Cross-Section vs Time Series Measures of Uncertainty. 
Using UK Survey Data 

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Cross-Section vs Time Series Measures of Uncertainty.
Using UK Survey Data

Abstract

This paper considers measures of uncertainty used in economic estimation. Our first contribution is to address the theoretical relationship between cross-section and time series measures, highlighting the reasons why these might diverge. In a subsequent empirical section, we compare measures of uncertainty, all of which are based on underlying data on optimism from an established UK survey database, managed by the main employers’ organization, the CBI. We measure uncertainty at industry level in three ways: by cross-section dispersion of optimism expectations, by a GARCH series based on the optimism data and by an unconditional volatility measure based on the same data.

J.E.L. Classification Numbers: C21, C22, C42, E22.
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1 Introduction

The issue of defining and proxing risk and uncertainty in economics has received attention for a long time in the economic literature. Several measures have been proposed, but the theoretical relationships and the empirical correlations among them are still to be fully explored. This problem is distinct from but complementary to the large body of literature that attempts to measure the effect of uncertainty in economic relationships such as the investment function (Carruth et al., 2000).

A key point of contention in the literature is the relationship between cross-section and time-series measures of uncertainty (Zarnowitz and Lambrinos, 1987). These authors used a unique database, which allowed the calculation of means (point estimates) and standard deviations for each respondent, and compared the dispersion of the point estimates (disagreement) with intra-personal variation (subjective uncertainty). Disagreement and uncertainty were found to be related although the disagreement statistics were found to understate the level of uncertainty and to overstate its variance. In subsequent work, Giordani and Soderlind (2003) analysed the Society of Professional Forecaster (SPF) data for a more recent period, finding that a measure of disagreement based on point forecasts was highly correlated (0.6) with averaged measures of individual perceived confidence bands. The authors conclude that “disagreement is a better proxy of inflation uncertainty than what previous literature has indicated…” Other authors have questioned the correspondence of dispersion to uncertainty on the grounds that clustering creates slow adjustment of forecasts and so consensus forecasts are likely to be biased. An implication then is that dispersion will not properly capture variance of forecast error (Gallo et al., 2002; Harvey et al., 2003). A different reason for suspecting the lack of correlation between disagreement and uncertainty is where agents are completely certain of their different beliefs. Both these cases raise questions about how subjective uncertainty evolves, a subject on which there has been little detailed work. Batchelor and Zarkesh (2000) argue that the subjective forecasts are not “variance rational”, in that respondents typically fail to adjust their subjective standard deviation estimates in the light of past performance and that they give too much weight to the size of recent errors in mean forecasts compared with their long-term accuracy. For related studies see also Batchelor and Dua (1993), Rich, Raymond and Butler (1992), Rich and Butler (1998), Bomberger (1996, 1999).

In this paper, we derive a theoretical model to capture the difference between cross-section and times series measures of uncertainty. Our enquiry
relates to the relation or lack of it between disagreement as reflected by the cross-sectional dispersion of point estimates and measures of time series volatility. The remainder of the paper is organised as follows. In Section 2 we propose the theoretical framework to investigate the relationship between cross-section and time series measures of uncertainty. In Section 3 we test the theory using survey industries data. Section 4 concludes.

2 Cross-section and time series measures of uncertainty

In this section, we propose a theoretical framework able to embody a coherent structure of the relationship between cross-section and time series measures of uncertainty.

The existence of such relationship is grounded on the consideration that private information creates dispersion across agents; hence, the conditional variance can be viewed as the sum of dispersion and volatility about the time-averaged mean (Engle, 1983).

Specifically, let us consider the process \( y_t \) whose data generating process (DGP) is:

\[
y_t = \beta y_{t-1} + \eta_t + \sum_{i=1}^{n} \alpha_i \varepsilon_{it},
\]

where \( \eta_t \) and \( \varepsilon_{it} \) are iid random processes with zero mean and variances \( \sigma^2 \) and \( 1 \) respectively, and the \( \alpha_i \)'s satisfy the square summability condition \( \sum_{i=1}^{n} \alpha_i^2 < \infty \) for \( n \to \infty \). The variance of \( y_t \) conditional on past information \( I_{t-1} \) is \( \text{Var} [ y_t | I_{t-1} ] = \sigma^2 + \sum_{i=1}^{n} \alpha_i^2 \), and each individual forecaster with inside information has conditional expectation \( y_i^t \) and forecast error \( \varepsilon_i^t \) given respectively by

\[
\begin{align*}
E [ y_t | y_{t-1}, \varepsilon_{it} ] & = y_i^t = \beta y_{t-1} + \alpha_i \varepsilon_{it} \\
\varepsilon_i^t & = y_t - y_i^t.
\end{align*}
\]

The MSE for any forecaster \( i \) is given by

\[
E ( \varepsilon_i^t )^2 = \sigma^2 + \sum_{j \neq i} \alpha_j^2
\]

and re-arranging this equation one gets a characterisation of the \( \alpha_i \)'s as the amount of private information

\[
\alpha_i^2 = \text{Var} ( y_t | I_{t-1} ) - E ( \varepsilon_i^t )^2.
\]
Large values of the $\alpha_i$s will reduce the individual forecaster’s MSE with respect to $\text{Var}(y_t|I_{t-1})$.

Following Engle (1983), the average MSE can be expressed as:

$$\frac{1}{n} \sum_{i=1}^{n} E(\varepsilon_i^2) = \sigma^2 + \frac{n-1}{n} \sum_{i=1}^{n} \alpha_i^2.$$  

This will be close to $\text{Var}(y_t|I_{t-1})$ either when $n \to \infty$, or when the amount of private information $\alpha_i$ is small for all individuals. The average MSE can also be decomposed as:

$$\frac{1}{n} \sum_{i=1}^{n} E(\varepsilon_i^2) = \frac{1}{n} \sum_{i=1}^{n} E \left[ \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y}_t)^2 \right] + E \left[ (y_t - \bar{y}_t)^2 \right], \quad (4)$$

where

$$\bar{y}_t = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

It is worth noticing that the second term on the right hand side of equation (4) is equal to:

$$E \left[ (y_t - \bar{y}_t)^2 \right] = \left( \eta_t + \frac{n-1}{n} \sum_{i=1}^{n} \alpha_i \varepsilon_i \right)^2 = \sigma^2 + \left( \frac{n-1}{n} \right)^2 \sum_{i=1}^{n} \alpha_i^2,$$

and hence

$$\frac{1}{n} \sum_{i} E(\varepsilon_i^2) = \frac{1}{n} \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y_i - \bar{y}_t)^2 \right] + \left( \frac{n-1}{n} \right)^2 \sum_{i} \alpha_i^2 + \sigma^2 =$$

$$= \frac{1}{n} \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y_i - \bar{y}_t)^2 \right] + \frac{n-1}{n} \left[ \frac{1}{n} \sum_{i} E(\varepsilon_i^2) \right] + \frac{1}{n} \sigma^2.$$

Re-arranging, one gets

$$\frac{1}{n^2} \sum_{i} E(\varepsilon_i^2) = \frac{1}{n} \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y_i - \bar{y}_t)^2 \right] + \frac{1}{n} \sigma^2.$$

And from equation (3) it follows that:

$$\text{Var}[y_t|I_{t-1}] = \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y_i - \bar{y}_t)^2 \right] + \sigma^2 - \frac{1}{n} \sum_{i} \alpha_i^2. \quad (5)$$
The first term in the right-hand side of equation (5) can be viewed as the cross-sectional measure of dispersion (CS henceforth); the left-hand side term is a time series measure of volatility (TS henceforth). Equation (5) can be written, employing this new notation, as

\[ TS = CS + \sigma^2 - \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 < CS + \sigma^2 \]  

where the upper bound

\[ \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 < CS + \sigma^2 \]

is needed in order for TS to be positive. Equation (6) shows that the discrepancy between CS and TS depends on three quantities: the variance of the unobservable component \( \eta_t \) (\( \sigma^2 \)), the number of individuals \( n \) and the amount of private information \( \alpha_i \). It is immediate to notice that as \( \sigma^2 \) grows large, the difference between TS and CS increases. Also, when the number of individuals \( n \rightarrow \infty \), ceteris paribus, TS will be larger than CS:

\[
\lim_{n \to \infty} TS = \lim_{n \to \infty} \left[ CS + \sigma^2 - \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 \right] = CS + \sigma^2 > CS,
\]

even though the non monotonicity of \( \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 \) with respect to \( n \) leads to ambiguous signs for finite \( n \). Last, for fixed values of \( n \), the difference between TS and CS is reduced as private information increases.

Equation (5) is derived under the assumption that estimation takes account of all information available to the agents. It may be of interest to model the case where some information is available to all agents, but unavailable to or ignored by the econometrician. Thus, let us consider a different specification for the DGP for \( y_t \)

\[ y_t = \beta y_{t-1} + x_t + \sum_{i=1}^{n} \alpha_i \varepsilon_{it} + \eta_t \]  

where with respect to (1) we add \( x_t \) as a random variable with variance \( \sigma_x^2 \), representing common information available to all agents (but not to the econometrician). The conditional variance of \( y_t \) is now given by \( \text{Var} [y_t | I_{t-1}] = \sigma_x^2 + \sum_{i=1}^{n} \alpha_i^2 + \sigma^2 \).

Each individual \( i \) will forecast the level of \( y_t \) employing both common information \( y_{t-1} \) and \( x_t \) and private information \( \varepsilon_{it} \), obtaining

\[ E [y_t | y_{t-1}, x_t, \varepsilon_{it}] \equiv y'_t = \beta y_{t-1} + x_t + \alpha_i \varepsilon_{it} \]
with forecast error $\varepsilon^i_t$ given as in (2). Similar calculations as before show that

$$E(\varepsilon^i_t)^2 = \sum_{j \neq i} \alpha^2_j + \sigma^2,$$

$$\frac{1}{n} \sum_{i=1}^{n} E(\varepsilon^i_t)^2 = \sigma^2 + \frac{n-1}{n} \sum_{i=1}^{n} \alpha^2_i.$$ 

Therefore

$$\text{Var} [y_t | I_{t-1}] - \frac{1}{n} \sum_{i=1}^{n} E(\varepsilon^i_t)^2 = \sigma^2_x - \frac{1}{n} \sum_{i=1}^{n} \alpha^2_i < \sigma^2_x. \quad (8)$$

Notice that when the number of individuals $n$ approaches infinity, the difference between the conditional variance and the average mean squared error will be positive:

$$\lim_{n \to \infty} \left\{ \text{Var} [y_t | I_{t-1}] - \frac{1}{n} \sum_{i=1}^{n} E(\varepsilon^i_t)^2 \right\} = \sigma^2_x.$$

Notice that for finite values of $n$, the difference between $\text{Var} [y_t | I_{t-1}]$ and the average mean squared error does not necessarily grow when $n$ increases, and it can also be negative.

Similar derivations as in equation (4) lead to

$$\frac{1}{n} \sum_{i} E(\varepsilon^i_t)^2 = \frac{1}{n} \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y^i_t - \bar{y}_t)^2 \right] + \frac{n-1}{n} \left[ \frac{1}{n} \sum_{i} E(\varepsilon^i_t)^2 \right] + \frac{1}{n} \sigma^2,$$

and finally

$$\frac{1}{n^2} \sum_{i} E(\varepsilon^i_t)^2 = \frac{1}{n} \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y^i_t - \bar{y}_t)^2 \right] + \frac{1}{n} \sigma^2.$$ 

Then from equation (8) it holds that

$$\text{Var} [y_t | I_{t-1}] = \sum_{i} E \left[ \frac{1}{n} \sum_{i} (y^i_t - \bar{y}_t)^2 \right] + \sigma^2_x - \frac{1}{n} \sum_{i} \alpha^2_i + \sigma^2. \quad (9)$$

Provided that $n^{-1} \sum_{i=1}^{n} \alpha^2_i < CS + \sigma^2_x + \sigma^2$, equation (9) can be rewritten with the same notation as in (5):

$$TS = CS + \sigma^2_x - \frac{1}{n} \sum_{i=1}^{n} \alpha^2_i + \sigma^2 < CS + \sigma^2_x + \sigma^2. \quad (10)$$
Equation (10) leads to the same conclusions as in equation (6) above. It is worth noticing that the impact of the variance of the unobserved component \( x_t, \sigma^2_{x_t} \), increases the discrepancy between \( TS \) and \( CS \).\(^1\)

### 3 Empirical results

In this section, we evaluate empirically the pair-wise relationship between cross-section and time series measures of uncertainty. Though the theoretical discrepancy exists, we have no a-priori view as to its magnitude as captured in equations (6) and (10). While alternative forms of relationships may be tested, we confine ourselves to a set of battery of causality tests.

In what follows, we first introduce the data set used, we then describe the alternative measures of uncertainty we can derive and finally we report the results from Granger causality analysis to characterise the relationship between those measures.

#### 3.1 Data set

We draw on the Industrial Trends Survey carried out by the main UK employers’ organisation, the Confederation of British Industry (CBI), consisting of approximately 1000 replies on average each quarter, and disaggregated into nearly fifty industries. The survey has been published on a regular basis since 1958, has a response rate of over 50%, and has been widely used by economists. Our panel data set covers the period from the first quarter of 1978 to the first one in 1999 for 47 industries. The responses in the survey, which in principle are seasonally adjusted, are weighted by net output with the weights being regularly updated. The survey sample is chosen to be representative and is not confined to CBI members. The data are publicly available and their properties have been extensively discussed (Bosworth and Heathfield, 1987; Dicks and Burrell, 1994).

All data in the CBI Survey are qualitative data based on answers, such as ‘up’, ‘down’, or ‘same’ regarding the trend in the economic variables. We consider a transformation of the qualitative data into quantitative data by using the ‘balance’ statistic that is the percentage of respondents replying ‘up’ minus those replying ‘down’. This may be shown to approximate a growth rate under restrictive assumptions.\(^2\)

\(^1\)Appendix I provides further derivations on equations (6) and (10).

The variable we use is Question 1 that records changes in the state of business confidence or optimism ($Opt_{it}$), as follows:

**Question 1**

Are you more, or less, optimistic than you were four months ago about the general business situation in your industry? MORE/LESS/SAME.

### 3.2 Alternative measures of uncertainty

We base our analysis on three possible subjective (i.e. relying upon agents’ answers) measures of uncertainty, one being a cross-sectional measure of dispersion and the remaining two being measures of time series volatility:

- the **cross-sectional entropy** across forecasting agents, given by:

  \[ s_{it} = -\sum_{j=1}^{3} S_{ijt} \log S_{ijt} \]  

  where $S_{ijt}$ is the share of the $j$-th reply category for unit (industry in our case) $i$ at time $t$ on the degree of being ‘more’ or ‘less’ optimistic about the general business situation compared with the situation four months earlier. When the answers are equally divided, $s_{it}$ reaches its maximum of $\log 3$;

- the **conditional volatility measures** for the balance of ups over downs in the Survey question on optimism, that will henceforth be referred to as $Opt_{it}$. To define it, we first estimate the following GARCH (1,1) equation using data at industry level:

  \[ Opt_{it} = a_0 + a_1 Opt_{it-1} + a_2^t Seas_t + e_{it} \]  

  where $e_{it} \sim N(0, h_{it}^2)$ and $h_{it}^2 = \alpha_0 + \alpha_1 e_{it-1}^2 + \beta_1 h_{it-1}^2$; $Seas_t$ is a vector of seasonal dummies, which is included only if they are jointly significant. Conditional variance $h_{it}^2$ is defined as the **conditional volatility measure**. We use the square root of this volatility measure, $h_{it}$.

---

3Dispersion across agents has been used to measure uncertainty in several studies e.g. Driver and Moreton (1991), Guiso and Parigi (1999). The entropy measure has been used in Fuchs, Krueger and Poterba (1998).

4Time-series volatility measures based on GARCH estimation are used in Price (1995, 1996) and Byrne and Davis (2002). Criticisms of this framework have been developed by Loungani (2001, 2002); Giordani and Soderlind (2003) interpret $h_{it}$ as an indicator of the occurrence of structural breaks.
• a time series volatility measure that doesn’t depend on the presence of GARCH effects. This is considered in order not to be confined to the case where significance can be established for an ARCH process. The indicator we consider will be referred to as $u_{it}$, and it is constructed as the standard deviation of the previous four residuals from a fourth order autoregressive model of the optimism variable and the vector of seasonal dummies$^5$:

$$Opt_{it} = b_0 + \sum_{j=1}^{4} b_j Opt_{it-j} + \gamma' Seas_t + u_{it}$$ (13)

The main objective of our analysis will be to investigate the relationship among these measures.

## 3.3 The analysis of the pair-wise relationships between uncertainty measures: Granger causality

In this section we report the results of the empirical investigations of the relationship between the three uncertainty measures described above$^6$.

We tested for Granger causality between the alternative measure, where the appropriate lag structure is selected using the AIC. The results found can be summarised as follows:

- causality between $h_{it}$ and $s_{it}$. Prior to evaluating this relationship, first we found that only 16 out of 47 industries presented significant conditional volatility ($h_{it}$) effects. Secondly, we checked the stationarity of the 16 series using standard ADF test and we found that five of the $h_{it}$ series were non-stationary (we also eliminated from our sample two other industries (23 and 29) which had a truncated sample). Since the non stationarity marks a contrast with the dispersion series, which are all stationary, Granger causality analysis was therefore carried out with respect to the remaining 9 industries (Table 1). Out of these,

$^5$The literature has considered several measures of uncertainty based on this framework. Usually such measures of uncertainty include volatility indices estimated as a moving standard deviation or as the variance of residuals from an ARMA model (see Campa, 1993; Campa and Goldberg, 1995; Goldberg, 1993; Darby et al. 1999; Ghosal and Loungani, 2000; Lensink et al., 2001). The relationship between such measures and the GARCH measure of the previous section is that the GARCH process implies a (declining weight) long memory of previous error variances whereas the “unconditional” variance measure filters out all memory before some interval and uses equal weights.

$^6$Graphs for series $h_{it}$ and $s_{it}$ are in Figures 1 and 2, and are relative to the industries for which an ARCH structure was identified.
we have two cases of significant positive Granger causation from $h_{it}$ to $s_{it}$ and one from $s_{it}$ to $h_{it}$ at the 10% level. This supplies only very weak evidence of association between the $h_{it}$ and $s_{it}$ series. A possible conclusion is that these series may represent different aspects of uncertainty or that the GARCH is reflecting structural shifts;

[Insert somewhere here Table 1]

- *causality between $h_{it}$ and $u_{it}$* (Table 2). For the majority of industries (8 out of 9) there is strong causal (positive) relationship running from the standard deviation of the residuals from an AR(4) model (unconditional volatility) and the GARCH measure with no causation in the opposite direction with bicausality noted in three cases. The direction of causality is not surprising since any shock in the preceding four periods will affect the GARCH. It is worth noticing that whilst the GARCH measure has an infinite memory, with declining weights on the past error variances, the “unconditional” measure truncates all shocks outside the four-quarter interval over which the variance is measured, assigning equal weight to the latter ones. Nonetheless, the strength of the relationship suggests that lagged value of unconditional volatility may often act as a reasonable proxy for the GARCH measure;

[Insert somewhere here Table 2]

- *causality between $u_{it}$ and $s_{it}$*. This was analysed for the full set of industries (=47), unlike the comparison between the GARCH measure $h_{it}$ and $s_{it}$. The Granger causality results, performed in the same way as in the last subsections but with the full set of 47 industries, showed five industries with 10% positive significant causation running from $u_{it}$ to $s_{it}$ and seven industries with positive causation running from $s_{it}$ to $u_{it}$ and only one industry common to both sets. This outcome again provides very weak evidence of a linear association between the series. The findings underscore the low significance found earlier for the relationship between dispersion and the GARCH measure for the smaller sample of available industries.

Together with causality analysis, to investigate a possible linear relationship between the $h_{it}$ and $s_{it}$ series, we also employed the same methodology as in Bomberger (1996) who tests whether the dispersion series $s_{it}$ contains information that can help predict $h_{it}$. We estimated for the 9 available series
a modified GARCH relationship as in equation (12) but including the current value of the dispersion variable $s_{it}$ as an extra regressor in the conditional variance equation:

$$Opt_{it} = a_0 + a_1 Opt_{it-1} + a_2 Sea_{it} + e_{it}$$

with now $e_{it} \sim N(0, h^2_{it})$ and $h^2_{it} = \alpha_0 + \alpha_1 e^2_{it-1} + \beta_1 h^2_{it-1} + \delta_1 s_{it}$. The coefficient on $s_{it}$ should be zero in the event of there being no association between the dispersion of optimism and the conditional volatility of the optimism series. For 6 of the 9 industries (53, 54, 59, 66, 67, 68), the GARCH coefficients remain stable and the $s_{it}$ term is insignificant when added to the conditional variance equation. For the remaining industries, the GARCH coefficients lose significance when $s_{it}$ is included, although the $\delta_1$ coefficient is significant at the 5% level only in one case (54). In sum, there is only weak evidence in favour of a relationship between dispersion and GARCH measures.

4 Concluding remarks

The main aim of the paper was to provide a coherent theoretical structure of the relationship between cross-section and time series measures of uncertainty. We show that the discrepancy between the two measures depends on the unconditional dispersion, the dispersion of the component of information available to all agents but unavailable to or ignored by econometricians and a measure of the extent of private information, available only to individual agents.

We examine for nine UK industries the proposition that there is a relationship between conditional volatility (GARCH) measures of optimism uncertainty and cross-section dispersion of recorded optimism. There is only weak evidence that the inclusion of cross-section measures in the conditional variance equations eliminates the GARCH effects. There is similarly weak support for significant Granger causation either way between the series. We have also found that “unconditional” volatility (in the sense of the variance of errors about trend) and the GARCH measure are causally related with the former more strongly affecting the latter. Finally, we examined the relationship between the unconditional volatility and dispersion for the full set of 47 industries. We again found only weak evidence for causal or contemporaneous relationships.

The general pattern of the results suggests that time series measures are not related in any simple way to the dispersion of expectations. Our interpretation of these results is that care should be used when representing
time-series volatility measures as indices of uncertainty, given that the dis-
persion across agents has been found in previous literature to be good proxy
for subjective uncertainty.
References


Appendix I

This Appendix is aimed at providing some further insight on equation (6), i.e. on the relationship between time series volatility (TS) and forecast dispersion across agents (CS). Consider the following assumption regarding the different weights $\alpha_i$ representing the use that individuals make of private information:

**Assumption 1:** Each $\alpha_i$ is given by

$$\alpha_i = \gamma(n) \beta_i$$

where $\gamma(n) = O\left(n^{-\frac{1}{2}+\delta}\right)$, $\delta \geq 0$ is a function of the number of individuals $n$ and $\beta_i$ is an iid random variable across $i$ with support $[a, b]$, $a > 0$ and finite second moment, $E(\beta_i^2) < \infty$.

Assumption 1 is needed in order for the square summability of the $\alpha_i$s to hold within this (stochastic) framework. In fact, the LLN assures that

$$p \lim n \sum_{i=1}^{n} \alpha_i^2 = p \lim n \sum_{i=1}^{n} \beta_i^2 = \begin{cases} 0 & \text{if } \delta = 0 \\ E(\beta_i^2) & \text{if } \delta > 0 \end{cases}$$

and in either case the result is a finite quantity. Moreover, the following theorem holds:

**Proposition 1** Under Assumption 1, the quantity $TS - CS$ will be:

- an almost surely positive quantity if
  $$b < \frac{\sigma}{\gamma(n)};$$

- an almost surely negative quantity if
  $$a > \frac{\sigma}{\gamma(n)}.$$

1 Results can be straightforwardly extended to the case represented by equation (10).
Proof. The proof will be derived with respect to the first statement, the second following immediately. Firstly, let us consider the following expressions:

\[ TS - CS = \sigma^2 - \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2 = \]

\[ = \sigma^2 - \frac{\gamma^2(n)}{n} \sum_{i=1}^{n} \beta_i^2. \]

The inequality \( TS > CS \) holds almost surely if

\[ \frac{\gamma^2(n)}{n} \sum_{i=1}^{n} \beta_i^2 < \sigma^2, \]

which in turn means

\[ \sum_{i=1}^{n} \beta_i^2 < \frac{n}{\gamma^2(n)} \sigma^2. \]

The random variable \( \sum_{i=1}^{n} \beta_i^2 \) will have support given by \([na^2, nb^2]\); hence, \( TS > CS \) almost surely means

\[ P \left[ \sum_{i=1}^{n} \beta_i^2 \geq \frac{n}{\gamma^2(n)} \sigma^2 \right] = 0, \]

which holds when

\[ nb^2 < \frac{n}{\gamma^2(n)} \sigma^2, \]

and therefore

\[ b < \frac{\sigma}{\gamma(n)}. \]

QED.

As a special application of Proposition 1, we could consider the case when \( \gamma(n) = n^{-1/2} \). This would imply that \( TS > CS \) if

\[ b < n^{1/2} \sigma. \]

This bounds every moment of the distribution of the \( \beta_i \)s. Particularly, it limits the mean (i.e. individuals are not able to use their private information too well) and the variance (i.e. the skills of individuals in using private information are not very different among each other). An even more particular application arises when considering the \( \beta_i \)s as uniformly distributed. In such case, the inequality relationship that makes \( TS \) larger than \( CS \) becomes:

\[ Var(\beta_i) < \frac{(n^{1/2} \sigma - a)}{12}. \]
Table 1: Granger Causality Test on entropy and square root of estimated conditional variance derived by GARCH(1,1).

Hypothesis 1: $s_{it}$ not causing $h_{it}$; Hypothesis 2: $h_{it}$ not causing $s_{it}$.

<table>
<thead>
<tr>
<th>Industry</th>
<th>Hypothesis 1</th>
<th>Hypothesis 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>27  Glass and ceramics</td>
<td>0.571291</td>
<td>0.095140</td>
</tr>
<tr>
<td>53  Instrument engineering</td>
<td>0.881202</td>
<td>0.4208</td>
</tr>
<tr>
<td>54  Food</td>
<td>0.277701</td>
<td>0.446139</td>
</tr>
<tr>
<td>55  Drink and Tobacco</td>
<td>0.7103</td>
<td>0.7103</td>
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<tr>
<td>59  Textile consumer goods</td>
<td>0.876852</td>
<td>0.216808</td>
</tr>
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<td>61  Footwear</td>
<td>0.480229</td>
<td>0.079264</td>
</tr>
<tr>
<td>66  Pulp, paper and board</td>
<td>0.106223</td>
<td>0.3368</td>
</tr>
<tr>
<td>67  Paper and board products</td>
<td>0.081495</td>
<td>0.658257</td>
</tr>
<tr>
<td>68  Printing and publishing</td>
<td>0.137111</td>
<td>0.437083</td>
</tr>
</tbody>
</table>

Table 2: Granger Causality Test on time series volatility and square root of estimated conditional variance derived by GARCH(1,1).

Hypothesis 1: $u_{it}$ not causing $h_{it}$; Hypothesis 2: $h_{it}$ not causing $u_{it}$.

<table>
<thead>
<tr>
<th>Industry</th>
<th>Hypothesis 1</th>
<th>Hypothesis 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>27 Glass and ceramics</td>
<td>0.51</td>
<td>0.66</td>
</tr>
<tr>
<td>53 Instrument engineering</td>
<td>0.01</td>
<td>0.69</td>
</tr>
<tr>
<td>54 Food</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>55 Drink and Tobacco</td>
<td>0.00</td>
<td>0.65</td>
</tr>
<tr>
<td>59 Textile consumer goods</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>61 Footwear</td>
<td>0.00</td>
<td>0.45</td>
</tr>
<tr>
<td>66 Pulp, paper and board</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>67 Paper and board products</td>
<td>0.00</td>
<td>0.29</td>
</tr>
<tr>
<td>68 Printing and publishing</td>
<td>0.06</td>
<td>0.25</td>
</tr>
</tbody>
</table>