Estimation of Future Initial Margins

Multi-Curve Interest Rate Framework

Marc Henrard

Advisory Partner - OpenGamma
Visiting Professor - University College London
Estimation of Future Initial Margins

1. Initial margin, multi-curve and collateral framework
2. Rational model
3. IM dynamics
4. Margin value adjustment
Margins

1. Initial margin, multi-curve and collateral framework
2. Rational model
3. IM dynamics
4. Margin value adjustment
Variation Margin - Initial Margin - MPR
Variation Margin - Initial Margin - MPR
Regulatory time table

- 2013 Mandatory clearing (USA)
  Front-loading as of 21 February 2016
- 2016 – 2019 Mandatory Bilateral margin
  Category 1: 1 September 2016 for VM and IM
Cash collateral pricing formula

\[ N_t^c = \exp \left( \int_0^t c_\tau d\tau \right) \]

Theorem (Collateral with cash price formula)

In presence of cash collateral with rate \( c \), the quote at time \( t \) of an asset with price \( V_u^c \) at time \( u \) is

\[ V_t^c = N_t^{c_\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[ (N_u^c)^{-1} V_u^c | \mathcal{F}_t \right] \]

for some measure \( \mathbb{Q} \) (identical for all assets, but potentially currency-dependent).

Note that the result refers to three “objects”: \( V_u \), \( c \) and \( \mathbb{Q} \). This formula is also called collateral account discounting.
Collateral: pseudo-discount factors

Definition (Collateral pseudo-discount factors)

The collateral (pseudo-)discount factors for the collateral rate $c$ paid in currency $X$ are defined by

$$P^c(t, u) = N^c_t E^Q \left[ (N^c_u)^{-1} \mid \mathcal{F}_t \right].$$

In the sequel we will work with OIS discounting and use the notation $D_u = (N^c_u)^{-1}$. Change of numeraire is still possible in this framework. In particular we will introduce a different measure, called $\mathbb{M}$, and use the notation

$$(D_t)^{-1} E^Q \left[ D_u V_u^c \mid \mathcal{F}_t \right] = (h_t)^{-1} E^\mathbb{M} \left[ h_u V_u^c \mid \mathcal{F}_t \right]$$
Multi-curve framework with collateral

The value of a $j$ floating coupon in currency $X$ with collateral at rate $c$ is an asset for each tenor $j$, each fixing date $\theta$ and each collateral rate $c$.

Definition (Forward index rate with collateral)

The forward curve $F^c_{t}^{c,j}(\theta, u, v)$ is the continuous function such that,

$$P^c(t, v)\delta F^c_{t}^{c,j}(t, u, v)$$

is the quote at time $t$ of the $j$-Ibor coupon with fixing date $\theta$, start date $u$, maturity date $v$ ($t \leq t_0 \leq u = \text{Spot}(t_0) < v$) and accrual factor $\delta$ collateralised at rate $c$. 
Margins

1. Initial margin, multi-curve and collateral framework
2. Rational model
3. IM dynamics
4. Margin value adjustment
**Rational model**


Formulas for the discounting and *forward* in the rational model are

\[
P^c(t, u) = \frac{P^c(0, u) + b_1(u)A_t^{(1)}}{P^c(0, t) + b_1(t)A_t^{(1)}}
\]

\[
L^{c,j}(t; u, v) = \frac{L^{c,j}(0; u, v) + b_2(u, v)A_t^{(1)} + b_3(u, v)A_t^{(2)}}{P^c(0, t) + b_1(t)A_t^{(1)}}
\]

where \(A_t^{(i)}\) is a martingale in a \(\mathbb{M}\)-measure with

\[
A_t^{(i)} = \exp \left( a_i X_t^{(i)} - \frac{1}{2} a_i^2 t \right) - 1
\]

where \(j\) is an Ibor index and \(v - u = \text{Tenor}(j)\).

Note: The \(L^{c,j}(t; u, v)\) in the rational model corresponds to the \(P^c(t, v)F^{c,j}(t, u, v)\) in the standard multi-curve framework.
Rational model - spread

Let’s define

\[ L^c(t; T_{i-1}, T_i) = \frac{1}{\delta_i} \left( \frac{P^c(t, T_{i-1}) - P^c(t, T_i)}{\delta_i A_t^{(1)}} \right) \]

with \( \delta_i \) the accrual factor for the period \([T_{i-1}, T_i]\. The dynamics for this quantity is described by

\[ L^c(t; T_{i-1}, T_i) = \frac{L^c(0; T_{i-1}, T_i) + (b_1(T_{i-1}) - b_1(T_i))}{\delta_i A_t^{(1)}} \]

\[ P^c(0, t) + b_1(t)A_t^{(1)} \]

\[ L^{c,j}(t; u, v) = \frac{L^c(0) + (b_1(u) - b_1(v))}{\delta_i A_t^{(1)}} \]

\[ P^c(0, t) + b_1(t)A_t^{(1)} \]

\[ + \frac{(L^{c,j}(0) - L^c(0)) + (b_2 - b_1)A_t^{(1)} + b_3(u, v)A_t^{(2)}}{P^c(0, t) + b_1(t)A_t^{(1)}} \]
Rational model - calibration

Calibration one-factor model, term structure.
Rational model - calibration

Calibration one-factor model, smile.

![Graph showing Black vol (%) vs Moneyness (%)](image)
Margins

1. Initial margin, multi-curve and collateral framework
2. Rational model
3. IM dynamics
4. Margin value adjustment
Initial margin

Initial margin are usually computed as Value-at-Risk (VaR) or Expected Shortfall (ES) using historical or Monte-Carlo approaches. Direct brute force calculation of future IM and MVA might be expensive.

- Historical VaR usually implies full revaluation
- Some simplifying methods neglect stochastic interactions between IM and market.
- Nested Monte Carlo iterations are very expensive.

Our approach: Characterize the initial margin process in terms of the dynamics of the underlying processes and a conditional risk measure.
Abstract formulation

Initial Margin

The initial margin process \( \{IM_t\}_{0 \leq t \leq T} \) associated to the portfolio is a process such that

\[
IM_t = \lambda_t(Z_{t,t+\delta})
\]

where

- \( Z_{t,t+\delta} \) are the cash flows given default associated with the portfolio between times \( t, t + \delta \) (loss given default)
- \( \delta \) is the margin period of risk (time to "close out")
- \( \{\lambda_t\}_{0 \leq t \leq T} \) is a family of conditional risk measures
Risk measure

**Conditional risk measures:** $\lambda_t$ maps any $\mathcal{F}$ measurable r.v. to an $\mathcal{F}_t$-measurable r.v. with finite expectation.

**Examples:** If $\mathbb{P}^*$ denotes the subjective probability observed by a CCP:

- $\text{VaR}^{\alpha,\mathbb{P}^*}_t[X] = \text{ess inf} \left\{ \Theta : \mathbb{P}^*[X \leq \Theta | \mathcal{F}_t] \leq \alpha \ a.s., \Theta \text{ is } \mathcal{F}_t\text{-measurable} \right\}$
- $\text{ES}^{\alpha,\mathbb{P}^*}_t[X] = \text{VaR}^{\alpha,\mathbb{P}^*}_t[X] + \frac{1}{1-\alpha} \mathbb{E}^{\mathbb{P}^*} \left[ (X - \text{VaR}^{\alpha,\mathbb{P}^*}_t[X])^+ | \mathcal{F}_t \right]$

**Notation:** $\bar{\lambda}_t(X) = -\lambda_t(-X)$
Risk measure

Properties: We use several properties satisfied by $\text{VaR}^{\alpha,\mathbb{P}^*}_t$ and $\text{ES}^{\alpha,\mathbb{P}^*}_t$; Let $X, Y$ be two $\mathcal{F}$—measurable r.v. and let $\Theta$ be an $\mathcal{F}_t$-measurable r.v. with $\mathbb{E}^{\mathbb{P}^*} [\Theta] < \infty$. We assume:

- Normalization: $\lambda_t(0) = 0$ $\mathbb{P}$-a.s.
- Monotonicity: $X \leq Y$ ($\mathbb{P}$ — a.s.) $\Rightarrow$ $\lambda_t(X) \leq \lambda_t(Y)$ ($\mathbb{P}$-a.s.)
- Conditional positive homogeneity: if $\Theta \geq 0$ ($\mathbb{P}$ — a.s.) $\Rightarrow$ $\lambda_t(\Theta X) = \Theta \lambda_t(X)$.
- Conditional translation: $\lambda_t(X + \Theta) = \lambda_t(X) + \Theta$
Risk measure

Cashflow decomposition:

\[ V_t = {1 \over D_t} \mathbb{E}^Q \left[ \sum_{i=1}^n D_{T_i} C_{T_i} \right| \mathcal{F}_t \]  

Value given default: (signs for member, assuming "closing out" exactly at \( \delta \)):

\[ Z_{t,t+\delta} = {1 \over \bar{D}(t, t+\delta)} V_{t+\delta} + \sum_{i=1}^n {1 \over \bar{D}(t, T_i)} C_{T_i} 1_{\{T_i \in [t,t+\delta)\}} - V_t \]

- \( \bar{D} \): discounting term associated to the funding of CCP (may differ from \( D \))

- "close out" value

- Variation margin at \( t \)

Cashflows missed in \([t, t+\delta)\)
Risk measure

Cashflow decomposition:

\[ V_t = \frac{1}{D_t} \mathbb{E}_Q^Q \left[ \sum_{i=1}^{n} D_{T_i} C_{T_i} \mid \mathcal{F}_t \right] \]

Value given default: (signs for member, assuming "closing out" exactly at \( \delta \)):

\[ Z_{t,t+\delta} = \frac{1}{\tilde{D}(t, t + \delta)} V_{t+\delta} + \sum_{i=1}^{n} \frac{1}{D(t, T_i)} C_{T_i} 1_{\{T_i \in [t,t+\delta)\}} - V_t \]

- \( \tilde{D} \): discounting term associated to the funding of CCP (may differ from \( D \))
Initial margin computation

\[
\text{IM}_t = \lambda_t \left( \frac{1}{\bar{D}(t, t+\delta)} V_{t+\delta} - V_t + \sum_{i=1}^{n} \frac{1}{\bar{D}(t, T_i)} C_{T_i} 1_{\{T_i \in [t, t+\delta)\}} \right)
\]

Rational framework leads to complexity reduction:

- Profit from the explicit/semi-explicit expressions available for most common IR derivatives with rational framework under the $\mathbb{M}$ measure: $V_t = V_t(A^{(1)}, A^{(2)})$; $C_t = C_t(A^{(1)}, A^{(2)})$
- Exploit numerically the simplify setup
- Introduce historically estimated change of measure between $\mathbb{M}$ and $\mathbb{P}^*$ (CCP scenario information)
Initial margin computation

\[
IM_t = \kappa \mathbb{E}S_t^{\alpha, \mathbb{P}^*} \left( \frac{h_{t+\delta}}{h_{t}} V_{t+\delta} - V_t + \sum_{i=1}^{n} \frac{h_{T_i}}{h_{t}} C_{t+\delta} 1_{\{T_i \in [t, t+\delta)\}} \right)
\]

Rational framework leads to complexity reduction:

- Profit from the explicit/semi-explicit expressions available for most common IR derivatives with rational framework under the \( \mathbb{M} \) measure: \( V_t = V_t(A^{(1)}, A^{(2)}) \); \( C_t = C_t(A^{(1)}, A^{(2)}) \)
- Exploit numerically the simplify setup
- Introduce historically estimated change of measure between \( \mathbb{M} \) and \( \mathbb{P}^* \) (CCP scenario information)
Initial margin computation

\[ IM_t = \kappa \mathbb{ES}^{\alpha, P^*}_t \left( \frac{h_{t+\delta}}{h_t} V_{t+\delta} - V_t + \sum_{i=1}^{n} \frac{h_{T_i}}{h_t} C_{t+\delta} 1\{T_i \in [t, t+\delta)\} \right) \]

**Rational framework leads to complexity reduction:**

- Profit from the explicit/semi-explicit expressions available for most common IR derivatives with rational framework under the \( \mathbb{M} \) measure: \( V_t = V_t(A^{(1)}, A^{(2)}) \); \( C_t = C_t(A^{(1)}, A^{(2)}) \)
- Exploit numerically the simplify setup
- Introduce historically estimated change of measure between \( \mathbb{M} \) and \( P^* \) (CCP scenario information)
Change of measure between $M$ and $P^*$

- Change of measure at time $\delta$: Comparing the model and historical distribution of P&L

- Change of measure after time $\delta$: Using independent increments assumption and iterating
### IM: Interest rate swaps

Assume no payments in \([t, t + \delta]\). Then:

\[
\text{IM}_t = \kappa \text{ES}^{\alpha, \mathbb{P}^*}_t \left( \text{Sw}_t - \frac{h_{t+\delta}}{h_t} \text{Sw}_{t+\delta} \right)
\]

Since the swap price processes satisfy

\[
\text{Sw}_t = \frac{c_0(t) + c_1(t)A_t^{(1)} + c_2(t)A_t^{(2)}}{P(0; t) + b_0(t)A_t^{(1)}}
\]

the properties of the risk measure and the assumed model imply

\[
\text{IM}_t = C_0^t + C_1^t \nu(R_t) + C_2^t \bar{\nu}(R_t)
\]

where \(C^0, C^1, C^2, R\) are \(\mathcal{F}_t\)-meas. (depend only on \(t, A^{(1)}, A^{(2)}\)),

\[
\nu(x) := \kappa \text{ES}^{\alpha, \mathbb{P}^*}_t \left( \gamma^{(1)}_{\delta} + x\gamma^{(2)}_{\delta} \right) ; \quad \gamma^{(i)}_{\delta} := A^{(i)}_{\delta} + 1
\]

for \(i = 1, 2\).
Functions $\nu, \bar{\nu}$

The functions $\nu, \bar{\nu}$ we just defined are smooth and increasing and can be approximated by a grid with few elements.

*Values obtained with $4 \cdot 10^7$ MC iterations
Numerical estimation (IM)

IM for a 5 years future swap with tenor of 6 months

Mean value (yellow) and 90% and 10% percentiles (red and blue)
Numerical tests: speed

CPU time in prototype implementation of $MC^2$ and the refined method

<table>
<thead>
<tr>
<th>Paths*</th>
<th>Grid† points</th>
<th>$MC^2$ (s)</th>
<th>Refined Method (s)</th>
<th>Accel. factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Initial</td>
<td>Evolution</td>
</tr>
<tr>
<td>1250</td>
<td>120</td>
<td>28.52</td>
<td>5.73</td>
<td>0.20</td>
</tr>
<tr>
<td>1250</td>
<td>520</td>
<td>126.32</td>
<td>5.76</td>
<td>0.76</td>
</tr>
<tr>
<td>2500</td>
<td>120</td>
<td>94.21</td>
<td>5.72</td>
<td>0.23</td>
</tr>
<tr>
<td>2500</td>
<td>520</td>
<td>423.37</td>
<td>5.73</td>
<td>0.92</td>
</tr>
</tbody>
</table>

* 1250 paths ≈ 5yrs history; 2500 ≈ 10yrs history.
† 120 ≈ monthly periodicity; 520 ≈ weekly periodicity.
Margins

1. Initial margin, multi-curve and collateral framework
2. Rational model
3. IM dynamics
4. Margin value adjustment
Cost of IM: Funding rate
Between times $u$ and $\delta_u$, a clearing member pays on average the funding cost:

\[
\widehat{r}_u^{f} \quad IM_u \quad \delta_u
\]

**Assumption:** The treasury of a clearing member funds all liquidity requirements by securing a basket of funds with best-matching maturities.

\[
r_t^f := \sum_{k=1}^{M} \gamma_k L \left( t, T_{i_k^*(t)}, T_{i_k^*(t)+1}^{\Delta_k} \right) + A_t^{(3)} - r_t
\]

where

- $\Delta_1, \ldots, \Delta_M \in \mathbb{R}_+$ are maturities
- $\gamma_1, \ldots, \gamma_M \in [0, 1]$ with $\sum_{k=1}^{M} \gamma_k = 1$ weights
- $A^{(3)}$ is an idiosyncratic factor.
MVA

Then, the (Follmer Schweizer) price of MVA can be modeled as

\[ \text{MVA}_t = \frac{1}{D_t} \mathbb{E}^Q \left[ \int_t^T D_u r_u^f \text{IM}_u \, du \middle| \mathcal{F}_t \right] + H_t \]

\[ = \frac{1}{h_t} \mathbb{E}^M \left[ \int_t^T h_u r_u^f \text{IM}_u \, du \middle| \mathcal{F}_t \right] + H_t \]

where \( H \) is a martingale orthogonal to the tradable assets.

- \( IM \) and \( r^f \) are Markovian: functions of the underlying factors.
- In the swap portfolio case and \( A^{(3)} \equiv 0 \), the MVA is hedgeable and \( H \equiv 0 \).
Numerical MVA
MVA for a 5 years future swap with tenor of 6 months ($A^{(3)} = 0$)

Mean value (yellow) and 90% and 10% percentiles (red and blue)
Conclusion

- Proposed a method based on explicit computation of the IM as a dynamic process by itself.

- Numerical implementation can become efficient as we remove one layer of numerical effort.

- The implementation is basically portfolio size invariant: only the cash flow description \( c_i(t) \) is portfolio composition dependent.

- The approach clearly differentiate between pricing and risk measures. In particular the IM computation includes the tail used in the actual CCP margin computation.

- The dynamic IM can be used as a base to MVA computations.
Presentation based on the paper


http://ssrn.com/abstract=2682727

Contact OpenGamma

Web: www.opengamma.com
Email: info@opengamma.com
Twitter: @OpenGamma

Europe
OpenGamma
185 Park Street
London SE1 9BL
United Kingdom

North America
OpenGamma
125 Park Avenue
New York, NY 10017
United States