Anticipating the new life market:

Dependence-free bounds for longevity-linked derivatives

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Some References

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- Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002).
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- Laurence and Wang (2008).
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- Laurence and Wang (2009).
  Insurance: Mathematics and Economics 44, 35–47.

Summary

- **Focus**: Longevity (trend) bonds.
- **Question**: How do multi-population models behave in the analysis of the payoff?
- **Answer**: We find some inconsistencies between the different models, especially in the *tail of the distribution*.
- **Solution**: Derive upper and lower bounds based on country-specific derivatives.

Coping with the systematic longevity risk

- The systematic risk is born by the **insurer**.
  - Natural Hedging.

- The systematic risk is born by the **individuals**.
  - Tontine schemes or survival funds.
  - Group-Self-Annuitization.
  - Updating mechanisms.

- The systematic risk is born by a **third party**.
  - Buy-Outs and Buy-Ins.
  - Longevity Swaps.
  - Longevity derivatives.

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Longevity derivatives

Blake *et al.* (2013)

- Mortality Forwards.
  - e.g. Lucida q-forward.
- (CAT) Mortality bonds.
  - e.g. Swiss Re Vita bonds.
- Longevity (trend) bonds.
  - e.g. EIB/BNP, Kortis bond,

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Swiss Re Kortis longevity bond

Annualized mortality improvements over n years:

- Age-specific index for EW population:

$$I_{EW}^{x}(t) = 1 - \left(\frac{m^{EW}(x,t)}{m^{EW}(x,t-n)} \right)^{\frac{1}{n}},$$

- Age-specific index for US population:

$$I_{US}^{y}(t) = 1 - \left(\frac{m^{US}(y,t)}{m^{US}(y,t-n)} \right)^{\frac{1}{n}}.$$

Swiss Re Kortis longevity bond

Annualized mortality improvement indices:

- For EW males aged 75-85:

$$I_{EW}(t) = \frac{1}{x_N - x_1 + 1} \sum_{x=x_1}^{x_N} I_{EW}^{x}(t),$$

- For US males aged 55-65:

$$I_{US}(t) = \frac{1}{y_N - y_1 + 1} \sum_{y=y_1}^{y_N} I_{US}^{y}(t).$$

Swiss Re Kortis longevity bond

Longevity Divergence Index

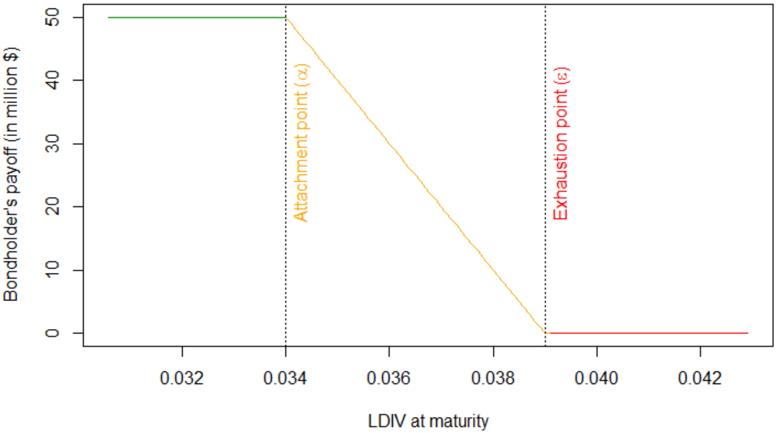
- **Longevity Divergence Index Value** at time t:

$$I(t) = I_{EW}(t) - I_{US}(t).$$

$\longrightarrow$ Hedging a portfolio of annuities from the EW cohort and life assurances from the US cohort.

Swiss Re Kortis longevity bond

Payoff



Source: Adapted from Blake *et al.* (2013).

Swiss Re Kortis longevity bond

Payoff

- The payoff of the Swiss Re Kortis bond:

$$\text{Payoff} = \begin{cases} 0, & \text{if } I(T) \geq \varepsilon. \\ B\left(1 - \frac{I(t)-\alpha}{\varepsilon-\alpha}\right), & \text{if } \varepsilon \geq I(T) \geq \alpha. \\ B, & \text{if } \alpha \geq I(T). \end{cases}$$

where α is the *attachment* point and ε is the *exhaustion* point.

Longevity trend bonds

Analyzing the payoff of longevity trend bonds requires a **multi-population model**.

Multi-population modeling

- Model 1 – Li and Lee (2005):

$$\log\left(m^i\left(x,t\right)\right) = \alpha^i(x) + \beta^i(x)\kappa^i(t) + \beta(x)\kappa(t).$$

- Model 2 – Common-Age-Effect, Kleinow (2015):

$$\log\left(m^i\left(x,t\right)\right) = \alpha^i(x) + \beta^1(x)\kappa^{1,i}(t) + \beta^2(x)\kappa^{2,i}(t).$$

- Model 3 – copula-Lee-Carter:

$$\log\left(m^i\left(x,t\right)\right) = \alpha^i(x) + \beta^i(x)\kappa^i(t),$$

Analysis of the Kortis bond payoff

	Li and Lee	CAE	Copula-Lee-Carter
BIC	150236	156977	150866
$\mathbb{P}[LDIV \geq 3.4\%]$	0.171%	0.003%	0.113%
$\mathbb{P}[LDIV \geq 3.5\%]$	0.129%	0.002%	0.085%
$\mathbb{P}[LDIV \geq 3.6\%]$	0.093%	0.001%	0.077%
$\mathbb{P}[LDIV \geq 3.7\%]$	0.071%	0.001%	0.053%
$\mathbb{P}[LDIV \geq 3.8\%]$	0.053%	0.000%	0.037%
$\mathbb{P}[LDIV \geq 3.9\%]$	0.038%	0.000%	0.031%
99.5 quantile	0.081	0.001	0.063
Conditional EL (Prob.)	47.368%	33.333%	55.752%
$\mathbb{E}$ [Payoff]	49.956	49.999	49.978

Table: Distribution of the LDIV and expected value of the payoff for the three models. The first row shows the BIC of the fitted models.

Analysis of the Kortis bond payoff

Figure: Fan charts of the simulated LDIV for the three models.

Analysis of the Kortis bond payoff

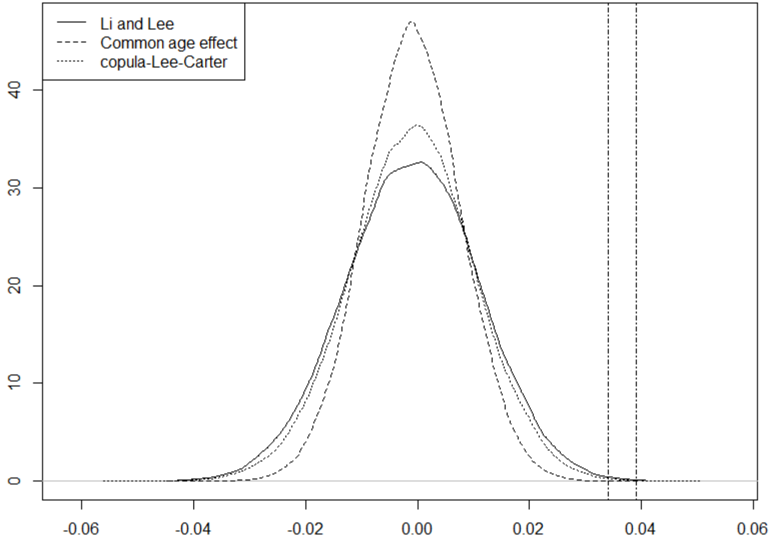


Figure: Densities of the simulated LDIV for the three models.

Analysis of the Kortis bond payoff

Modeling the dependence given the marginal distributions

	Gaussian	Gumbel	Galambos
BIC	-14.939	-14.025	-13.951
$\mathbb{P}\left[LDIV \geq 3.4\%\right]$	0.113%	0.085%	0.103%
$\mathbb{P}\left[LDIV \geq 3.5\%\right]$	0.052%	0.063%	0.081%
$\mathbb{P}\left[LDIV \geq 3.6\%\right]$	0.077%	0.048%	0.060%
$\mathbb{P}\left[LDIV \geq 3.7\%\right]$	0.053%	0.032%	0.042%
$\mathbb{P}\left[LDIV \geq 3.8\%\right]$	0.037%	0.024%	0.029%
$\mathbb{P}\left[LDIV \geq 3.9\%\right]$	0.031%	0.019%	0.019%
99.5 quantile	0.063	0.035	0.050
Conditional EL (Prob.)	55.752%	41.176%	48.543%
$\mathbb{E}\left[\text{Payoff}\right]$	49.978	49.977	49.973

Table: Distribution of the LDIV and expected value of the payoff for the copula-Lee-Carter model with 3 different copulas.

- Step 1: Upper and lower bounds for spread options.
  - Theory of comonotonicity.

- Step 2: Bounds in term of country-specific derivatives.
  - Application of Step 1.

- **Remark**: The payoff is **not convex** !

Longevity trend bonds and spread options

- The payoff of the Swiss Re Kortis bond:

$$\text{Payoff} = \begin{cases} 0, & \text{if } I(T) \geq \varepsilon. \\ B\left(1 - \frac{I(t)-\alpha}{\varepsilon-\alpha}\right), & \text{if } \varepsilon \geq I(T) \geq \alpha. \\ B, & \text{if } \alpha \geq I(T). \end{cases}$$

where α is the *attachment* point and ε is the *exhaustion* point.

$$\boxed{\mathcal{K}(\alpha, \varepsilon) = \frac{B}{\varepsilon-\alpha}\left(\varepsilon - \alpha - \left((I(T) - \alpha)_+ - (I(T) - \varepsilon)_+\right)\right).}$$

Spread options

Upper and lower bounds

The price C_X of a call option on $X = X_1 - X_2$ admits the following bounds:

$$C_{X^c} \leq C_X \leq C_{X^l},$$

where:

- $X^l = F_{X_1}^{-1}(U) - F_{X_2}^{-1}(1-U)$, i.e. the **Fréchet <u>lower</u> bound.**
- $X^c = F_{X_1}^{-1}(U) - F_{X_2}^{-1}(U)$, i.e. the **Fréchet <u>upper</u> bound.**

Spread options
Upper bound

- The **counter-monotonic upper bound**:

$$C_{X^l}[K] = C_{X_1}\left[F_{X_1}^{-1}\left(F_{X^l}(K)\right)\right] + P_{X_2}\left[F_{X_2}^{-1}\left(1 - F_{X^l}(K)\right)\right],$$

  with

$$F_{X_1}^{-1}\left(F_{X^l}(K)\right) - F_{X_2}^{-1}\left(1 - F_{X^l}(K)\right) = K.$$

- Proof: See Dhaene *et al.* (2000).

Spread options

Lower bound

- Consider the function $g(p) = F_{X_1}^{-1}(p) - F_{X_2}^{-1}(p)$ and let p^K and $p_1^K, p_2^K, ..., p_{n-1}^K$ be n solutions of $g(p) = K$.
- The **comonotonic lower bound**:

$$C_{X^c}\left[K\right] = \max\left\{\mathcal{S}_1\left(K\right), \mathcal{S}_2\left(K\right)\right\},$$

where

$$\begin{cases} \mathcal{S}_1\left(K\right) = C_{X_1}\left[F_{X_1}^{-1}\left(p^K\right)\right] - C_{X_2}\left[F_{X_2}^{-1}\left(p^K\right)\right] - \mathcal{B}_n \\ \mathcal{S}_2\left(K\right) = P_{X_2}\left[F_{X_2}^{-1}\left(p^K\right)\right] - P_{X_1}\left[F_{X_1}^{-1}\left(p^K\right)\right] + \mathcal{B}_n, \end{cases}$$

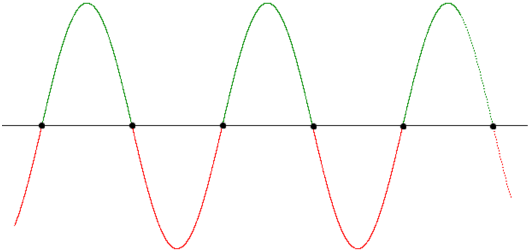
and

$$\mathcal{B}_n = \sum_{i=1}^{n-1}(-1)^{i+1}\left(C_{X_1}\left[t, F_{X_1}^{-1}\left(p_i^K\right)\right] - C_{X_2}\left[t, F_{X_2}^{-1}\left(p_i^K\right)\right]\right).$$

Spread option

Lower bound - Heuristic proof

$$\int_0^{p_0} (g(u) - K)_+ \, du + \sum_{i=0}^{n-2} \int_{p_i}^{p_{i+1}} (g(u) - K)_+ \, du + \int_{p_{n-1}}^{1} (g(u) - K)_+ \, du.$$



$$\mathcal{S}_1 (K) = \color{red}{0} + \int_{p_0}^{p_1} (g(u) - K) \, du + \color{red}{0} + \int_{p_2}^{p_3} (g(u) - K) \, du + \ldots$$

Longevity trend bounds

- An upper bound is given by:

$$\begin{aligned}\mathcal{K}^{+}(\alpha,\varepsilon) &= \frac{B}{\varepsilon-\alpha}\Bigg((\varepsilon-\alpha)\mathrm{e}^{-r(T-t)}-\Bigg(\max\left\{\mathcal{S}_1(\alpha),\mathcal{S}_2(\alpha)\right\}\\ &- C_{I_{EW}}\left[F_{I_{EW}}^{-1}\left(F_{I^l}(\varepsilon)\right)\right]-P_{I_{US}}\left[F_{I_{US}}^{-1}\left(1-F_{I^l}(\varepsilon)\right)\right]\Bigg)\Bigg).\end{aligned}$$

- A lower bound is given by:

$$\begin{aligned}\mathcal{K}^{-}(\alpha,\varepsilon) &= \frac{B}{\varepsilon-\alpha}\Bigg((\varepsilon-\alpha)\mathrm{e}^{-r(T-t)}+\Bigg(\max\left\{\mathcal{S}_1(\varepsilon),\mathcal{S}_2(\varepsilon)\right\}\\ &- C_{I_{EW}}\left[F_{I_{EW}}^{-1}\left(F_{I^l}(\alpha)\right)\right]-P_{I_{US}}\left[F_{I_{US}}^{-1}\left(1-F_{I^l}(\alpha)\right)\right]\Bigg)\Bigg).\end{aligned}$$

Longevity trend bounds

- The bounds $\mathcal{K}^{+}(\alpha, \varepsilon)$ and $\mathcal{K}^{-}(\alpha, \varepsilon)$ **cannot be reached**.

- **Question**: Can we derive sharp bounds for longevity trend bonds from their comonotonic and counter-monotonic transforms?

Longevity trend bounds

Expected payoff as a function of the Kendall tau

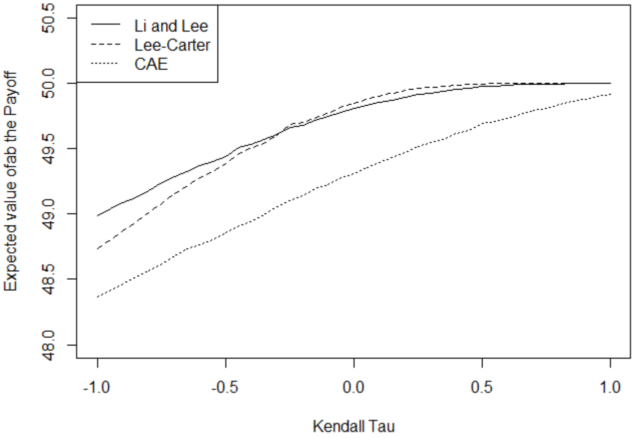


Figure: Expected value of the payoff as a function of the Kendall tau.

Longevity trend upper bound

- The comonotonic expected value is a **sharp upper bound**:

$$\mathcal{K}^c(\alpha,\varepsilon) = \mathbb{E}\left[B\left(1-\max\left\{\min\left(\frac{I^c(T)-\alpha}{\varepsilon-\alpha},1\right),0\right\}\right)\right].$$

- Expression in terms of country-specific derivatives:

$$\begin{aligned}\mathcal{K}^c(\alpha,\varepsilon) &= \frac{B}{\varepsilon-\alpha}\left(\varepsilon-\alpha-\left(\max\left\{C^{(1)}_{I^c}\left[\alpha\right],C^{(2)}_{I^c}\left[\alpha\right]\right\}\right.\right.\\ &- \left.\left.\max\left\{C^{(1)}_{I^c}\left[\varepsilon\right],C^{(2)}_{I^c}\left[\varepsilon\right]\right\}\right)\right).\end{aligned}$$

Longevity trend lower bound

Sub-replicating strategy for the intrinsic value

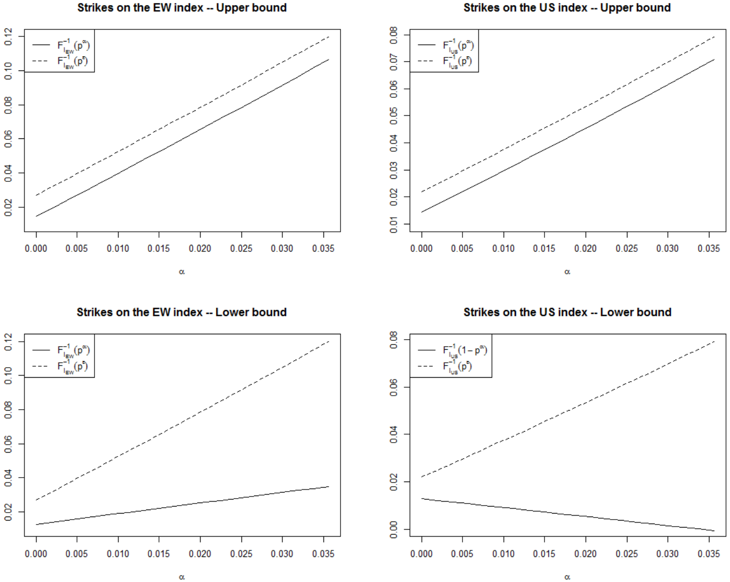
- The counter-monotonic expected value is a **sharp lower bound**:

$$\mathcal{K}^l(\alpha, \varepsilon) = \mathbb{E}\left[B\left(1 - \max\left\{\min\left(\frac{I^l(T) - \alpha}{\varepsilon - \alpha}, 1\right), 0\right\}\right)\right].$$

- Expression in terms of country-specific derivatives:

$$\begin{aligned}\mathcal{K}^l(\alpha, \varepsilon) &= \frac{B}{\varepsilon - \alpha}\Bigg(\varepsilon - \alpha \\ &- \left(C_{I_{EW}}\left[F^{-1}_{I_{EW}}\left(F_{I^l}\left(\alpha\right)\right)\right] - C_{I_{EW}}\left[F^{-1}_{I_{EW}}\left(F_{I^l}\left(\varepsilon\right)\right)\right]\right. \\ &+ \left.P_{I_{US}}\left[F^{-1}_{I_{US}}\left(1 - F_{I^l}\left(\varepsilon\right)\right)\right] - P_{I_{US}}\left[F^{-1}_{I_{US}}\left(1 - F_{I^l}\left(\alpha\right)\right)\right]\right)\Bigg)\end{aligned}$$

Illustration of the strikes



Conclusions

- Focus on longevity (trend) bonds.
- Highlight the inconsistencies between multi-population projections in the analysis of the payoff.
- Propose a *safeguard* against multi-population model risk, based on:
  - the well-developped single-population models, or
  - observed country-specific derivative prices.

Thank You