Optimal Longevity Hedge with Basis Risk

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Joint work with Ken Seng Tan and Chengguo Weng

Outline of the Presentation:

1. Question
   - Background
   - Problem setup
2. Bellman Equation and Solution
3. An example: Canada vs UK

Longevity risk management

Longevity risk is any potential risk attached to the unpredicted increasing life expectancy of pensioners and policy holders. Popular solutions include:

- Buy-out and buy-in: completely transfers plan liabilities and assets to another company.
- Asset-liability management.
  - customized hedigng instruments: reflects the particular characteristics of pension plan's demographics and benefit structure.
  - standardized hedigng instruments: linked to a relevant longevity index: basis risk arises.

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Literature review

Existing literature on longevity hedge in the presence of basis risk:

- static hedge (Li and Hardy, 2011; Li and Luo, 2012; Cairns, 2013)
  - vulnerable to future market change
  - requires long-dated securities, high cost, high counterparty risk
- dynamic hedge for continuous-time models (Dahl, Melchior and Moller, 2008)
  - inapplicable to commonly used stochastic mortality models, such as Lee-Carter model, Cairns-Blake-Dowd model, etc
- dynamic "delta" hedge (Cairns, 2011; Zhou and Li, 2016)
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- dynamic "delta" hedge (Cairns, 2011; Zhou and Li, 2016)
  - no guarantee on the optimality
- Our goal: optimal dynamic longevity hedge based on commonly used stochastic models

Longevity model example: Lee-Carter

Two-population augmented common factor (ACF) model, (Li and Lee, 2005),

$$\ln(m_{x,t}^{(i)}) = a_x^{(i)} + B_x K_t + b_x^{(i)} k_t^{(i)} + \epsilon_{x,t}^{(i)}$$

- $i \in \{H, R\}$, and H and R are two different populations,
- $a_x^{(i)}$, B_x and $b_x^{(i)}$ represent the age effect,
- K_t and $k_t^{(i)}$ represent the time effect, and are further modeled by time series models (e.g., an AR(1) model),
- $\epsilon_{x,t}^{(i)}$ denotes the residual term.

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Longevity model example: Cairns-Blake-Dowd (CBD)

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\begin{aligned}
\text{logit}(q_{x,t}^{(i)}) &:= \ln\left(\frac{q_{x,t}^{(i)}}{1-q_{x,t}^{(i)}}\right) \\
&= \kappa_{1,t}^{c} + \kappa_{2,t}^{c}(x-\bar{x}) + \kappa_{1,t}^{(i)} + \kappa_{2,t}^{(i)}(x-\bar{x}) + \epsilon_{x,t}^{(i)},
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Some notations

- $q_{x,t}^{(i)}$ denotes the probability that an individual aged x at time $t-1$ (alive) from population i dies between time $t-1$ and t.
- $$S_{x,t}^{(i)}(T) := \prod_{s=1}^{T}(1 - q_{x+s-1,t+s}^{(i)})$$

  denotes the probability that an individual from population i aged x at time t (alive) will survive to time $t+T$.
- $$p_{x,u}^{(i)}(T, K_t, k_t^{(i)}) := \mathrm{E}(S_{x,u}^{(i)}(T)|K_t, k_t^{(i)}), \quad u \geq t$$

  is the expected survival probability given information up to time t.
  - Referred as spot survival probability if $u = t$,
  - Referred as forward survival probability if $u > t$.

Pension Liability

We assume that the pension plan sponsor is managing a pension plan.

- It involves a single cohort of n pensioners all age x_0 at time 0 from population H.
- The total notional amount at time 0 is \$1.
- Size of pension plan is large enough such that there is no sample risk.
- Time-0 liability, denoted by FL_0, is expressed by:

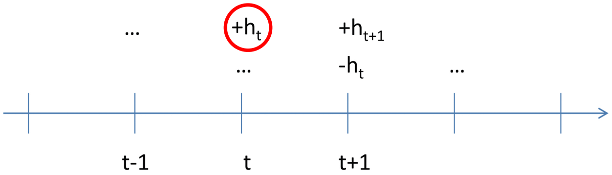
$$FL_0 = \sum_{s=1}^{\infty}(1+r)^{-s} S_{x_0,0}^{(H)}(s) = \sum_{s=1}^{\omega - x_0}(1+r)^{-s} S_{x_0,0}^{(H)}(s) \doteq \sum_{s=1}^{Y}(1+r)^{-s} S_{x_0,0}^{(H)}(s).$$

q-forwards

A q-forward (linked to population R) is basically a single-payment zero-coupon swap,

- Floating leg: $q^{(R)}_{x_f, t_0+T^*}$
  - x_f is a pre-specified reference age;
  - t_0 is the issuing date and $t_0 + T^*$ is the maturity date.
- Fixed leg: q^f is predetermined at t_0.
- As the hedger, we are the fixed rate receiver.

Hedging strategy



A "rolling strategy":

- At time t, $t = 0, 1, 2, ...$, we writes a q-forward contract as the fixed leg receiver with a notional amount of h_t (with no initial cost).
- At time $t + 1$, first we close out the position written at time t (with profit or loss) and then write a new contract with notional amount h_{t+1}.

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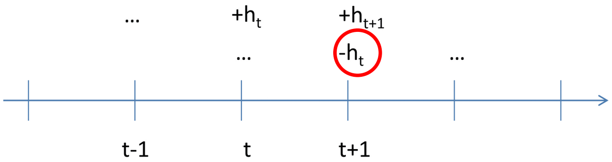

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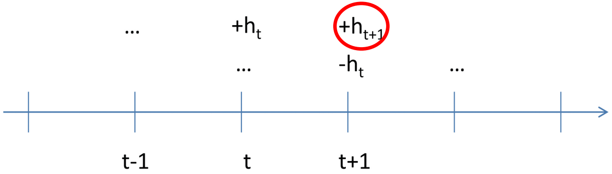

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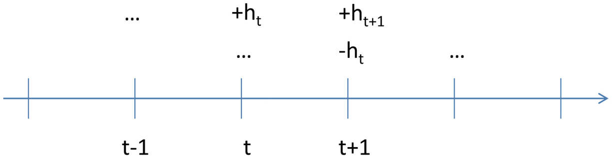
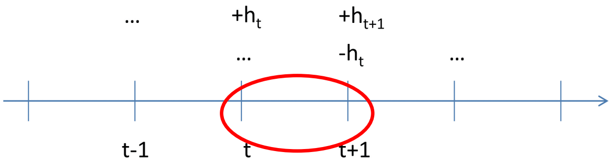


A "rolling strategy":

Denote $Q_{t_2}(t_1)$, $t_2 \geq t_1$ as the time-t_2 value of q-forward contract that was written at time t_1,

- $h_t \cdot Q_t(t) = 0$
- $h_t \cdot Q_{t+1}(t) = h_t \cdot (1+r)^{-(T^*-1)}(p^{(R)}_{x_f,t+T^*-1}(1, \mathcal{F}_{t+1}) - p^{(R)}_{x_f,t+T^*-1}(1, \mathcal{F}_t))$



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Hedging objective

Our goal is to minimize the hedging error:

- The time-t hedging error:

$$F(t) := h_{t-1} \cdot Q_t(t-1) - S_{x_0,0}^{(H)}(t) \quad t = 1, 2, 3, \ldots$$

- Then the (truncated) total hedging error is expressed by:

$$X(Y) = \sum_{t=1}^{Y} (1+r)^{-t} F(t).$$

- Optimization problem:

$$\min_{\{h_t\}_{t=0,1,2,\ldots,Y-1}} \mathrm{Var}_0[X(Y)]$$

$\{h_t\}_{t=0,1,2,\ldots,Y-1}$ is an adapted process.

Bellman equation and optimal strategy

$$
\begin{aligned}
V_t = & \inf_{h_t} \mathrm{E}_t \Bigg\{ V_{t+1} + (1+r)^{-2(t+1)} h_t^2 Q_{t+1}^2(t) + (1+r)^{-2(t+1)} S_{x_0,0}^{(H)}(t+1)^2 \\
& -2(1+r)^{-2(t+1)} h_t Q_{t+1}(t) S_{x_0,0}^{(H)}(t+1) \\
& +2(1+r)^{-(t+1)} h_t Q_{t+1}(t) \mathrm{E}_{t+1} \left[\sum_{s=t+2}^{Y} (1+r)^{-s} F^*(s) \right] \\
& -2(1+r)^{-(t+1)} S_{x_0,0}^{(H)}(t+1) \mathrm{E}_{t+1} \left[\sum_{s=t+2}^{Y} (1+r)^{-s} F^*(s) \right] \Bigg\}.
\end{aligned}
$$

$\Longrightarrow$

$$
h_t^* = \frac{\mathrm{E}_t \left\{ Q_{t+1}(t) \sum_{s=t+1}^{Y} (1+r)^{-[s-(t+1)]} S_{x_0,0}^{(H)}(s) \right\}}{\mathrm{E}_t \left[Q_{t+1}^2(t) \right]}
$$

Optimal solution

$$h_t^* = \frac{\mathrm{E}_t\left\{Q_{t+1}(t)\sum_{s=t+1}^{Y}(1+r)^{-[s-(t+1)]}S_{x_0,0}^{(H)}(s)\right\}}{\mathrm{E}_t\left[Q_{t+1}^2(t)\right]},$$

where

$$Q_{t+1}(t) = (1+r)^{-(T^*-1)}\left(p_{x_f,t+T^*-1}^{(R)}(1,\mathcal{F}_{t+1}) - p_{x_f,t+T^*-1}^{(R)}(1,\mathcal{F}_t)\right)$$

. To calculate h_t^*,

- Direct “nested” Monte Carlo.
- Approximation to forward survival rates (Cairns, 2011; Zhou and Li, 2016).

Data and assumptions

Model parameters are calibrated from the mortality data of Canadian unisex population (as population H) and United Kingdom unisex population (as population R) aged 60 to 89 over the period of 1966 to 2005.

- The pension liability we hedge against is from a single cohort of individuals aged $x_0 = 60$ at time 0 from population H.
- The total notional amount is \$1.
- The scale of the pension plan is large enough.
- The hedging horizon is $Y = 30$ years.
- The q-forwards used as hedging instruments are linked to population R. The reference age for all the contracts is fixed to be $x_f = 75$ and the time to maturity at inception is $T^* = 10$ years.
- The risk free rate is $r = 4\%$.

Performance evaluation and benchmark

- Hedge Effectiveness (HE):

$$\text{HE} = 1 - \frac{\text{Var}[\text{time-0 value of hedged portfolio}]}{\text{Var}[\text{time-0 value of unhedged portfolio}]}$$

  - HE $\in [0, 1]$
  - The higher HE, the better hedging performance

- "Dynamic delta" method (Cairns, 2011; Zhou and Li, 2016):

$$\frac{\partial \sum_{s=1}^{Y} (1+r)^{-s} p_{x_0+t,t}^{(H)}(s, K_t, k_t^{(H)})}{\partial K_t} = h_t^{**} \cdot \frac{\partial Q_t(t-1)}{\partial K_t},$$

where K_t and$k_t^{(H)}$ are model parameters in Lee-Carter model

Baseline results

Based on $N = 2,000$ generated path and a bootstrapping of $N_b = 100,000$,

HE	Method 1	Method 2	Method 3	Method 4	Method 5
mean	0.9149	0.9147	0.9213	0.8938	0.9153
variance	1.5812e-005	1.3876e-005	1.2142e-005	2.3656e-005	1.4084e-005
Min	0.8962	0.8979	0.9035	0.8699	0.8968
Q1	0.9123	0.9122	0.9190	0.8906	0.9128
Q2	0.9150	0.9148	0.9214	0.8939	0.9154
Q3	0.9177	0.9172	0.9237	0.8971	0.9179
Max	0.9299	0.9301	0.9341	0.9127	0.9300
95% C.I.	(0.9069,0.9225)	(0.9072,0.9218)	(0.9143,0.9279)	(0.8839,0.9030)	(0.9077,0.9224)
time (hrs)	0.27	1.56	12.18	0.16	117.51

- Method 1: First approximate $Q_{t+1}(t)$, then simulate $M = 1,000$ paths to estimate hedging strategy h_t^*.
- Method 2: Same as Method 1 but increase the number of paths generated in the second step to $M = 10,000$.
- Method 3: Same as Method 1 but further increase the number of paths generated in the second step to $M = 100,000$.
- Method 4: "Delta" method developed by Cairns (2011), and Zhou and Li (2016).
- Method 5: A direct simulation with $M = M_1 = 10,000$ sample paths.

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Robustness to q-forwards' time to maturity

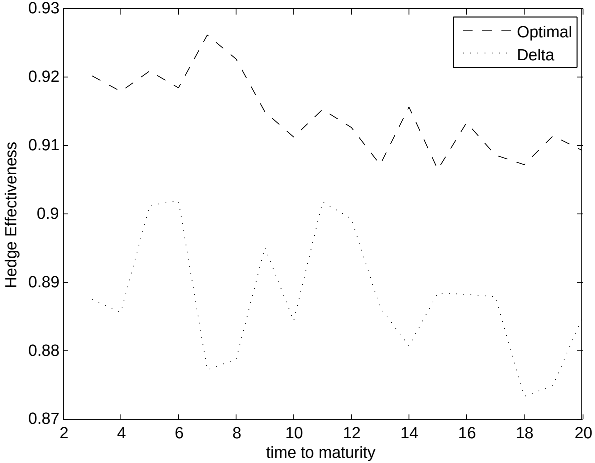

Figure: Robustness to q-forwards' time to maturity

Robustness to q-forwards' reference age

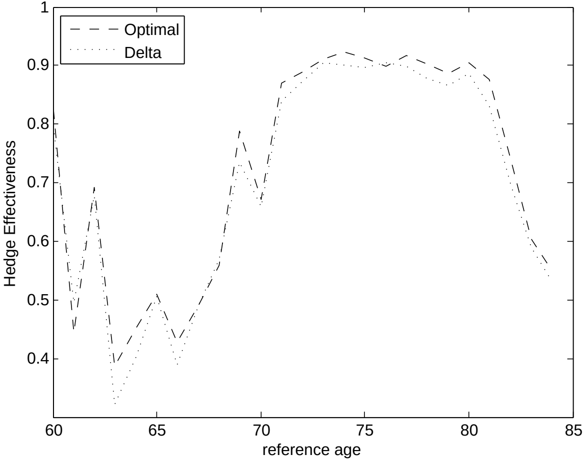

Figure: Robustness to q-forwards' reference age

Robustness to model risk

We fix the q-forward contract with reference age $x_f = 75$ and time to maturity $T^* = 10$.

"True"/"assumption" model	mean	variance	min	max	95% C.I.
Scenario 1 (CBD/ACF)	0.6437	1.7231×10^{-4}	0.5828	0.6942	(0.6174,0.6687)
Scenario 2 (ACF/CBD)	0.8712	3.2003×10^{-5}	0.8416	0.8952	(0.8597,0.8819)
Scenario 3 (CBD/CBD)	0.6492	1.7890×10^{-4}	0.5839	0.7029	(0.6222,0.6747)
Scenario 4 (ACF/ACF, baseline)	0.9213	1.2142×10^{-5}	0.9035	0.9341	(0.9143,0.9279)

- "True" model refers to the mortality generator in the simulation study.
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Conclusion and future work

Conclusion:

- By resorting to the dynamic programming framework, we obtain an analytical solution for the optimal dynamic hedging problem using q-forward contracts linked to a longevity index.

Future work:

- To extend the hedging problem to multi-dimensional case.
- To obtain explicit hedging strategies using other type of hedging instruments such as longevity swaps.

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Thank you