The Locally-Linear Cairns-Blake-Dowd Model: A Note on Delta-Nuga Hedging of Longevity Risk

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Presenter: Yanxin Liu

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2 Locally-Linear Cairns-Blake-Dowd Model

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   • In-sample Forecast
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   • Basic Set Up
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Importance of Managing Longevity Risk

As a result of various factors including:
1. a low interest rate environment
2. changes in regulatory regimes (e.g., Solvency II which is scheduled to come into effort in 2013).

→ Longevity risk has become a high profile risk in recent years.

- Pension plans and annuities providers are paying more attention to managing longevity risk.

- Two main aspects in managing longevity risk:
  - modelling mortality patterns
  - hedging longevity risk
Issues Related to the Current Stochastic Mortality Models

Sensitivity of estimation and forecast to different sample periods:

- For example, the drift terms of the bivariate random walk encompassed in the original Cairns-Blake-Dowd (CBD) model are highly sensitive to the first and last data points.

- What length of calibration window should be used?
  - a subjective judgement
    - Cairns(2013) regarded it as a source of "Knightian Uncertainty".

- main driven factor: the deterministic trend assumption encompassed in the stochastic model

- possible solution:
  - allow the drift terms to be stochastic
    - The Locally-linear CBD model is introduced.
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States Space Model

Suppose at each time $t$, we have an observation of $N$-dimensional multivariate time series $\vec{y}_t$, which is driven by some unobservable $m \times 1$ hidden states vector $\vec{\alpha}_t$.

**Measurement equation:**

$$y_t = B_t \alpha_t + \vec{\varepsilon}_t.$$

To model the unobservable states process, $\vec{\alpha}_t$ itself is assumed to follow a first-order Markov process.

**Transition equation:**

$$\alpha_t = A_t \alpha_{t-1} + \vec{\eta}_t.$$

$\vec{\varepsilon}_t \sim MVN(0, R)$ and $\vec{\eta}_t \sim MVN(0, Q)$ represent the error terms in measurement equations and transition equations, respectively.
Denote by
\[ y_{x,t} = \logit(q_{x,t}) = \log\left(\frac{q_{x,t}}{1 - q_{x,t}}\right). \]

Suppose the dataset we use include in total \( K \) ages.

For \( i = 1, 2 \), \( \kappa_i(t) \) follows a random walk with drift:
\[ \kappa_i(t) = C_i(t) + \kappa_i(t - 1) + \eta_i(t), \]
where \( C_i(t) \) itself is stochastic.

The underlying states vector becomes \( (\kappa_1(t), \kappa_2(t), C_1(t), C_2(t))' \).
The corresponding measurement equation can be written as:

\[ y_t = B_t \alpha_t + \vec{\epsilon}_t \]

and the transition equation can be shown as:

\[ \alpha_t = A_t \alpha_{t-1} + \vec{\eta}_t, \]

where

\[ y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{K,t} \end{pmatrix}, \quad \alpha_t = \begin{pmatrix} \kappa_1(t) \\ \kappa_2(t) \\ C_1(t) \\ C_2(t) \end{pmatrix}, \]

and both \( B_t \) and \( A_t \) can be expressed as time-invariant matrices:

\[
B = \begin{pmatrix}
1 & x_1 - \bar{x} & 0 & 0 \\
1 & x_2 - \bar{x} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & x_K - \bar{x} & 0 & 0 \\
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]
As a whole, the locally-linear Cairns-Blake-Dowd model (LLCBD) can be written as:

**Measurement equations:**

\[
\begin{align*}
\logit(q_{1},t) &= \kappa_1(t) + (x_1 - \bar{x})\kappa_2(t) + \varepsilon_{1,t} \\
\logit(q_{2},t) &= \kappa_1(t) + (x_2 - \bar{x})\kappa_2(t) + \varepsilon_{2,t} \\
\vdots \\
\logit(q_{K},t) &= \kappa_1(t) + (x_K - \bar{x})\kappa_2(t) + \varepsilon_{K,t}
\end{align*}
\]

**Transition equations:**

\[
\begin{align*}
\kappa_1(t) &= \kappa_1(t - 1) + C_1(t) + \eta_{1,t} \\
\kappa_2(t) &= \kappa_2(t - 1) + C_2(t) + \eta_{2,t} \\
C_1(t) &= C_1(t - 1) + \eta_{3,t} \\
C_2(t) &= C_2(t - 1) + \eta_{4,t}
\end{align*}
\]
Properties of the LLCBD Model Specification

1. Stochastic nature in the drift terms
   ⇒ More flexibility in capturing the longevity trend

2. State Space Form
   ▶ Generalization to other time series structures
     - by adjusting $A_t$ or $B_t$
   ▶ Generalization to multi-population
     - by adjusting the structure in state vector $\alpha_t$
   ▶ Generalization to other mortality models
     - by adjusting the measurement equations
     - e.g. Hári et al. (2008) consider the Lee-Carter Model in SSM

3. Unconstrained $Q$ matrix
   ⇒ Availability to account for the correlation between different states
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Robustness Study:

1. compare the in-sample forecasting performance between the original CBD model and the locally-linear CBD model;
2. investigate the robustness with respect to different look-back windows.

Data for the in-sample forecast:

- Population: English and Welsh (EW) male population
- Sample Period: 1951 to 1996 versus 1951 to 2011
- Age Range: 50 to 89
In-Sample Forecast Study in Terms of $q_{x,t}$

(a) Original CBD Age 65

(b) Original CBD Age 70

(c) Original CBD Age 75

(d) Original CBD Age 80
In-Sample Forecast Study in Terms of $q_{x,t}$

(a) LL CBD Model Age 65

(b) LL CBD Model Age 70

(c) LL CBD Model Age 75

(d) LL CBD Model Age 80
Robustness: Different Look-back Windows

Paradox in using different sample period

- It is reasonable to use the most up-to-date data;
- One paradox in using the original CBD model: longer sample period $\rightarrow$ less capacity in capturing the latest trend;
- The locally-linear CBD model provides a solution!
- Consider the following sample periods:
  - 1951 to 2011
  - 1961 to 2011
  - 1971 to 2011
  - 1981 to 2011
Different Look-Back Windows in Terms of $q_{x,t}$

(a) Original CBD Age 65
(b) Original CBD Age 70
(c) Original CBD Age 75
(d) Original CBD Age 80

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Different Look-Back Windows in Terms of $q_{x,t}$

(a) LL CBD Model Age 65

(b) LL CBD Model Age 70

(c) LL CBD Model Age 75

(d) LL CBD Model Age 80
Comments:

- **Original CBD Model:**
  - The use of different sample periods lead to very different median forecasts;
  - Relatively small forecast errors.

- **Locally-linear CBD Model:**
  - The median forecasts are substantially more robust;
  - Relatively large forecast errors as it incorporates the stochastic nature of drift terms.

What about the unobservable states?
Different Look-Back Windows in Terms of Unobservable States

(a) Random Walk $\kappa_1$

(b) Random Walk $\kappa_2$

(c) Random Walk $C_1$

(d) Random Walk $C_2$
Different Look-Back Windows in Terms of Unobservable States

(a) State Space Model $\kappa_1$

(b) State Space Model $\kappa_2$

(c) State Space Model $C_1$

(d) State Space Model $C_2$

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Key states: $C_1(t)$ and $C_2(t)$.

LL CBD model:
- the difference in the median forecasts of $C_1(t)$ over different sample periods is much smaller than that from the original CBD model;
- great similarity in the patterns of drift processes $C_1(t)$ and $C_2(t)$ under different look-back windows;

→ the median forecast is substantially more robust than the CBD model.
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Basic Set Up for Hedging Longevity Risk

Notation:
- $L_0$: the time-0 value of the liability being hedge
- $H_j$: the time-0 value of the $j$th hedging instrument
- $N_j$: the number of the $j$th hedging instrument
- $P_0$: the time-0 value of the constructed hedging portfolio

Setting:
- Suppose at time 0 we have a liability of 15 years term life annuity which is sold to age 65.
- The present value of this liability can be expressed as:

$$L_0 = \sum_{u=1}^{15} e^{-ru} \prod_{t=1}^{u} p_{x_0+t-1,t},$$

where

$$p_{x,t} = 1 - q_{x,t} = \frac{1}{1 + \frac{e^{\kappa_1(t) + (x-x)\kappa_2(t) + \varepsilon_{x,t}}}{\varepsilon_{x,t}}},$$

with $
\begin{align*}
\kappa_1(t) &= \frac{\lambda x}{2} - \frac{\mu x}{2} + \frac{\sigma^2}{4} - \frac{\sigma^2}{4} t, \\
\kappa_2(t) &= \frac{\lambda x}{2} - \frac{\mu x}{2} + \frac{\sigma^2}{4} - \frac{\sigma^2}{4} t, \\
\varepsilon_{x,t} &= \frac{\rho(x-x)\sigma_x\sigma_t}{\sqrt{\tau}} - \frac{\kappa_1(t) - \kappa_2(t)}{2} \sigma_x \sigma_t, \\
\tau &= \frac{\rho^2 \sigma_x^2 \rho \sigma_t^2}{\sqrt{\tau}}, \\
\varepsilon_{x,t} &= \frac{\rho(x-x)\sigma_x\sigma_t}{\sqrt{\tau}} - \frac{\kappa_1(t) - \kappa_2(t)}{2} \sigma_x \sigma_t.
\end{align*}$

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To hedge against this liability, we use q-forward contracts as the hedging instruments.

The present value of the $j$th q-forward contract can be expressed as:

$$e^{-rt_j}(q_{x_j,t_j} - q^f_{x_j,t_j}),$$

where $q^f_{x_j,t_j}$ is the forward mortality rate and $r$ is the interest rate. In particular, we set $r = 0.01$. 
**Objective:**

- To stabilize the present value of the unexpected cash flow.

**Main Idea:**

- To use the hedging instrument to construct a hedging portfolio s.t. the variance of the portfolio is minimized.

**Evaluation:**

- Define Hedge Effectiveness (HE) in terms of the amount of risk reduction:

\[
HE = 1 - \frac{Var(P_0)}{Var(L_0)}.
\]

A higher value of \(HE\) represents a greater amount of risk reduction.
At time 0, we use the best estimate of $q_{x,t}$ at time 0 to approximate the value for different contracts. The hedging strategy is based on this approximation.

The value of the liability and the $j$th hedging instrument becomes:

$$
\hat{L}_0 = \sum_{u=1}^{15} e^{-ru} \prod_{t=1}^{u} \hat{p}_{x_0 + t-1, t}
$$

and

$$
\hat{H}_j = e^{-rt_j} (\hat{q}_{x_j,t_j} - q_{x_j,t_j})
$$

where

$$
\hat{p}_{x,t} = 1 - \hat{q}_{x,t} = \frac{1}{1 + e^{\kappa_1(0) + C_1 \times t + (x-\bar{x}) \left( \kappa_2(0) + C_2 \times t \right)}}.
$$
Using the delta-hedging approach, the hedging portfolio would be constructed with at least two different q-forwards and should satisfy the following system of equations:

\[
\begin{align*}
\frac{\partial \hat{L}_0}{\partial \kappa_1(0)} &= N_1 \times \frac{\partial \hat{H}_1}{\partial \kappa_1(0)} + N_2 \times \frac{\partial \hat{H}_2}{\partial \kappa_1(0)} \\
\frac{\partial \hat{L}_0}{\partial \kappa_2(0)} &= N_1 \times \frac{\partial \hat{H}_1}{\partial \kappa_2(0)} + N_2 \times \frac{\partial \hat{H}_2}{\partial \kappa_2(0)}
\end{align*}
\]

In matrix form, we have

\[
\begin{pmatrix}
\frac{\partial \hat{L}_0}{\partial \kappa_1(0)} \\
\frac{\partial \hat{L}_0}{\partial \kappa_2(0)}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial \hat{H}_1}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_1(0)} \\
\frac{\partial \hat{H}_1}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_2(0)}
\end{pmatrix}
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
\]

The present value of the constructed portfolio \( P \) can be expressed as

\[ P = L - N_1 \times H_1 - N_2 \times H_2. \]
Nuga-Hedging

- The treatment of the drift terms as random variables naturally calls for Nuga-hedging, a technique proposed by Cairns (2013) to hedge the risk associated with changes in drifts.

- Using the Delta-Nuga-hedging approach, we need to match all four derivatives:

\[
\begin{pmatrix}
\frac{\partial \hat{L}_0}{\partial \kappa_1(0)} \\
\frac{\partial \hat{L}_0}{\partial \kappa_2(0)} \\
\frac{\partial \hat{L}_0}{\partial C_1(0)} \\
\frac{\partial \hat{L}_0}{\partial C_2(0)} \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial \hat{H}_1}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_3}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_4}{\partial \kappa_1(0)} \\
\frac{\partial \hat{H}_1}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_3}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_4}{\partial \kappa_2(0)} \\
\frac{\partial \hat{H}_1}{\partial C_1(0)} & \frac{\partial \hat{H}_2}{\partial C_1(0)} & \frac{\partial \hat{H}_3}{\partial C_1(0)} & \frac{\partial \hat{H}_4}{\partial C_1(0)} \\
\frac{\partial \hat{H}_1}{\partial C_2(0)} & \frac{\partial \hat{H}_2}{\partial C_2(0)} & \frac{\partial \hat{H}_3}{\partial C_2(0)} & \frac{\partial \hat{H}_4}{\partial C_2(0)} \\
\end{pmatrix}
\begin{pmatrix}
N_1 \\
N_2 \\
N_3 \\
N_4 \\
\end{pmatrix}
\]

- And the corresponding present value of the constructed portfolio is

\[P = L - N_1 \times H_1 - N_2 \times H_2 - N_3 \times H_3 - N_4 \times H_4.\]
Issues Related to the Current Hedging Strategy:

- The information underlying the correlation between different states is not fully utilized.

- In the previous literature, main focus is the correlation of the states across different populations (basis risk).

- The correlation of the states within the same population is not the main concern, if there are sufficient instruments to hedge all states. For example
  - Cairns (2011) considers the probit transform of the survival probabilities and derives the dynamic hedging strategies where both states $\kappa_1$ and $\kappa_2$ are hedged;
  - Cairns (2013) investigates the robust hedging strategy where, in addition to the hedging of $\kappa_t$, the trend state $\nu_{\kappa}$ would be hedged.
Issues Related to the Current Hedging Strategy: Con’t

- The hedging strategies based on highly correlated states may reduce the hedge effectiveness.

- More importantly, for a market when only a handful of hedging instruments can be used, the utilization of each instrument becomes one of the decisive factors when deriving the hedging strategies.

* We consider a new hedging technique that is able to
  
  1. account for the correlation between different states within the same population and across different populations simultaneously;
  
  2. obtain higher hedge effectiveness when only a few instruments can be used.
Recall that in the setting of LLCBD model, the $Q$ matrix which represents the covariance matrix of different states is unconstrained.

⇒ In other words, the states vector is correlated.

To further improve the delta- and nuga-hedging techniques, we consider a LDL transform to the $Q$ matrix.

The objective of this method is to decompose $Q$ into the product of three matrix, denoted as $KQ^*K'$, where $Q^*$ is a diagonal matrix with diagonal elements in decreasing order.

Similar transformation techniques include the singular value decomposition.
Under the LDL transformation, the transition equation then follows

\[ \alpha_t = A \alpha_{t-1} + \eta_t \]

\[ \Rightarrow K^{-1} \alpha_t = K^{-1} AK K^{-1} \alpha_{t-1} + K^{-1} \eta_t \]

\[ \Rightarrow \alpha^*_t = A^* \alpha^*_{t-1} + \eta^*_t, \]

where

\[ \alpha^*_t = K^{-1} \alpha_t, \quad A^* = K^{-1} AK \quad \text{and} \quad \eta^*_t \sim \text{MVN}(0, Q^*). \]

The partial derivatives under the transformed states are then calculated as:

\[ \frac{\partial \hat{L}}{\partial \alpha^*_t} = K' \frac{\partial \hat{L}}{\partial \alpha_t}, \]

where \( \partial \hat{L}/\partial \alpha_t = (\partial \hat{L}/\partial \kappa_1(t), \partial \hat{L}/\partial \kappa_2(t), \partial \hat{L}/\partial C_1(t), \partial \hat{L}/\partial C_2(t))' \).
Illustration I: Simulation Study

- EW male population
- Sample period: 1951 to 2011
- Assume no basis risk
- Data source: HMD (2014)

Hedging Instrument:

<table>
<thead>
<tr>
<th>$j_{th}$ Hedging Instrument</th>
<th>$x_j$</th>
<th>$t_j$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>73</td>
<td>8</td>
<td>approx. the median age of the cohort in the liability</td>
</tr>
<tr>
<td>$H_2$</td>
<td>68</td>
<td>3</td>
<td>approx. the 1/4 age of the same cohort</td>
</tr>
<tr>
<td>$H_3$</td>
<td>78</td>
<td>13</td>
<td>approx. the 3/4 age of the same cohort</td>
</tr>
<tr>
<td>$H_4$</td>
<td>65</td>
<td>5</td>
<td>selected to avoid singular problem</td>
</tr>
</tbody>
</table>
Why the singularity problem occurs when four q-forwards linked to the same cohort are used?

\[
\begin{align*}
\frac{\partial \hat{H}_j}{\partial \kappa_1(0)} &= \hat{p}_{x_j,t_j} \hat{q}_{x_j,t_j} \\
\frac{\partial \hat{H}_j}{\partial \kappa_2(0)} &= (x_j - \bar{x}) \hat{p}_{x_j,t_j} \hat{q}_{x_j,t_j} \\
\frac{\partial \hat{H}_j}{\partial C_1(0)} &= t_j \hat{p}_{x_j,t_j} \hat{q}_{x_j,t_j} \\
\frac{\partial \hat{H}_j}{\partial C_2(0)} &= t_j (x_j - \bar{x}) \hat{p}_{x_j,t_j} \hat{q}_{x_j,t_j}
\end{align*}
\]

\[
\begin{pmatrix}
\frac{\partial \hat{H}_1}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_3}{\partial \kappa_1(0)} & \frac{\partial \hat{H}_4}{\partial \kappa_1(0)} \\
\frac{\partial \hat{H}_1}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_2}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_3}{\partial \kappa_2(0)} & \frac{\partial \hat{H}_4}{\partial \kappa_2(0)} \\
\frac{\partial \hat{H}_1}{\partial C_1(0)} & \frac{\partial \hat{H}_2}{\partial C_1(0)} & \frac{\partial \hat{H}_3}{\partial C_1(0)} & \frac{\partial \hat{H}_4}{\partial C_1(0)} \\
\frac{\partial \hat{H}_1}{\partial C_2(0)} & \frac{\partial \hat{H}_2}{\partial C_2(0)} & \frac{\partial \hat{H}_3}{\partial C_2(0)} & \frac{\partial \hat{H}_4}{\partial C_2(0)}
\end{pmatrix}
\]
### Illustration I: Empirical Result

<table>
<thead>
<tr>
<th>Hedge Effectiveness</th>
<th>1 State</th>
<th>2 States</th>
<th>3 States</th>
<th>4 States</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBD</td>
<td>70.4416%</td>
<td>91.8240%</td>
<td>89.2252%</td>
<td>97.5299%</td>
</tr>
<tr>
<td>LLCBD</td>
<td>70.8834%</td>
<td>91.8779%</td>
<td>90.1934%</td>
<td>98.1765%</td>
</tr>
<tr>
<td>LLCBD*</td>
<td>77.4641%</td>
<td>92.3658%</td>
<td>95.3269% (97.1192%)</td>
<td>98.1765%</td>
</tr>
</tbody>
</table>

- In LLCBD*, the hedging strategies are obtained by matching the transformed states.
- (97.1192%) represents the H.E. from hedging the first three transformed states.
Illustration II: Real Data Analysis

Illustration II:
- EW male population
- Assume no basis risk
- Data source: HMD (2014)
- Same hedging instruments as Illustration I

Sample period: 1951 to 1996
Whole sample: 1951 to 2011
Data source: HMD (2014)

Evaluation of the Actual Hedging Performance (AHP)

- Define $AHP$ as the absolute value of the actual cash flow discounted to time 0:

$$AHP = |L_0^{(\text{actual})} - \hat{L}_0 - \sum_{j=1}^{m} N_j H_j^{(\text{actual})}|,$$

where $m$ is the number of states being hedged.

- A smaller value in $AHP$ represents a better hedging performance.
## Actual Hedging Performance

<table>
<thead>
<tr>
<th></th>
<th>1 State</th>
<th>2 States</th>
<th>3 States</th>
<th>4 States</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBD</td>
<td>0.0812</td>
<td>0.0782</td>
<td>0.1049</td>
<td>0.0213</td>
</tr>
<tr>
<td>LLCBD</td>
<td>0.0660</td>
<td>0.0584</td>
<td>0.0775</td>
<td>0.0162</td>
</tr>
<tr>
<td>LLCBD*</td>
<td>0.0424</td>
<td>0.0072</td>
<td>0.0156</td>
<td>0.0162</td>
</tr>
</tbody>
</table>

*Note: The values in parentheses represent standard errors.*
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Conclusions

- By allowing the drift term to be random, Locally-linear CBD is capable in capturing the latest trend in the sample period.

- The median forecast of Locally-linear CBD model is substantially more robust than that from the original CBD model.

- Without considering basis risk, the hedging strategies derived from the transformed states of LLCBD model has the ability to retrieve more hedge effectiveness.

- In the real data analysis, the hedging performance from the transformed technique also shows to be more efficient.
Future Extensions

- By adapting the measurement equation, Locally-linear CBD model can be easily generalized to model multiple populations.

- Under the multi-population LLCBD model, we are able to account for the basis risk in longevity hedging.

- When deriving the hedging strategies, the LDL transform can also be applied to the multi-population LLCBD model to achieve higher hedge effectiveness.