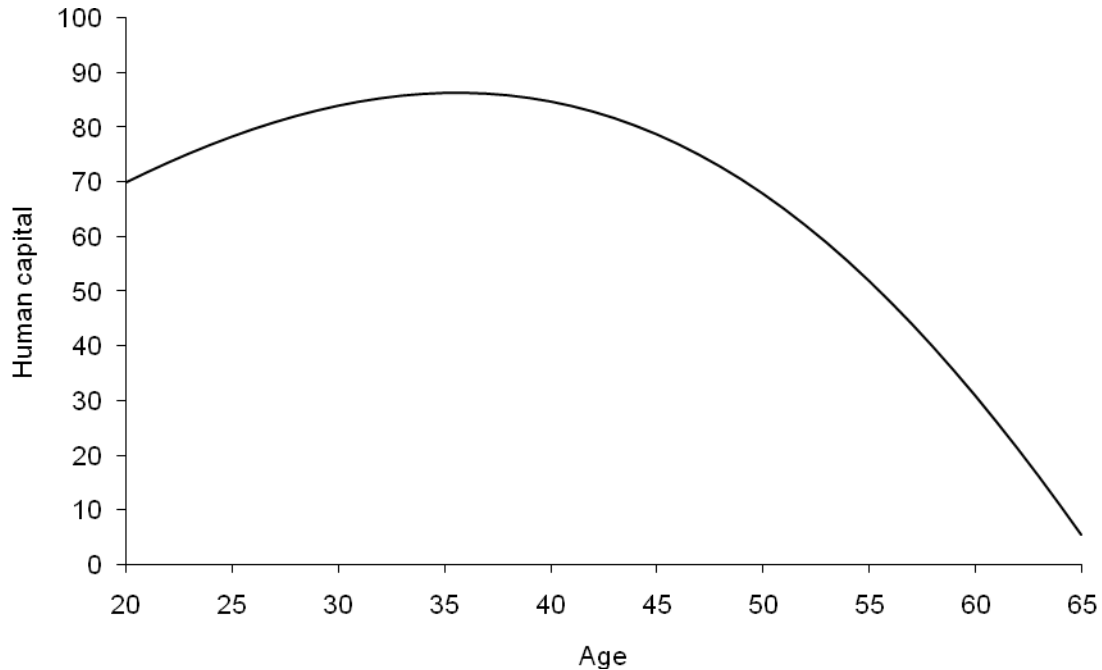


Figure 3.5 shows six possible optimal asset allocation profiles for equities at each age before retirement,  $\alpha_{20+t}$ . Each profile coincides with a particular quantile from the distribution of outcomes from 10,000 simulations. These profiles are consistent with the investment strategy called “stochastic lifestyling”, first outlined in Cairns et al (2006), with a high equity weighting at younger ages and a gradual switch from equities to the risk-free financial asset as the retirement age approaches. Prior to around age 35, the member should invest all financial wealth in the risky asset, because the implicit holding in the non-tradable riskless asset (i.e., human capital) is increasing, as illustrated in Figure 3.4. However, after age 35, human capital starts to decline. The member should then begin to rebalance the financial wealth portfolio towards the risk-free financial asset

to compensate for the decline in human capital. This is because the risk-free financial asset and human capital are substitutes, with the degree of substitutability inversely related to the correlation between labour income growth and equity returns,  $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$ .

The investment strategy is known as “stochastic lifestyling” because the optimal equity weighting over the life cycle depends on the realised outcomes for the stochastic processes driving the state variables, namely labour income and the risky financial asset. The profiles have a similar shape which can be characterised as three connecting (and approximately) linear segments. The first is a horizontal segment involving a 100% equity weighting (approximately) from age 20 to an age somewhere in the range of 40-47. The second is a steep downward segment that involves a reduction in equities to somewhere between 10-40% over a seven year period. The third is a more gentle downward sloping segment that reduces the equity weighting to somewhere between 0-10% by the retirement age. It is important to note that the profiles in Figure 3.5 are not, however, consistent with the more traditional “deterministic lifestyling” strategy, which involves an initial high weighting in equities with a predetermined linear switch from equities to cash in the period leading up to retirement (typically the preceding 5 or 10 years).

Figure 3.5 – Optimal equity allocation prior to retirement

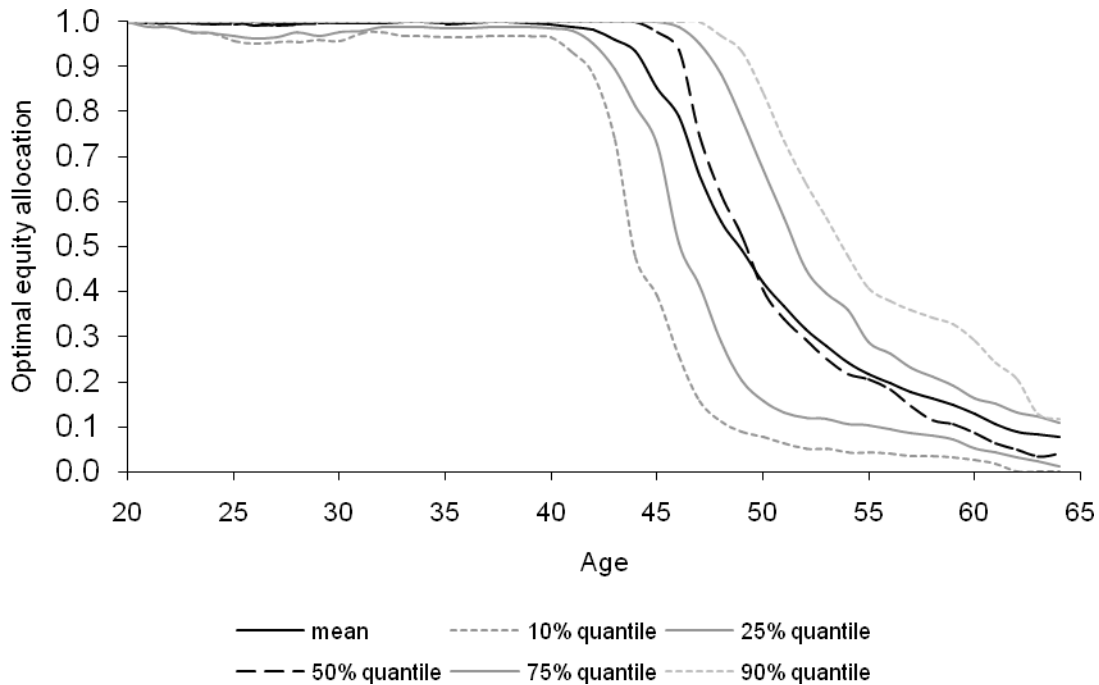
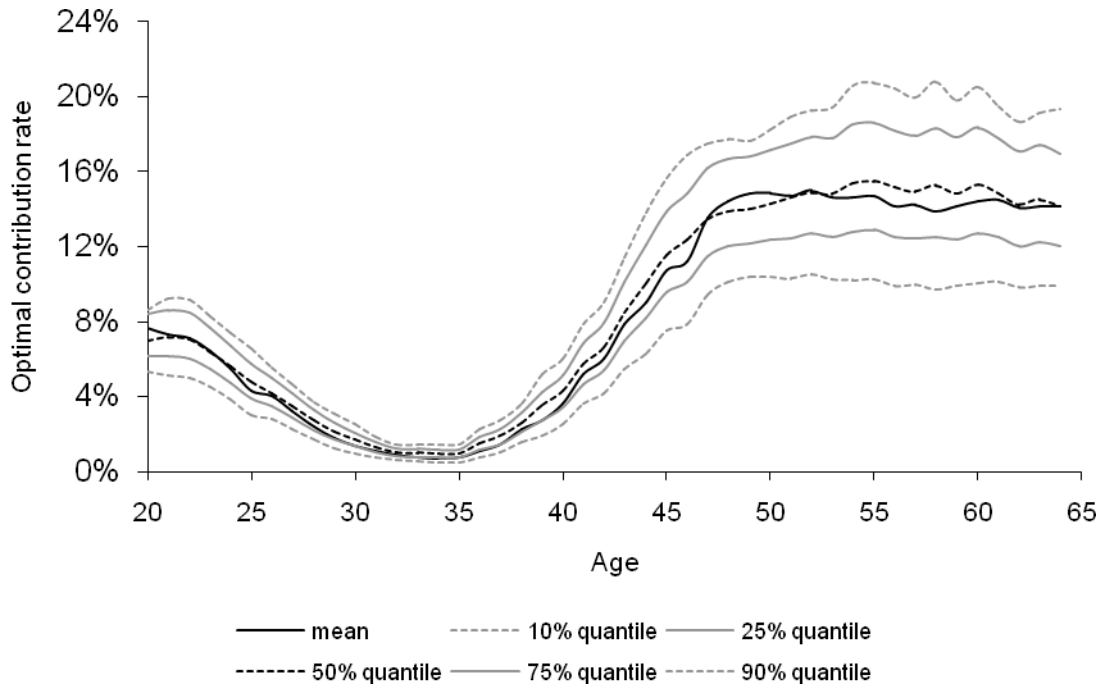


Figure 3.6 plots six quantiles from the distribution of optimal contribution rates, corresponding to the optimal asset allocation strategies shown in Figure 3.5. Considering the profile corresponding to the mean, the initial annual contribution rate at age 20 is just under 8% p.a.. It then decreases steadily to below 1% by age 35. This fall reflects a trade-off between the bequest motive and risk aversion, on the one hand, and the increase in human capital, on the other. Prior to age 35, when labour income is growing very rapidly and human capital is increasing, the member wishes to increase consumption and does so by reducing the contribution rate into the pension plan, despite being both risk averse and having a bequest motive.<sup>21</sup> After age 35, however, labour income growth slows down and human capital begins to decline, and the retirement savings motive starts to become important. The contribution rate then increases steadily to almost 15% p.a. by age 48. Labour income flattens out after age 48 (see Figure 2.1) and the contribution rate then remains roughly constant until retirement.

<sup>21</sup> Note that pension contributions will fall by less than the contribution rate since income is growing.

Figure 3.6 – Optimal contribution rate



The most interesting finding from Figure 3.6 is that the optimal contribution rate in a DC pension plan is age-related, rather than constant. It exhibits a U-shaped pattern between age 20 and age 48 and it is only (approximately) constant after age 48. As a consequence, the contribution rate is very variable, ranging from, in the case of the mean profile, below 1% (at age 35) to almost 15% (at ages 48 and above). The other quantiles have a very similar shape: they are relatively close to the mean during the U-shaped phase, but have a wider range of post-age-48 constant rates (ranging between 10 and 20%).

An age-related pattern of contribution rates is not common in real-world DC plans: for example, in the UK, there is typically a fixed standard (combined employer and employee) contribution rate varying between 8 and 10% (GAD (2006, Table 8.2)). Although age-related contribution rates are not common, minimum contributions are more so. Figure 3.7 illustrates the mean optimal contribution rate over the life cycle when a lower limit of 5% p.a. is imposed on the contribution rate (the original unconstrained mean contribution rate profile is shown for comparison). In this case, the member accumulates greater pension wealth when young and, therefore, can afford a lower

contribution rate as retirement approaches. Further, because of the higher accumulated pension wealth, the member switches to the risk-free asset earlier, as shown in Figure 3.8 (the original unconstrained mean optimal asset allocation is shown for comparison).. Nevertheless, there is a cost from imposing this constraint: expected utility at age 20 drops by around 1% (from 2.349 to 2.329).

Figure 3.7 – Mean optimal contribution rate (with lower limit on contribution rate of 5% per annum, shown as dotted line)

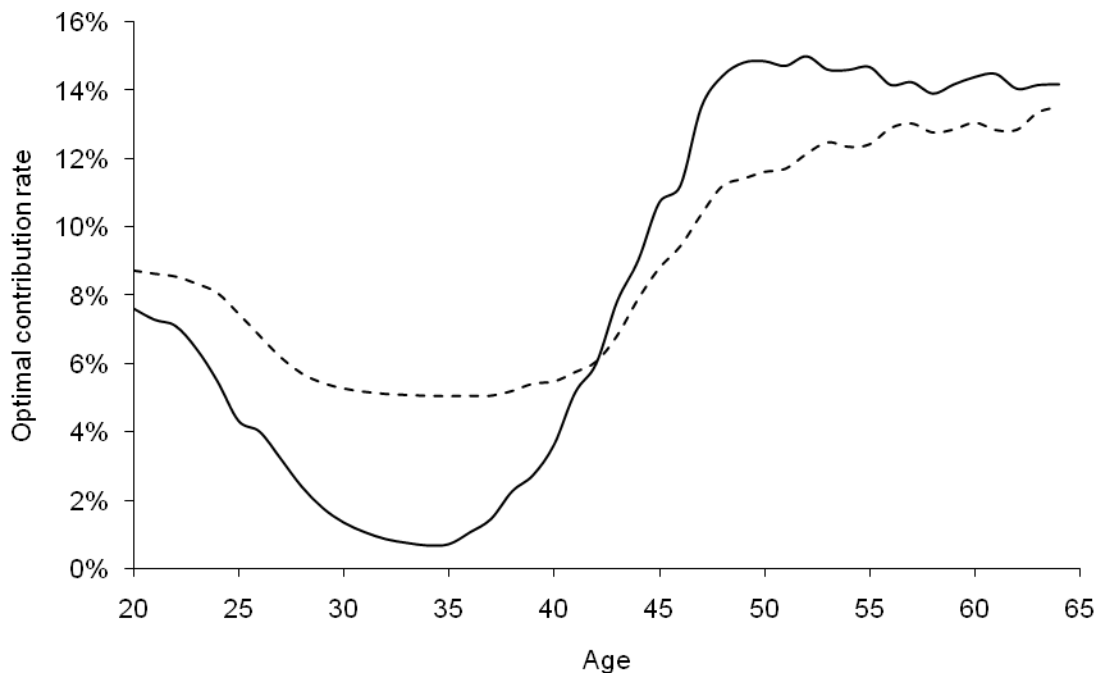
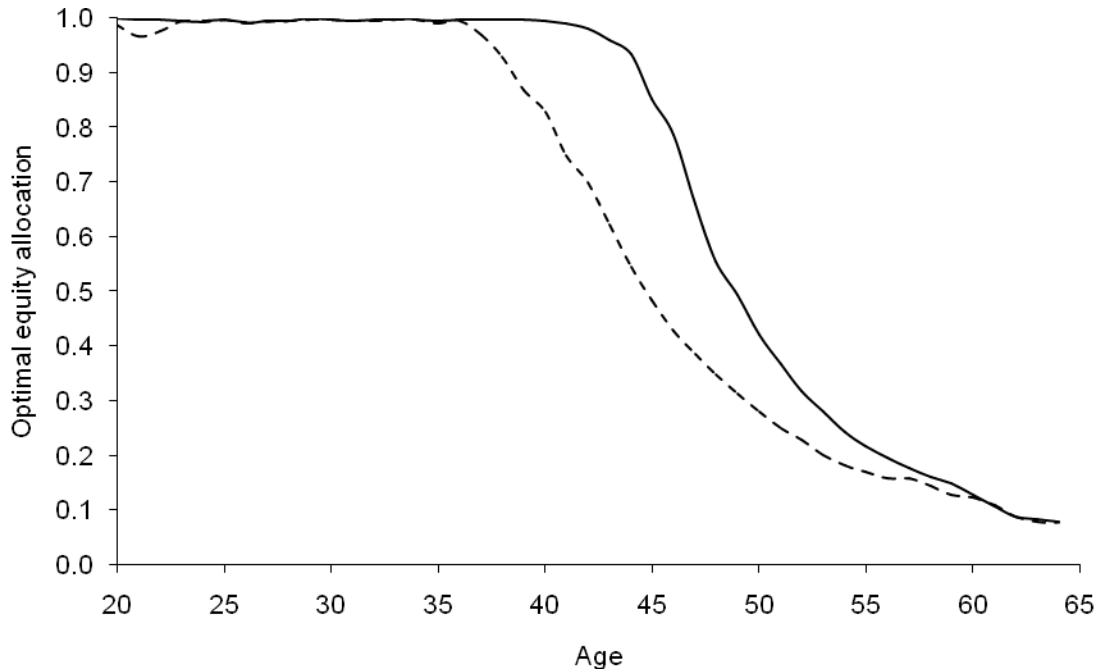


Figure 3.8 – Mean optimal equity allocation (with lower limit on contribution rate of 5% per annum, shown as dotted line)



## 3.2 Sensitivity analysis

In this section, we conduct a sensitivity analysis on the key parameters in the model.

### 3.2.1 RRA and EIS

We begin by examining simulation results for plan members with different RRA and EIS.

As shown in Table 3.1 (see also Table 1.1), we have four types:

- low RRA and low EIS (Type 1):
  - risk-tolerant member who dislikes consumption volatility over time
  - e.g., low income member without dependants
- low RRA and high EIS (Type 2):
  - risk-tolerant member who does not mind consumption volatility over time
  - e.g., high income member without dependants



- high RRA and low EIS (Type 3):
  - risk-averse member who dislikes consumption volatility over time
  - e.g., low income member with dependants
- high RRA and high EIS (Type 4):
  - risk-averse member who does not mind consumption volatility over time
  - e.g., high income member with dependants

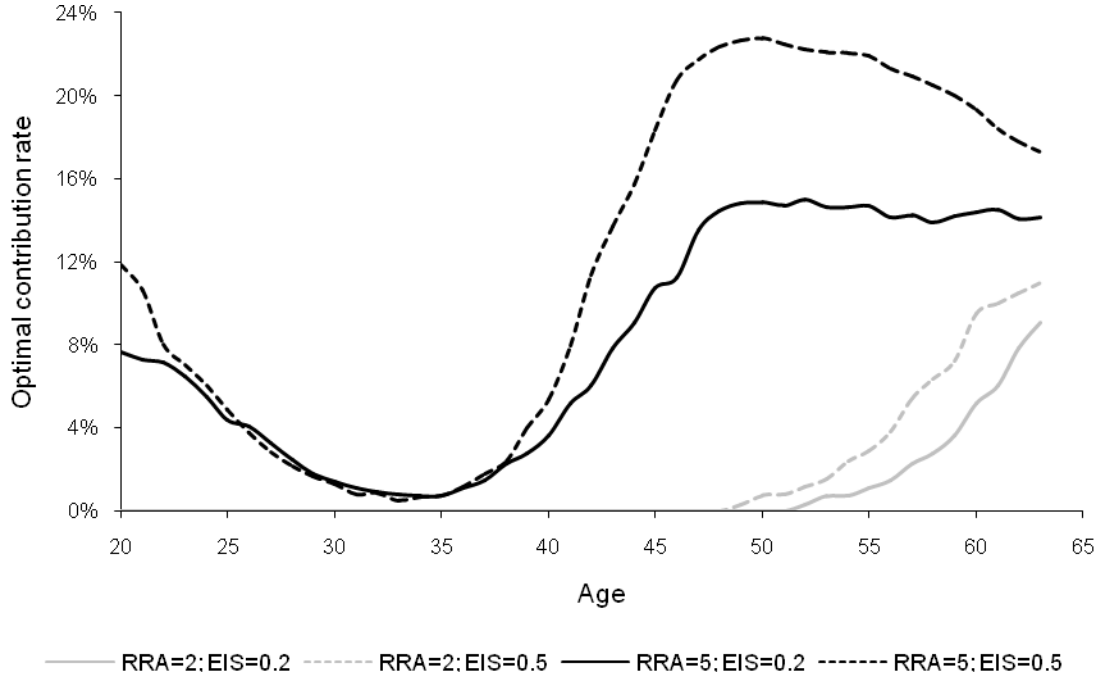
The baseline case in Section 3.1 dealt with Type 3 (highlighted in Table 3.1), a member with a high RRA and a low EIS (i.e.,  $\gamma = 5$  and  $\varphi = 0.2$ ).

**Table 3.1 RRA and EIS values for the different types of plan member**

	RRA, $\gamma$	EIS, $\varphi$
Type 1	2	0.2
Type 2	2	0.5
Type 3	5	0.2
Type 4	5	0.5

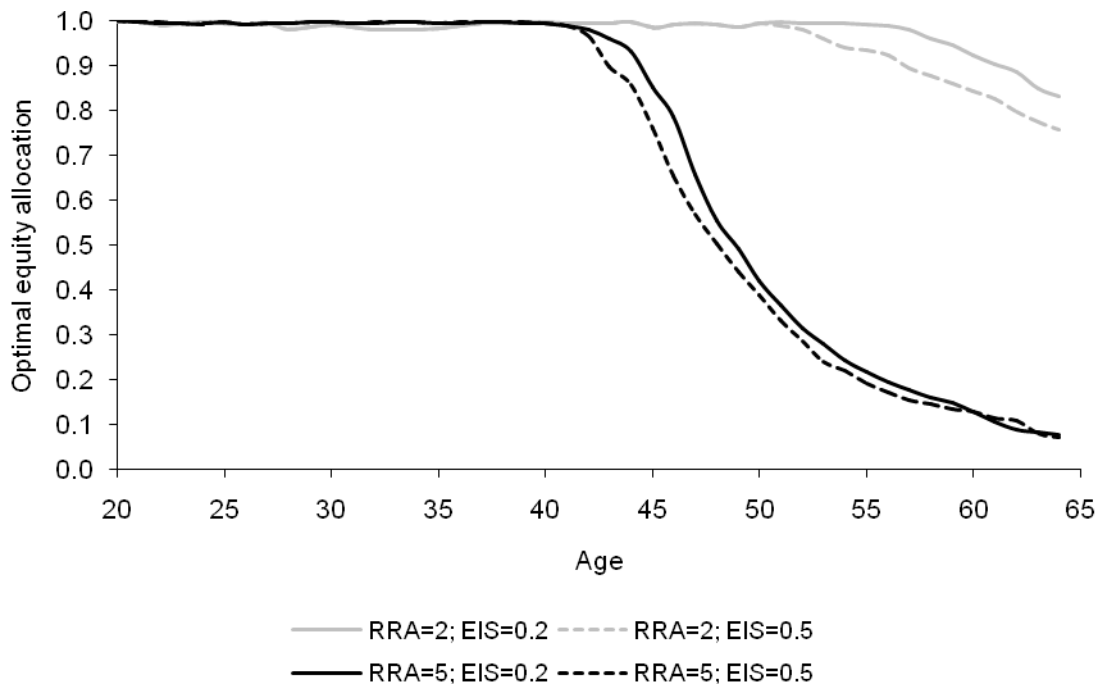
Figure 3.9 shows the different patterns of optimal contribution rates corresponding to these four types. For risk-tolerant members with low RRA (i.e., Types 1 and 2), contributions prior to age 50 are negligible. From age 50 or so, the retirement savings motive kicks in and contributions into the pension plan begin. The contribution rate is lower for Type 1 (low EIS) than for Type 2 (high EIS) members (by approximately 3-4% p.a.). This lower retirement savings intensity is the result of a stronger aversion to cutting consumption: due to the lower EIS level, cuts in consumption needed to fund the pension plan are more heavily penalised in the utility function of Type 1 members than of Type 2 members.

Figure 3.9 – Mean optimal contribution rate (for different RRA/EIS combinations)



As a result of the lower mean contribution rates (particularly at younger ages), risk-tolerant members, *ceteris paribus*, accumulate a lower level of pension wealth. They therefore need (and are, of course, willing to accept) a much higher average equity weighting (and the corresponding equity premium) in the financial portfolio in an attempt to produce the desired level of retirement savings. As shown in Figure 3.10, the mean equity allocation decreases only gradually and still exceeds 70% at retirement. The risk-tolerant member with a low EIS level (i.e., Type 1) chooses a higher equity weighting at retirement (by approximately 10%) than the risk-tolerant member with a high EIS level (i.e., Type 2). This is because, as discussed above, the annual contribution rate is lower and the reliance on the equity premium correspondingly greater.

Figure 3.10 – Mean optimal equity allocation (for different RRA/EIS combinations)



For risk-averse members with high RRA (i.e., Types 3 and 4), Figure 3.9 shows that the bequest motive leads to much larger initial contributions (in excess of 7% p.a. at age 20 on average) than for members with low RRA, and the retirement savings motive leads to a significant rise in contributions after age 40, again compared with low RRA members. For those risk-averse members with low EIS (i.e., Type 3), the variability of the contribution rate over the working life is much less pronounced than is the case for those risk-averse members with high EIS (i.e., Type 4). The contribution rate is also lower particularly at very young and very old ages (e.g., by around 5% p.a. at both age 20 and 60 on average). The explanation is the same as given above (i.e., the reluctance of members with a lower EIS level to tolerate cuts in current consumption to fund future consumption in retirement). The fall in the contribution rate to an average of around 1% between age 20 and age 35 for risk-averse members is explained by the increase in human capital over this period which increases the desire for current consumption at the expense of saving for retirement.

As a result of higher contribution rates, risk-averse members accumulate a much higher level of pension wealth and, as can be seen in Figure 3.10, switch away from equities much earlier (from about age 40) and hold less than 10% of the pension fund in equities at retirement on average.

For a given level of risk aversion, Figure 3.10 shows that a lower EIS leads to a higher equity weighting during the later stages of the accumulation phase of the pension plan. At first sight, this finding seems counterintuitive: surely individuals with a lower IES would prefer a lower equity weighting in their pension fund and hence more stable contributions over time? Constantinides (1990), for example, appeals to habit formation (i.e., the complementarity between consumption in adjacent periods) for inducing a strong desire for stable consumption and a correspondingly low demand for equities. However, in our model, the retirement saving motive is important, since the only source of post-retirement consumption in our model comes from the pension plan. During the accumulation phase, a member with a low EIS is not willing to reduce current consumption in order to increase plan contributions (to meet the retirement saving motive), but is willing to use a higher equity weighting and the corresponding equity premium in an attempt to generate the necessary retirement savings. Figure 3.10 shows that the retirement savings effect dominates the habit formation effect in determining the equity weighting in the later years of the accumulation phase (Gomes and Michaelides (2005) derived the same result).

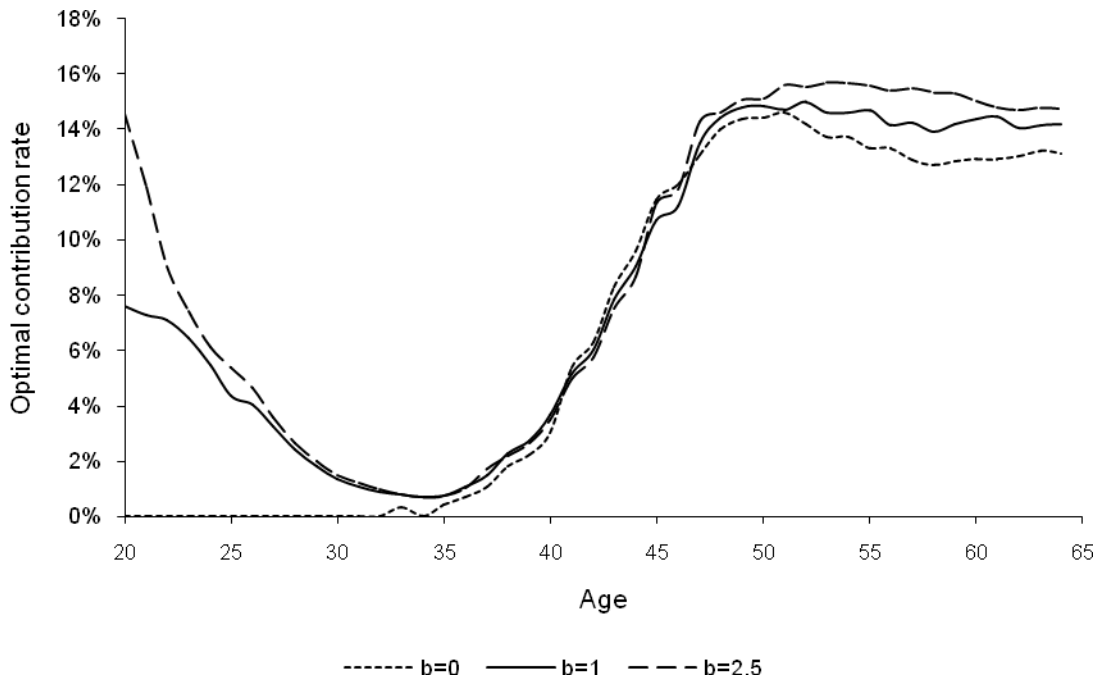
### **3.2.2 Bequest motive**

Investors with a desire to bequest wealth to their dependants on death would be expected to save more than those who do not. Figure 3.11 shows the mean optimal pattern of contribution rates for the baseline case of  $\gamma = 5$  and  $\varphi = 0.2$  and different bequest intensities. When the bequest motive is absent ( $b = 0$ ), members consume almost all of their earnings in the early years of their working lives, and have very negligible contributions into their pension plans. By contrast, members with a high bequest intensity ( $b = 2.5$ ) make very high contributions in the early years (14% at age 20). They also

make higher contributions throughout their working lives,<sup>22</sup> but the differences after age 35 are much less.

Figure 3.12 shows that the optimal equity weight in the portfolio is lower, the higher the bequest intensity. However, the effect is small because: first, the mortality rate does not vary much before retirement, and second, and more importantly, the annuitisation ratio ( $k$ ) is a control variable in our model and, thus, a member with a high bequest intensity could choose to annuitise a smaller ratio of the pension fund at retirement, instead of assuming significant equity risk in an attempt to accumulate more wealth prior to retirement.<sup>23</sup>

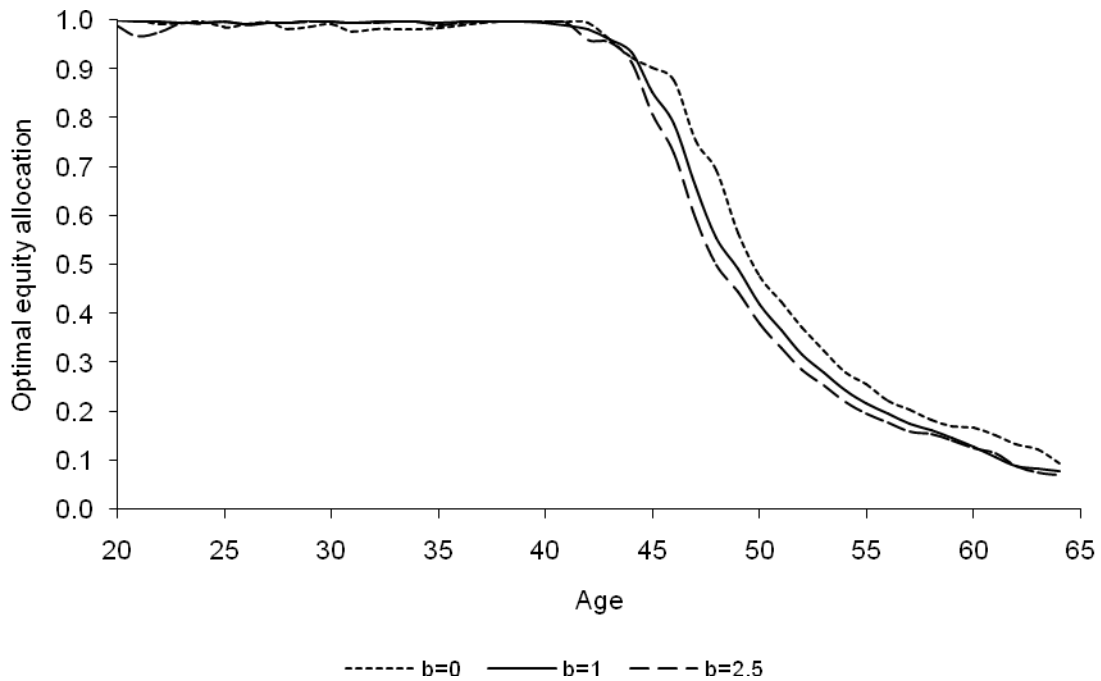
Figure 3.11 – Mean optimal contribution rate (for different levels of bequest motive)



<sup>22</sup> Although it is still pulled down to below 1% between ages 20 and 35 by the effect of the increase in human capital.

<sup>23</sup> This is confirmed in Table 3.2 below.

Figure 3.12 – Mean optimal equity allocation (for different levels of bequest motive)



### 3.2.3 Personal discount factor

Figures 3.13 and 3.14 show the outcome from conducting a sensitivity analysis on  $\beta$ , the individual's personal discount factor, on the mean optimal contribution rate and asset allocation.

Individuals with a low personal discount factor (or high personal discount rate) value current consumption more highly than future consumption in comparison with individuals with a high personal discount factor. This will lead, *ceteris paribus*, to a lower contribution rate into the pension plan as shown in Figure 3.13. There will be a correspondingly slower accumulation of financial wealth and therefore a higher ratio of human to financial wealth throughout the working life. This, in turn, leads to an optimal lifestyle strategy with a higher portfolio allocation to the risky asset throughout the working life, together with a shorter switching period, as shown in Figure 3.14.

Figure 3.13 – Mean optimal contribution rate (for different levels of personal discount factor,  $\beta$ )

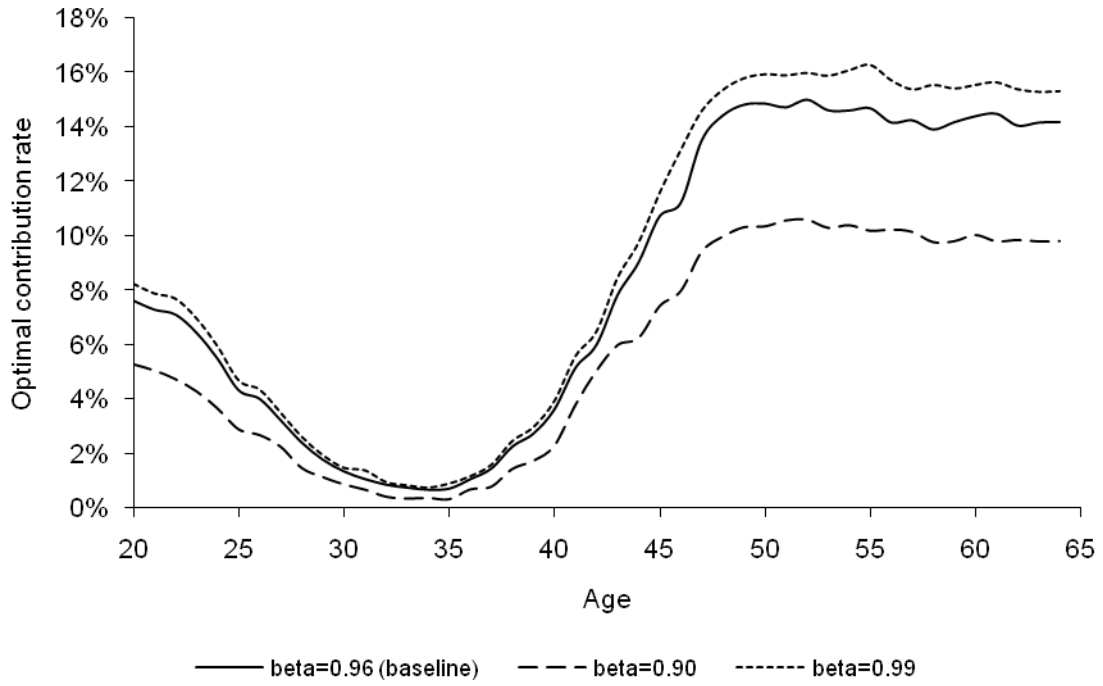
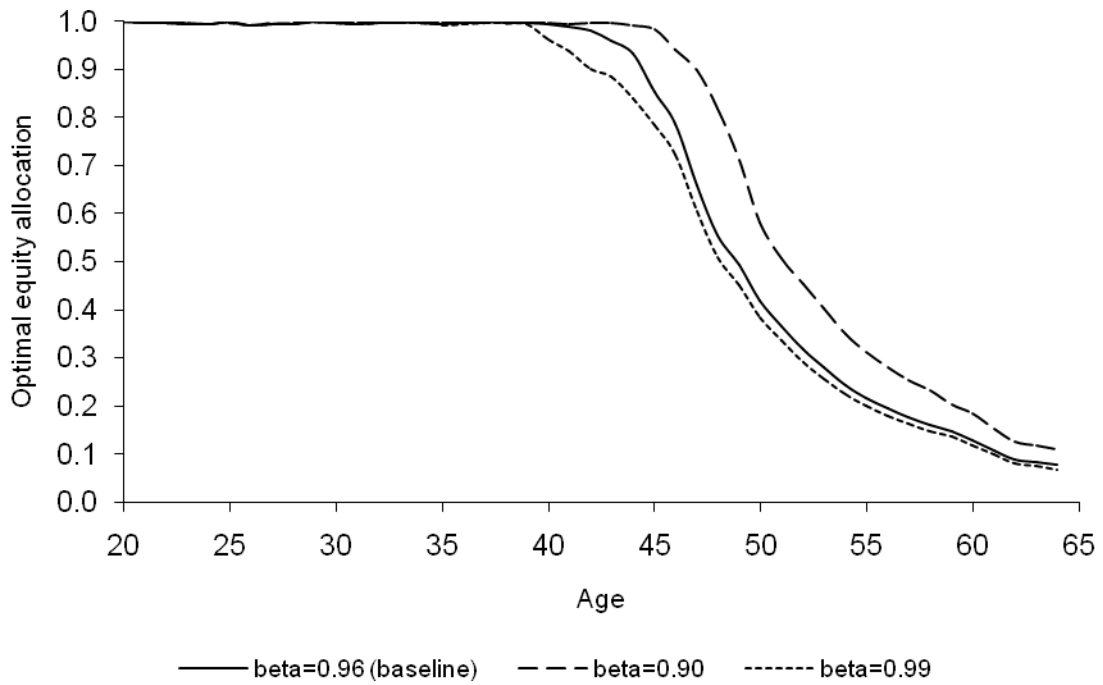


Figure 3.14 – Mean optimal equity allocation (for different levels of personal discount factor,  $\beta$ )



### 3.2.4 Correlation between labour income growth and equity returns

In our model, the degree of correlation between labour income growth and equity returns is determined by  $\sigma_1 / \sqrt{(\sigma_1^2 + \sigma_2^2)}$ . When this correlation coefficient is high, human capital and financial wealth will also be highly correlated. Although the ratio of human capital to financial wealth will fall over the life cycle, it will do so more gradually when the correlation coefficient is high than when it is low. As discussed earlier, this ratio is a crucial determinant of portfolio composition over the life cycle: as it falls, so does the optimal weight in equities. A more gradual decline in the ratio of human capital to financial wealth will lead to a more gradual switch from equities to the risk-free financial asset.

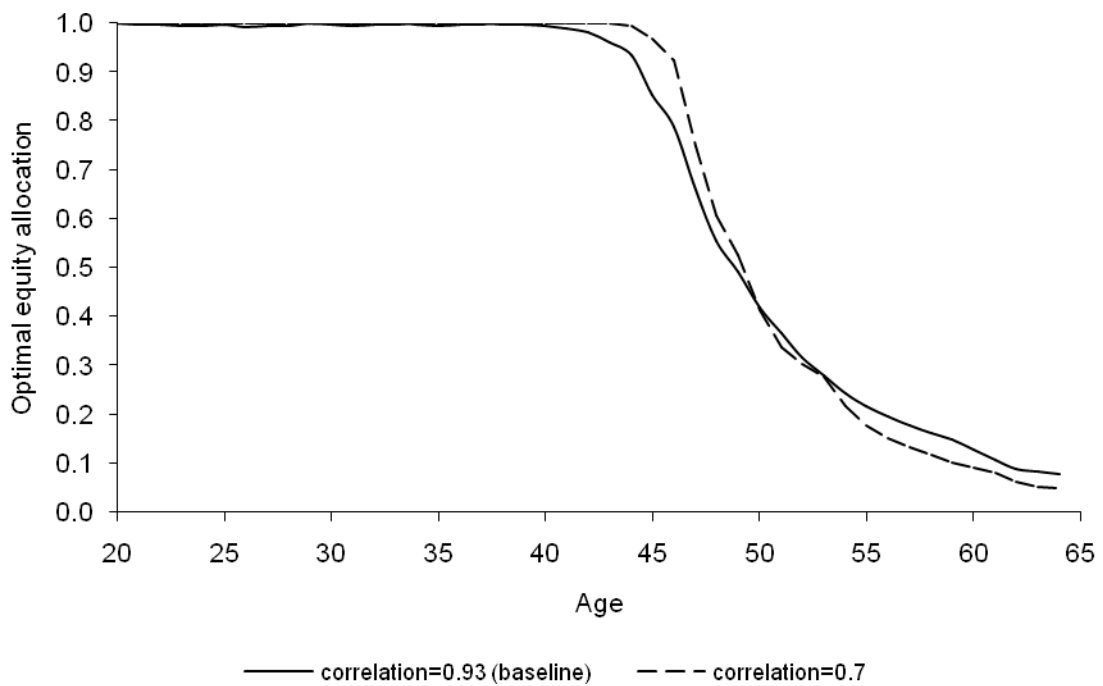
We assume a high correlation of 0.93 in our baseline case. Figures 3.15 and 3.16 show the outcomes from conducting a sensitivity analysis on this correlation coefficient: we lowered it to 0.70. The lower correlation coefficient has a negligible effect on the optimal contribution rate (Figure 3.15), but leads to a later and steeper switch away from equities (Figure 3.16).



Figure 3.15 – Mean optimal contribution rate (for different levels of correlation between labour income and equity returns)



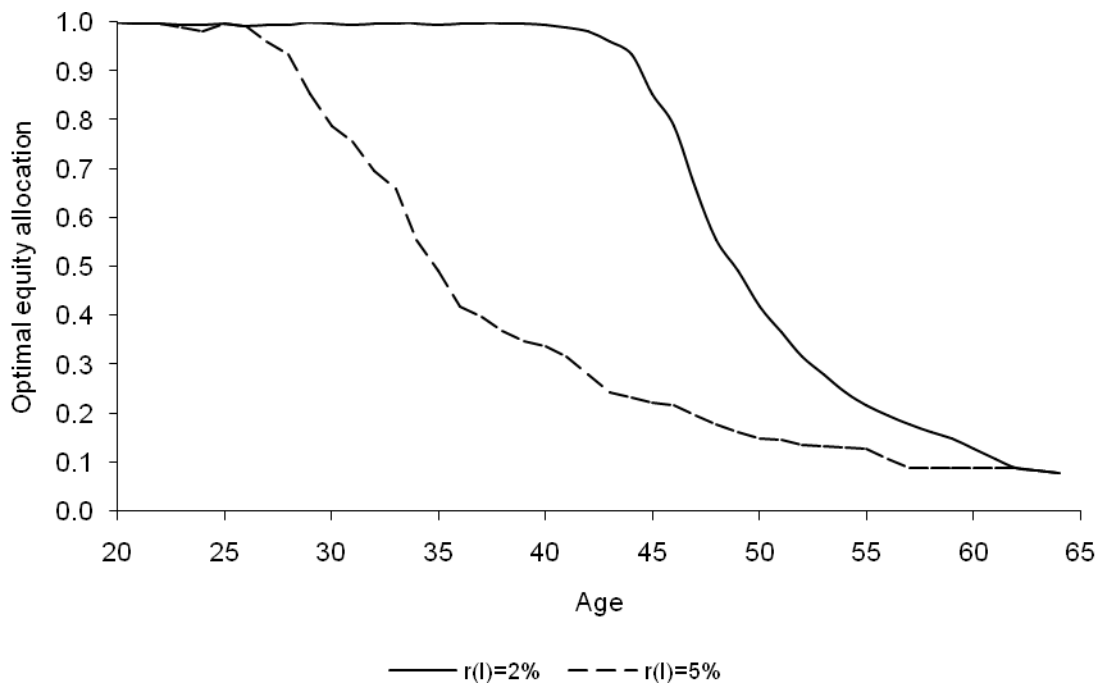
Figure 3.16 – Mean optimal equity allocation (for different levels of correlation between labour income and equity returns)



### 3.2.5 Salary growth rate

Finally, Figure 3.17 shows the effect on the mean optimal equity allocation of increasing the average annual real rate of salary growth,  $r_l$ , from 2% to 5%. The optimal equity weighting is lower at most ages.

Figure 3.17 – Mean optimal equity allocation (different average annual real rate of salary growth rates)



### 3.2.6 Annuitisation ratio

Although the annuitisation ratio,  $k$ , is part of the budget constraint not the utility function, it is still a control variable in our model, and is chosen to maximise  $E(V_{64})$ , conditional on the values for RAA, EIS and the personal discount factor (see Section 2.1.5 above). Table 3.2 shows the mean optimal values of  $k$  corresponding to different values for RAA, EIS and the personal discount factors.

There is a positive relationship between RAA and the annuitisation ratio,  $k$ , and also between EIS and  $k$ , although the relationship in each case is not very strong. There is, unsurprisingly, a much stronger relationship between the bequest intensity,  $b$ , and  $k$ . When there is no bequest motive, the member annuitises the largest ratio (99.35%) of his pension wealth.<sup>24</sup> According to Yaari (1965), when there is no bequest motive, the member should annuitise the entire accumulated pension fund at retirement. Davidoff et al (2005) extend Yaari's framework to include imperfect credit markets and habit formation (i.e., a low IES) and show it is optimal to annuitise less than 100% of wealth, but the optimal proportion is still very high. We find that this is the case even when we impose a significant reduction in the discount factor,  $\beta$ . A discount factor of 0.9, implying a very high personal discount rate of 10% (and very strong preference for current over future consumption) only reduces the optimal annuitisation ratio to 88%.

<b>Preference parameter</b>	<b>Optimal annuitisation ratio (mean %)</b>
<b>RRA, <math>\gamma</math></b>	
2	93.17
5	94.72
<b>EIS, <math>\phi</math></b>	
0.2	94.72
0.5	95.88
<b>Bequest intensity, <math>b</math></b>	
0.0	99.35
1.0	94.72
2.5	90.03
<b>Discount factor, <math>\beta</math></b>	
0.90	87.66
0.96	94.72
0.99	96.43
<b>Note: All parameters are at their baseline values (see Table 2.1) unless otherwise stated</b>	

<sup>24</sup> We obtain a value of 99.35% when the bequest motive is zero (rather than the expected 100%) due to the numerical methods used to solve the problem and the estimation error caused by interpolation approximations.

## 4 Conclusion

In this paper, we have examined optimal funding and investment strategies in a DC pension plan using a life-cycle model that has been extended in three significant ways:

- (i) the assumption of Epstein-Zin recursive preferences by the plan member which enables a separation between the RRA and EIS,
- (ii) the introduction of human capital as an asset class along with financial assets, such as equities and cash, and
- (iii) endogenising the decision about how much to annuitise at retirement.

We also considered two important motives for saving: bequest and retirement. In addition, a plan member's personal discount rate influenced the optimal strategies.

We found that the optimal funding strategy typically exhibits a U-shaped pattern for the contribution rate in early working life and then stabilises in the period leading up to retirement. The initial high level is explained by a high bequest intensity combined with a high degree of risk aversion. The falling part of the U is explained by the increase in human capital inducing an increase in consumption at the expense of savings, while the rising part of the U is explained by the retirement savings motive. A higher personal discount rate lowers contributions at each age, while preserving the general shape of the optimal contribution profile. A lower bequest intensity reduces contributions to the pension plan in early career and in the extreme case of a zero bequest intensity completely eliminates them until mid-career when the retirement savings motive kicks in. The effect of lower risk aversion is to delay the start of contributions into the pension plan: contributions do not begin until late in the working life, but then increase steadily until the retirement age. The effect of lower EIS is to reduce contributions into the plan at all ages, since members with low EIS are reluctant to accept cuts in consumption to “pay for” the pension contributions.

We found that the optimal investment strategy is stochastic lifestyling rather than the more conventional deterministic lifestyling. While the optimal weighting in equities is initially very high and subsequently declines as the retirement date approaches, it does not do so in a predetermined manner as in the case of deterministic lifestyling. Instead the optimal equity weighting over the life cycle depends on the realisations of the stochastic processes determining labour income and equity returns. Stochastic lifestyling is justified by recognising the importance of human capital and interpreting it as a bond-like asset which deteriorates over the working life. An initial high weighting in equities is intended to counterbalance human capital in the combined “portfolio” of human capital and pension wealth. In time, the weighting in equities falls, while that in bonds rises as human capital decays over time. When the correlation between labour income and equity returns is high, human capital and pension wealth will also be more highly correlated; the optimal switch away from equities over the life cycle will therefore be later and steeper.

The size of the pension fund is a crucial determinant of the optimal asset allocation. The greater the pension wealth accumulated, the more conservative the optimal asset allocation strategy will be, for a given RAA, EIS and discount rate. Also, the higher the contribution rate, the earlier the switch out of equities can be made. In our model, risk-tolerant members who value both current consumption over future consumption as smooth consumption profiles over time (i.e., have a low RRA, a high discount rate and a low EIS) accumulate the lowest pension wealth levels and therefore need to adopt the most aggressive investment strategy.

Finally, we found that the optimal annuitisation ratio was typically very high, suggesting that longevity protection is a hugely valuable feature of a well-defined DC pension plan to a rational consumer. It was negatively related to the bequest intensity and the discount rate, but not very sensitive to RRA or EIS.

The results in this paper have some important implications for the optimal design of DC pension plans:

- An investment strategy involving a switch from equities to bonds as members approach retirement is appropriate for DC pension plans, even when equity returns are not mean reverting. However, the switch out from equities is not predetermined, but depends on what happens to equity returns. Nevertheless, the switch should typically be made earlier than in traditional lifestyle strategies (i.e., from age 45 or so rather than age 55, which is more common in practice). Also, the optimal equity weight in the portfolio typically never reduces to zero (even immediately prior to retirement), as is common in traditional lifestyle strategies.
- It is very important to incorporate the salary process in the optimal design of a DC pension plan.<sup>25</sup> For most people, their human capital will be bond-like in nature and this will have a direct impact on the optimal contribution rate and asset allocation decisions, in particular, justifying a high weight for equities in the pension plan. However, for senior plan members whose salary levels (including bonus and dividends from their own stock holdings) may have a strong link with corporate profitability, their labour income growth rate may be much higher than the risk-free rate of return and also be more volatile. In this case, their human capital asset will be more equity-like in nature and so the optimal investment strategy will be more geared towards the risk-free asset (as shown in Figure 3.17).
- The results provide some justification for age-related contribution rates in DC plans. Because members tend to prefer relatively smoothed consumption growth, a plan design involving a lower contribution rate (e.g., 5% p.a. or less) when members are young, and a gradually increasing contribution rate as members get older (reaching on average 15% p.a. in the period prior to retirement) offers higher expected utility than fixed age-independent contribution rates. Greater contribution rate flexibility will be especially welcomed by members with high EIS. Such members are more sensitive to

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<sup>25</sup> This was first pointed out by Blake et al (2007).

changes in financial incentives and thus are more desirous of flexible contribution rates. If they were allowed to make additional voluntary contributions (AVCs), the optimal asset allocation strategy would actually become more conservative as they approach retirement.

- An annuity is an important component of a well-designed plan. The optimal annuitisation ratio is very high, even when there is a strong bequest motive and the plan member values current consumption highly. This is true despite the well-known aversion to annuitisation documented in Friedman and Warshawsky (1990) and Mitchell et al (1999).

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## Appendix 1 - Numerical solution for the dynamic programming and integration process

The Epstein-Zin utility function at time  $t$  is as follows:

$$V_{20+t} = \left\{ (1 - \beta p_{20+t}) C_{20+t}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \left[ E_{20+t} \left( p_{20+t} \times V_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \left( \frac{W_{20+t+1}}{b} \right)^{1-\gamma} \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\varphi}}} \right\}$$

In the last period  $(\omega-1, \omega)$ , where  $p_{\omega-1} = 0$ , the terminal value function is given by:

$$J_{\omega-1}(W_{\omega-1}) = \max_{0 \leq C_{\omega-1} \leq W_{\omega-1}, 0 \leq \alpha_{\omega-1} \leq 1} \left\{ C_{\omega-1}^{\frac{1-\frac{1}{\varphi}}{\varphi}} + \beta \times E_{20+T} \left[ b \times \left( \frac{W_{\omega}}{b} \right)^{1-\gamma} \right]^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

The optimisation problem is then:

$$\max_{\alpha_{20+t}, C_{20+t}, k} E(V_{20+t})$$

subject to the constraints given by:

- $W_{20+t+1} = (W_{20+t} + \pi_{20+t} Y_{20+t}) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})]$ , when  $0 \leq t \leq 43$ ;
- $W_{65} = (1 - k) \times ((W_{64} + \pi_{64} Y_{64}) \times [1 + r + \alpha_{64} (\mu + \varepsilon_{64})])$ , when  $t = 44$ ;
- $W_{20+t+1} = \left( W_{20+t} + \frac{k \times W_{65}^-}{\ddot{a}_{65}} - C_{20+t} \right) \times [1 + r + \alpha_{20+t} (\mu + \varepsilon_{20+t})]$ , when  $t \geq 45$ ;
- $0 \leq \alpha_{20+t} \leq 1$ ; and
- $0 \leq k \leq 1$ .

The Bellman equation is given by:

(a) for  $0 \leq t \leq 43$ , we have:

$$\begin{aligned}
 & J_{20+t}(W_{20+t}, Y_{20+t}) \\
 = & \max_{\alpha_{20+t}, C_{20+t}} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{1-\frac{1}{\varphi}} + \beta \left[ E_{20+t} \left( p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left( \frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}
 \end{aligned}$$

(b) for  $t = 44$ , we have:

$$\begin{aligned}
 & J_{64}(W_{64}, Y_{64}) \\
 = & \max_{\alpha_{64}, C_{64}, k} \left\{ (1 - \beta p_{64}) C_{64}^{1-\frac{1}{\varphi}} + \beta \left[ E_{64} \left( p_{64} \times J_{65}^{1-\gamma} + (1 - p_{64}) \times b \times \frac{\left( \frac{W_{65}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}
 \end{aligned}$$

(c) and, for  $t \geq 45$ , we have:

$$\begin{aligned}
 & J_{20+t}(W_{20+t}, Y_{20+t}) \\
 = & \max_{\alpha_{20+t}, C_{20+t}} \left\{ (1 - \beta p_{20+t}) C_{20+t}^{1-\frac{1}{\varphi}} + \beta \left[ E_{20+t} \left( p_{20+t} \times J_{20+t+1}^{1-\gamma} + (1 - p_{20+t}) \times b \times \frac{\left( \frac{W_{20+t+1}}{b} \right)^{1-\gamma}}{1-\gamma} \right) \right]^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}
 \end{aligned}$$

To solve the non-linear expectation part in the Bellman equation above, i.e.,  $E_{20+t} \left( J_{20+t+1}^{1-\gamma} \right)$ , Gauss-Hermite quadrature is used to discretise  $Z_1$  and  $Z_2$  into 9 nodes, and the procedure of discretising  $Z_1$  and  $Z_2$  is to substitute  $\sqrt{2}Z_{1,m}$  and  $\sqrt{2}Z_{2,n}$  for them respectively.

So, we have:

$$E_{20+t} \left( J_{20+t+1}^{1-\gamma} \right) \approx \pi^{-1} \sum_{m=1}^9 \sum_{n=1}^9 w_{Z_{1,m}} w_{Z_{2,n}} \left[ J_{20+t+1} \left( \sqrt{2}Z_{1,m}, \sqrt{2}Z_{2,n}; W_{20+t+1}, Y_{20+t+1} \right) \right]^{1-\gamma}$$

where  $w_{Z_{1,m}}$ ,  $w_{Z_{2,n}}$  and  $Z_{1,m}$ ,  $Z_{2,n}$  are the Gauss-Hermite quadrature weights and nodes, and  $\pi$  is the mathematical constant.

To avoid choosing a local maximum, we discretise both control variables (i.e., consumption and asset allocation) into 20 equally-spaced grids and optimise using a standard grid search. Wealth and labour income level are discretised in 30 and 10 equally-spaced grids, respectively.

Substituting the expectation with the Gauss-Hermite approximated function in the Bellman equation at time  $t$ , we compute the maximum value function and optimal variables at each grid point. We then iterate the procedure back to age 20. For a point in the state space other than grid points, cubic spline interpolation is employed to approximate the optimal results.