

Asset Prices in General Equilibrium with Transactions Costs and Recursive Utility*

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Abstract

In this paper, we study the effect of proportional transactions costs on asset prices and liquidity premia in a *general equilibrium* economy with multiple agents who are *heterogeneous*. The agents in our model have Epstein-Zin-Weil utility functions and can be heterogeneous with respect to endowments and all three characteristics of their utility functions—time preference, risk aversion, and elasticity of intertemporal substitution. The securities traded in the financial market include a one-period bond and *multiple* risky stocks. We show how the problem of identifying the equilibrium can be characterized in a recursive fashion even in the presence of transactions costs, which make markets incomplete. We find that transactions costs on stocks or the bond lead investors to reduce the magnitude of their positions in financial assets. The holding of each stock is very sensitive to its own transactions cost, but relatively insensitive to the transaction cost for the other stock. Transactions costs also reduce the frequency of trading of the stock; however, the effect on the frequency of trading the bond is much smaller. Our main finding is that even in the presence of non-tradable labor income, the effect of transactions costs on the liquidity premium and expected returns is smaller in general equilibrium than in partial equilibrium: for a proportional transactions cost of 1%, the difference in the expected return on a stock that incurs this cost and one that does not is only 0.25% in general equilibrium.

Keywords: Incomplete markets, heterogeneous agents, portfolio choice, equity risk premium, liquidity risk premium

JEL: G11, G12

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1 Introduction

In a recent article, Lynch and Tan (2011) find that if asset returns are allowed to be predictable and agents have wealth shocks calibrated to labor income, then transactions costs lead to liquidity premia that are of the same order of magnitude as transactions costs. However, their analysis is carried out in a partial equilibrium setting, and the conclusion (p. 36) of their article states:

One important limitation of our analysis is that it is a partial equilibrium analysis. Therefore, it says nothing about how transaction costs affect equilibrium prices by limiting the ability of agents to share risk. More work is needed to understand how transaction costs affect prices and returns in a general equilibrium setting.

Our objective is to fill this gap by studying the effect of proportional transactions costs on asset prices in a general equilibrium economy with multiple agents who are heterogeneous.

In the general equilibrium model we consider, agents have Epstein and Zin (1989) and Weil (1990) utility functions and can be heterogeneous with respect to endowments (nontradable labor income) and all three characteristics of their utility functions—time preference, risk aversion, and elasticity of intertemporal substitution. We consider a financial market in which the traded securities consist of a one-period bond and multiple risky stocks, and these securities, even in the absence of transactions costs, may not be sufficient to span the market because agents have nontraded labor income. We show how the problem of identifying equilibrium in this incomplete-markets economy can be characterized in a recursive fashion even in the presence of costs for transacting in stocks and bonds. We then study the effect of transactions costs on the interest rate, the stock price, the expected return, equity risk premium, and liquidity risk premium on the stock, and the volatility of stock returns in the general equilibrium model, and compare the results to the partial equilibrium version of the model.

We find that transactions costs on either stocks or the bond lead investors to reduce the magnitude of their positions in these financial assets. The holding of each stock is very sensitive to its own transactions cost, but relatively insensitive to the transaction cost for the other stock. And, as one increases the transactions costs on either stock, the magnitude of bond

holding declines because of the decrease in the holding of the stock whose transaction cost has increased. As one would expect, agents use the stock with the lower transactions cost to share risk and smooth consumption over time. Transactions costs also reduce the frequency of trading the stock with the transaction cost. However, the effect on the frequency of trading the bond is much smaller; for even relatively large transactions costs in the bond market, the investors continue to trade the bond.

Transactions costs make it less attractive to hold financial assets, and therefore, require an increase in expected returns (that is, a decrease in prices). Our key finding is that, while the increase in expected returns as a consequence of introducing transactions costs is large in a partial equilibrium setting in which there are shocks to labor income, the effect is much smaller in general equilibrium. For example, for two identical stocks where one can be traded without cost and the other incurs a 1% transaction cost, the difference in returns is about half as large as the transactions cost in partial equilibrium, but only about a quarter in general equilibrium. The reason why the result in our general-equilibrium model is different from that in the partial-equilibrium model of Lynch and Tan (2011) is that in our setting prices for the bond and stock are allowed to change, and thus, they absorb some of the effect of the transactions costs so that the effect on stock returns is smaller. The second reason for the smaller effect is that in general equilibrium risk sharing between the heterogeneous agents reduces the impact of the transactions costs.

Our paper makes two major contributions. One, our paper contributes to the asset-pricing literature by extending the results of existing models examining the effects of transactions costs to a general equilibrium setting with heterogeneous investors: in particular, the model we study allows for an endogenous interest rate, recursive utility functions, and agents who are heterogeneous with respect to their endowments and/or preferences. And, as discussed above, the extension to a general equilibrium setting leads to economic insights that are different from those in a partial equilibrium setting, such as the one considered in Lynch and Tan (2011), where there is a single agent with power utility function, prices are given exogenously, and the interest rate is constant.

Two, we demonstrate how to identify the equilibrium in an economy where there are heterogeneous agents who have recursive utility, even in the presence of transactions costs and incomplete financial markets. There are two problems that arise in identifying the equilibrium when markets are incomplete. The first is that one can no longer use a “central planner” to identify the equilibrium, which, in a complete-market setting, can be done conveniently in two steps: first, allocate consumption optimally across agents, and then, determine the asset prices and portfolio policy for each investor that supports this allocation. The reason why one cannot use the central-planner’s approach in markets that are incomplete is that the consumption allocation that one chooses must lie in the span of the traded assets. Thus, when markets are incomplete one must solve for the consumption and portfolio policies *simultaneously*. This makes it difficult to implement a recursive scheme, because the portfolio chosen at the current date depends on asset prices in the future, but these asset prices depend on consumption at the next date, which is already fixed when solving the model backwards. The second problem is that in the presence of transactions costs, the problem of each investor becomes path dependent: whether or not to trade depends not just on exogenous state variables, but also on the current portfolio of the investor.

We show how both these problems can be resolved, and hence, our solution method can be applied to study other problems in general equilibrium with incomplete markets. For example, it allows us to extend to incomplete markets the complete-markets analysis of Dumas, Uppal, and Wang (2000), who show how to characterize equilibrium in a setting with multiple heterogeneous agents with recursive utility, and the analysis of Bhamra and Uppal (2010), who identify asset prices in a model where agents are heterogeneous with respect to their time-additive utility functions and their beliefs.

The rest of the paper is organized as follows. In Section 2, we discuss the existing literature that is related to our work. In Section 3, we describe the general model. In Section 4, we characterize the equilibrium and explain how it can be described by a system of path-independent backward-only (recursive) equations instead of a system of backward-forward equations. In Section 5, we analyze the quantitative effect of transactions costs on consumption and portfolio choices, asset prices, the equity risk premium, and the liquidity risk premium. We conclude in Section 6.

2 Related Literature

Our work is related to five strands of the literature, and in this section we describe how our paper extends existing work in these five areas.

The first strand of the literature consists of *general-equilibrium models with incomplete markets and transactions costs*. One set of models in this strand introduces transactions costs and time-additive utility but no idiosyncratic labor income; see, for example, Buss and Dumas (2011), which is the paper that is closest to our work. Just like our work, this paper also studies a model of a general equilibrium economy with transactions costs. However, its focus is on the microstructure effects of transactions costs, and so it considers a model with only a single risky asset. In contrast, the focus of our work is on the effect of transactions costs on the cross section of asset returns, and so we consider a model with multiple risky assets; in addition, we allow for idiosyncratic labor income and our investors have Epstein-Zin-Weil utility rather than power utility. On the technical side, both papers use the insights in Dumas and Lyasoff (2010) to solve the model; however, we solve the primal problem, which has the advantage of yielding directly quantities of interest such as the region of no trade, while Buss and Dumas (2011) solve the dual problem. Vayanos and Vila (1999) also study the effects of transactions costs in a model with no labor income but two assets, *both* of which are risk-free, but where one asset has transactions costs while the other does not.

A second set of models in this strand, such as Heaton and Lucas (1996), allows for both idiosyncratic labor income *and* transactions costs, but with time-additive utility functions. In the model studied in Heaton and Lucas (1996), heterogeneity across agents arises because of idiosyncratic labor income shocks and there is a quadratic transaction cost for trading the stock.¹ They find that the model can produce a sizable equity premium only if transactions costs are large or the assumed quantity of tradable assets is limited and borrowing constraints are binding. In contrast to this model, we allow for multiple risky assets, proportional transactions costs, and investors who have Epstein-Zin-Weil utility functions.

¹Heaton and Lucas (1996, Equation (19)) also consider a specification where the transaction cost function is quadratic for small transactions and linear for larger transactions.

A second strand of the literature to which our paper is related is the work studying *general-equilibrium models with incomplete markets but without transactions costs*. One set of models in this strand studies investors with time-additive utility who are constrained or prohibited from holding some of the financial assets; see, for example, Basak and Cuoco (1998), Kubler and Schmedders (2002, 2003), Garleânú and Pedersen (2011), and especially Dumas and Lyasoff (2010), who propose an elegant solution method that is recursive; we will use many of the insights in this paper for solving our model. Guvenen (2009) extends these models to allow for Epstein-Zin preferences and finds that “heterogeneity in risk aversion plays no essential role, whereas heterogeneity in the EIS (and especially the low EIS of non-stockholders) is essential” for explaining empirically observed features of asset prices.

A second set of models within this strand considers a setting where the source of market incompleteness is idiosyncratic labor income. For example, Lucas (1994), Telmer (1993), and Krusell and Smith (1998) examine asset prices in a model with agents who have time-additive utility functions and transitory idiosyncratic income shocks and find that market incompleteness has only a small effect on equilibrium prices. On the other hand, Mankiw (1986), Constantinides and Duffie (1996), and Krueger and Lustig (2010) allow for permanent idiosyncratic shocks and identify the condition under which market incompleteness will have a substantial effect on equilibrium prices. Storesletten, Telmer, and Yaron (1998) argue that life-cycle effects can be important and extend the analysis to a stationary overlapping-generations model. Gomes and Michaelides (2005) extends these models to allow for recursive utility functions. Our work extends the models in this strand of the literature by allowing for transactions costs.

The third strand of the literature to which our analysis contributes consists of *general equilibrium models with heterogeneous investors but complete financial markets*. This includes models with time-additive preferences, such as Dumas (1989), Dumas, Kurshev, and Uppal (2009), and Bhamra and Uppal (2010), and models where agents have recursive utility, as in Dumas, Uppal, and Wang (2000) and Dumas and Uppal (2001). We extend these models to the setting where markets are incomplete because of the presence of idiosyncratic labor income and transactions costs.

The fourth strand of the literature consists of *partial equilibrium models that study the effects of transactions costs on asset prices*. Amihud and Mendelson (1986) consider a single-period model in which agents are risk-neutral and must exit the market at which time they sell stock to newly arriving agents; they find that the excess return on a stock equals the product of the asset's turnover and the proportional transaction cost. Constantinides (1986) shows that because the agent chooses when to trade optimally, the effect of transactions costs on asset prices is much smaller than suggested by Amihud and Mendelson.

Vayanos (1998) considers an overlapping-generations model with multiple stocks and also finds that transactions costs have a small impact on prices. One of the strengths of his paper is that the model has a closed-form solution, which can be used to obtain several interesting insights. However, to obtain a closed-form solution, several restrictive assumptions need to be made. For instance, the interest rate is assumed to be exogenous and constant, which can have an important bearing on results, as shown by Loewenstein and Willard (2006), and as we also find in our model; agents are assumed to have exponential utility functions, which do not allow for the study of wealth effects; dividends follow an Ornstein-Uhlenbeck, so they are normal, instead of being lognormal; transactions costs are proportional to the number of shares rather than the value of shares; the model is one of overlapping-generations, with risk aversion increasing with age; and, it is assumed (in his Section 5) that the shortsale constraint is binding. A consequence of these assumptions is that stock prices are linear in dividends, and the stock holdings of agents are deterministic. In contrast to these modeling assumptions, we allow for an endogenous interest rate, recursive utility functions (which nest exponential utility as a special case), a process for dividends that is not restricted to be normal, proportional transactions costs that depend on the value of shares traded, and agents who can be heterogeneous with respect to their endowments and/or preferences.

Lo, Mamaysky, and Wang (2004) consider a setting with *fixed* transaction costs and *high-frequency transaction needs*; they find that the effect of transactions costs in such a setting is larger, and of the same order as the transaction costs. Just as Vayanos (1998), they also assume a constant (exogenous) interest rate and exponential utility, but in contrast to Vayanos (1998), they consider fixed transactions costs for stocks and no transactions costs for bonds. The motivation for trading in the model is heterogeneous nontraded (labor) income, which in

aggregate sums to zero; that is, there is no aggregate risk. Moreover, it is assumed that the risk in the nontraded asset is perfectly correlated with the stock, which implies that the non-traded income is marketed. These assumptions allow one to get a closed-form solution for the special case where agents can trade at only the first date or for the case where transactions costs are small. As in Lo, Mamaysky, and Wang (2004), Lynch and Tan (2011) also find that the effect of transactions costs is large: liquidity premia are of the same order of magnitude as transactions costs in their model. They obtain this result by considering a model where asset returns are specified exogenously to be predictable and agents have nontradable wealth shocks.² Our work extends the analysis of these partial-equilibrium models to a general-equilibrium setting.

A fifth strand of the literature consists of *partial-equilibrium models that focus on the effect of transactions costs on portfolio policies*.³ In particular, these papers identify the “region of no-trade” where, because of transactions costs, an investor finds it optimal not to rebalance her portfolio even though asset prices have changed. We, too, identify the region of no trade, but in contrast to these papers, our agents have recursive utility, asset prices are endogenous, and we allow for transactions costs on not just risky assets but also the bond.

3 The General Model

In this section, we describe the features of the model we study. In our model, there is a single consumption good. Time is assumed to be discrete. We denote time by t , with the first date being $t = 0$ and the terminal date being $t = T$. In our model we will allow for $K = 2$ agents, who are indexed by k and who have recursive utility functions. We assume that there are multiple sources of uncertainty, with the number of sources of uncertainty denoted by M . There are $N + 1$ risky assets that are indexed by $n = \{0, 1, \dots, N\}$, where the first asset, $n = 0$, is assumed to be a one-period bond; the remaining N assets are assumed to be

²There is also a large literature studying the effect of liquidity on asset prices; see, for example, Gârleanu (2009) and the references therein. In contrast to that literature, our focus is on the effect of trading costs on asset prices.

³This includes the work in Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Gennotte and Jung (1994), Atkinson and Wilmott (1995), Morton and Pliska (1995), Korn (1998), Bertsimas and Lo (1998), Schroder (1998), Balduzzi and Lynch (1999), Lynch and Balduzzi (2000), Akian, Sulem, and Taksar (2001), Liu and Loewenstein (2002), Liu (2004), Muthuraman and Kumar (2006), and Garleânú and Pedersen (2009). There is also the literature that uses partial-equilibrium models to study how life-cycle considerations influence portfolio selection; see, for example, Campbell, Cocco, Gomes, Maenhout, and Viceira (2001), Gomes and Michaelides (2003), Gomes and Michaelides (2005), Cocco, Gomes, and Maenhout (2005). In our work, we do not focus on life-cycle issues.

stocks. We allow for the possibility that the number of risky assets traded in financial markets is strictly less than the number of sources of uncertainty, that is, $N < M$. The main feature of our model is that there is a proportional transactions cost for trading financial assets. We allow for transactions costs on both the bond and the N stocks, with the possibility that these transactions costs are different for different assets.⁴ We are interested in examining the effect of the transactions costs on the trading of financial assets by the two agents, and the effect of this on asset prices. In the rest of this section, we give the details of the model.

3.1 Uncertainty

Time is assumed to be discrete, with $t = \{0, 1, \dots, T\}$. Uncertainty is represented by a σ -algebra $\mathcal{F}$ on the set of states Ω . The filtration $\mathbb{F}$ denotes the collection of σ -algebras $\mathcal{F}_t$ such that $\mathcal{F}_t \in \mathcal{F}_s, \forall s > t$, with the standard assumptions that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In addition to time being discrete, we will also assume that the set of states is finite, and so the filtration can be represented by a tree, with each node on the tree representing a particular state of nature, $\omega(t, s)$. The probability measure on this space is represented by $P : \mathcal{F} \rightarrow [0, 1]$ with the usual properties that $P(\emptyset) = 0, P(\Omega) = 1$, and for a set of disjoint events $A_i \in \mathcal{F}$ we have that $P(\cup_i A_i) = \sum_i P(A_i)$.

In our implementation of the model, we will assume that uncertainty is generated by a M -dimensional multinomial process, as described in He (1990), which is an extension of the binomial process that is often used for pricing options in a discrete-time and discrete-state framework (see Cox, Ross, and Rubinstein (1979)).⁵

3.2 Financial assets

We assume that there are $N + 1$ assets that are traded in financial markets. The first asset is a one-period discount bond in zero net supply. The other N stocks are assumed to be in unit supply and have a dividend $d(n, t)$, which is assumed to be $\mathcal{F}_t$ measurable. Aggregate dividends at any node are then given by $\sum_{n=1}^N d(n, t)$. The ex-dividend price of each asset n as

⁴The transactions costs could differ also across agents.

⁵Given that we allow for incomplete financial markets, the exact process used to generate uncertainty could be more general; for instance, we could allow for jumps.

perceived by agent k at date t , $S(n, k, t)$, is determined in equilibrium; note that in the presence of transactions costs, agents may choose *not* to trade a particular asset at a particular date, in which case agents will not agree on the price of this asset: $S(n, 1, t) \neq S(n, 2, t)$. The ex-dividend price on the terminal date for these assets is zero. The number of units of a particular asset n held by investor k at date t is denoted by $\theta(n, k, t)$.

In the special case where one assumes $M = N$, then each component of the multinomial process could be interpreted as the exogenous dividend from the n^{th} “tree”.⁶ In the general case where $N < M$, one could interpret N components of the multinomial process as the exogenous dividends for the N trees, and the remaining $M - N$ processes as nontradable labor income received by the agents.

3.3 Labor Income

The labor income of Agent k is denoted by $Y(k, t)$. We adopt the same process for labor income as in Lynch and Tan (2011), who, following Carroll (1996, 1997), specify the logarithmic of labor income, $\log Y(k, t) = y(k, t)$, to have both permanent and temporary components:

$$y(k, t) = y^P(k, t) + \varepsilon(k, t) \quad (1)$$

$$y^P(k, t) = y^P(k, t-1) + \bar{g}(k) + b_g b(t) + u(k, t), \quad (2)$$

where ε_t and u_t are uncorrelated i.i.d. processes that have normal distributions, $\bar{g}(k)$ is a constant representing the average growth rate for the labor income of Agent k , b_g is a constant, and $b(t)$ is a predictive variable that generates predictability in labor income; for most of our analysis, we will set this term to zero. Just as in Lynch and Tan (2011), we also turn off the temporary component because, when calibrated to data, the temporary component has a negligible impact on liquidity premia. Thus, throughout our analysis, we consider the case in which $y(k, t) = y^P(k, t)$ and $\varepsilon(k, t) = 0$ for all t .

⁶Note that, because of the presence of transactions costs, financial markets are incomplete even for the case in which $M = N$.

3.4 Preferences

We assume that the preferences of agents are of the Kreps and Porteus (1978) type. These utility functions nest the more standard time-separable utility functions, and in particular, the constant relative risk aversion power utility function, but have the well-known advantage that the risk aversion parameter, which drives the desire to smooth consumption across states of nature, is distinct from the elasticity of intertemporal substitution parameter, which drives the desire to smooth consumption over time. We adopt the Epstein and Zin (1989) and Weil (1990) specification of this utility function, in which lifetime utility $V(k, t)$ is defined recursively:

$$V(k, t) = \left[(1 - \beta_k) c(k, t)^{1 - \frac{1}{\psi_k}} + \beta_k E_t [V(k, t + 1)^{1 - \gamma_k}]^{\frac{1}{\phi_k}} \right]^{\frac{\phi_k}{1 - \gamma_k}}. \quad (3)$$

In the above specification, E_t denotes the conditional expectation at t , $c(k, t) > 0$ is the consumption of agent k at date t in state $\omega(t, s)$,⁷ β_k is the subjective rate of time preference, $\gamma_k > 0$ is the coefficient of relative risk aversion, $\psi_k > 0$ is the elasticity of intertemporal substitution, and $\phi_k = \frac{1 - \gamma_k}{1 - 1/\psi_k}$. The above specification reduces to the case of constant relative risk aversion if $\phi_k = 1$, which occurs when $\psi_k = 1/\gamma_k$. The index k for the parameters β_k , γ_k , and ψ_k indicates that the agents may differ along all three dimensions of their utility functions.

3.5 Transactions costs

We assume that agents pay a proportional cost for trading financial assets. The transaction cost at t depends on the value of assets being traded.⁸ We denote this transaction cost by $\tau(\theta(n, k, t), \theta(n, k, t - 1))$. We assume that this is a deadweight cost for making a transaction, and hence, this amount does not flow to any agent.

4 Characterization of Equilibrium

In this section, we first describe the optimization problem of each agent. We then impose market clearing to obtain a characterization of equilibrium, which is given in terms of a backward-

⁷To simplify notation, we do not write explicitly the dependence on the state $\omega(t, s)$.

⁸We could also consider the case where the transaction cost depends on the number of shares being traded, which is the specification studied in Vayanos (1998).

forward system of equations. Finally, we show how this backward-forward system of equations can be transformed into a recursive (backward-only) system of equations.

4.1 The optimization problem of each agent

The objective of each investor k is to maximize lifetime utility given in (3) by choosing consumption, $c(k, t)$ and the portfolio positions in each of the financial assets, $\theta(n, k, t), n = \{0, 1, \dots, N\}$. This optimization is subject to a dynamic budget constraint:

$$c(k, t) + \sum_{n=0}^N \theta(n, k, t) S(n, k, t) + \sum_{n=0}^N \tau(\theta(n, k, t), \theta(n, k, t-1)) \leq \quad (4)$$

$$Y(k, t) + \sum_{n=0}^N \theta(n, k, t-1) (S(n, k, t) + d(n, t)),$$

where the left-hand side of the above equation is the amount of wealth allocated to consumption and the purchase of assets at date t , and the right-hand is the sum of labor income and the value, using the prices prevailing at date t , of shares that were purchased at date $t-1$, and the “dividends” received from these assets; this sum can be interpreted as the investor’s wealth at t . We assume that each agent is endowed with some shares of the risky assets at the start of time. Note that in the above formulation we have not imposed constraints on short selling or borrowing; if one wished, constraints on portfolio positions could be imposed on the trading strategy of the agent.

Thus, the Lagrangian for the utility function in (3) that is to be maximized subject to the (4) is:

$$\begin{aligned} \mathcal{L}(k, t) = & \sup_{c(k, t), \theta(n, k, t)} \inf_{\lambda(k, t)} \left[(1 - \beta_k) c(k, t)^{1 - \frac{1}{\psi_k}} + \beta_k E_t [V(k, t+1)^{1 - \gamma_k}]^{\frac{1}{\phi_k}} \right]^{\frac{\phi_k}{1 - \gamma_k}} \quad (5) \\ & + \lambda(k, t) \left[Y(k, t) + \sum_{n=0}^N \theta(n, k, t-1) (S(n, k, t) + d(n, t)) \right. \\ & \left. - c(k, t) - \sum_{n=0}^N \theta(n, k, t) S(n, k, t) - \sum_{n=0}^N \tau(\theta(n, k, t), \theta(n, k, t-1)) \right], \end{aligned}$$

where $\lambda(k, t)$ is the Lagrange multiplier for the dynamic budget constraint.

Based on the above Lagrangian, the first-order conditions with respect to $c(k, t)$, $\theta(n, k, t)$, and $\lambda(k, t)$ are:

$$0 = \frac{\partial V(k, t)}{\partial c(k, t)} - \lambda(k, t), \quad (6)$$

$$0 = \lambda(k, t) \left[S(n, k, t) + \frac{\partial \tau(\theta(n, k, t), \theta(n, k, t-1))}{\partial \theta(n, k, t)} \right] \quad (7)$$

$$- E_t \left[\frac{\partial V(k, t+1)}{\partial c(k, t+1)} \left(S(n, k, t+1) + d(n, t+1) - \frac{\partial \tau(\theta(n, k, t+1), \theta(n, k, t))}{\partial \theta(n, k, t)} \right) \right],$$

$$0 = Y(k, t) + \sum_{n=0}^N \theta(n, k, t-1) (S(n, k, t) + d(n, t)) \\ - c(k, t) - \sum_{n=0}^N \theta(n, k, t) S(n, k, t) - \sum_{n=0}^N \tau(\theta(n, k, t), \theta(n, k, t-1)). \quad (8)$$

Equation (6) is the first order condition for consumption and it equates the marginal utility of consumption to $\lambda(k, t)$, the shadow price for relaxing the budget constraint. Equation (7) equates the benefit from holding the stock versus selling the stock, net of transactions costs. Equation (8) is the budget constraint that the optimal consumption and portfolio policies must satisfy. One can substitute for $\lambda(k, t)$ in Equation (7) using Equation (6):

$$0 = \frac{\partial V(k, t)}{\partial c(k, t)} \left[S(n, k, t) + \frac{\partial \tau(\theta(n, k, t), \theta(n, k, t-1))}{\partial \theta(n, k, t)} \right] \quad (9) \\ - E_t \left[\frac{\partial V(k, t+1)}{\partial c(k, t+1)} \left(S(n, k, t+1) + d(n, t+1) - \frac{\partial \tau(\theta(n, k, t+1), \theta(n, k, t))}{\partial \theta(n, k, t)} \right) \right].$$

After this substitution, we need to solve only for optimal consumption and the optimal portfolio at each node in order to identify the optimal policies for each agent. Thus, the solution of the problem of maximizing the lifetime utility in (3) subject to the budget constraint in (4) is characterized by the system of equations given in (8) and (9), which must hold for each date and state on the tree.

4.2 Market-Clearing Conditions

In the economy we are considering, there are financial markets for the risk-free asset and the N risky securities, and a commodity market for the consumption good. The market-clearing condition for the bond is that the aggregate demand for bonds must net to zero:

$$0 = \sum_{k=1}^K \theta(0, k, t). \quad (10)$$

The market-clearing condition for equity is that the aggregate demand for each stock must add up to the number of shares outstanding, which we normalize to one:

$$1 = \sum_{k=1}^K \theta(n, k, t), \quad \forall n = \{1, 2, \dots, N\}. \quad (11)$$

Finally, aggregate dividends and labor income should be equal to aggregate consumption and transactions costs:⁹

$$0 = \left(\sum_{k=1}^K Y(k, t) + \sum_{n=0}^N d(n, t) \right) - \left(\sum_{k=1}^K c(k, t) + \sum_{k=1}^K \sum_{n=0}^N \tau(\theta(n, k, t), \theta(n, k, t-1)) \right). \quad (12)$$

4.3 Equilibrium in the Economy

Equilibrium in this economy is defined as a set of consumption policies, $c(k, t)$, and portfolio policies, $\theta(n, k, t)$, along with the resulting price processes for the financial assets, $S(n, k, t)$, such that the consumption policy of each agent maximizes her lifetime utility; that this consumption policy is financed by the optimal portfolio policy; financial markets clear, and the market for the consumption good clears.

4.4 Solving for the Equilibrium in Markets that are Incomplete

When financial markets are *complete*, one can divide the task of identifying the equilibrium into two distinct steps by exploiting the condition that in complete markets agents can achieve perfect risk sharing. Consequently, at each date and state, the marginal utility of consumption must be the same across all agents. This condition can be used to identify the optimal allocation

⁹Note that we use the consumption good as the numeraire, and therefore, its price is equal to unity. This also implies that the final market-clearing condition is satisfied simply because of Walras' law.

of aggregate consumption across agents, which is often referred to as the solution to the “central planner’s problem.” Once we know the allocation of consumption across agents, we can use this to determine asset prices and also the portfolio policy of each investor that supports this allocation.

However, when financial markets are *incomplete*, one cannot divide the task of identifying the equilibrium into two steps because the consumption allocation one chooses must lie in the span of traded assets. Thus, when markets are incomplete, one must solve for the consumption and portfolio policies *simultaneously*.

In principle, one can identify the equilibrium by solving simultaneously the set of nonlinear first-order conditions for the two agents in (8) and (9), along with the market-clearing conditions in (10) and (11), for all the states across all dates. This is the approach proposed in Cuoco and He (2001). Dumas and Lyasoff (2010), who consider the problem of identifying the equilibrium in an economy with incomplete markets but no transactions costs, call this the “global method.” The problem in implementing this approach is that the number of equations grows exponentially with the number of periods. In the presence of transactions costs, the number of equations grows even faster because the optimal policies of the agents are path dependent and so the decision tree is not recombining. For example, even for a problem with only ten dates, the number of equations to be solved is in the millions. An additional complication that arises when one has transactions costs is that whether a particular security is traded or not at a given node is determined endogenously; this makes it more difficult to solve the large number of nonlinear equations, because the system of equations to be solved changes depending on whether one is inside or outside the no-trade region.

In order to simplify the task of identifying the equilibrium in markets that are incomplete, Dumas and Lyasoff (2010) propose a “recursive method.” In this method, one determines the equilibrium at each date in a recursive fashion; that is, at date t one solves for the equilibrium having already solved for the equilibrium at date $t + 1$. Thus, at each node in the tree one needs to solve only a small number of equations.

There are two problems in solving the system of equations in (8)–(11) recursively in a general-equilibrium setting. The first problem is that the current consumption and portfolio

choices depend on the prices of assets, which from Equation (9) we see depend on *future* consumption. But, in a general-equilibrium setting, when the agent attempts to solve for the optimal consumption and portfolio policies at date t , asset prices need to adjust in order for markets to clear; but, when one is solving the system of equations backward, these prices *cannot* adjust because they depend on future consumption, which has already been determined in the previous steps. Thus, to solve these equations, one would need to iterate backwards *and* forwards until the equations for all the nodes on the tree are satisfied. Dumas and Lyasoff (2010) address this problem by proposing a “time-shift” whereby at date t one solves for the optimal portfolio for date t but the optimal consumption for date $t + 1$, instead of the optimal consumption for t . Using this insight allows one to write the system of equations so that it is recursive.

In partial equilibrium, the recursive approach for determining the optimal portfolio policy of an investor requires one to introduce the agent’s wealth as an additional endogenous state variable, in addition to other exogenous state variables that characterize the investment opportunity set. Dumas and Lyasoff (2010) show that in a general-equilibrium setting with incomplete markets, using the distribution of consumption across agents as the additional state variable, instead of individual wealth, has two advantages: one, it is a variable that is bounded, and two, the recursive problem becomes path-independent, so that the decision tree is recombining.

In the presence of transactions costs, we show that it is not sufficient to include only consumption as an additional state variable; one needs to include also the portfolio composition of the investor as an endogenous state variable. The reason for this is that, because of transactions costs, the choice of the optimal portfolio at date t will depend also on the composition of the portfolio at date $t - 1$. Moreover, as Equation (12) shows, in the presence of transactions costs the consumptions of the two agents do not add up to aggregate endowment and labor income; thus, the state variables need to include also the consumption of the other agent.

The second problem in solving the system of equations in (8) and (9) recursively arises because of transactions costs. If agents choose to trade all assets, then they will agree on the prices of these assets. However, if agents find it optimal not to trade some of the assets, then

agents will disagree on the prices of the assets that are not traded at that node. Consequently, the number of unknowns to be solved for, and the system of equations characterizing the solution, depends on whether or not agents choose to trade all assets or only some of the assets. We explain below how this problem can be addressed.

Note that the past portfolio holdings enter the system of equations only through the condition (9), as a first partial derivative of the transaction cost function $\tau(\cdot)$ with respect to the current portfolio investment. Under the assumption that the transaction costs are a constant proportion κ_n of the value of an asset n being traded, we observe that there are only three possibilities for the form of this derivative. It is equal to zero when an agent decides not to trade; it is equal to $\kappa_n \times S(n, k, t)$ when the agent decides to increase the position in the asset; or, it is equal to $-\kappa_n \times S(n, k, t)$ when the agent sells the asset. Consequently, all the $\theta(n, k, t - 1)$ values for which the agent decides to buy an asset at time t result in the same solution $\theta(n, k, t)$ for a given value of current consumption $c(k, t)$. Similarly, all $\theta(n, k, t - 1)$ values for which the agent decides to sell an asset at t result in the same solution $\theta(n, k, t)$ for a given value of current consumption. And all other values of past portfolio holdings will result in no trading at t . In other words, instead of solving the problem over the wide grid of portfolio holdings at $t - 1$ that is difficult to determine, we can solve it first for the two trading decisions—sell or buy—at time t ; that is, over the two values of the derivative of the transaction cost function. The solution to this provides us with the bounds of the no-trade region, for which the portfolio investment from $t - 1$ to t does not change. Knowing the bounds of the no-trade region, we solve the system of equations for future consumption $c(k, t + 1)$ only, explicitly restricting current portfolio holdings within the no-trade region to be equal to the past portfolio holdings $\theta(n, k, t - 1)$. It is important to recognize that within the no-trade bounds the agents can disagree on the prices of the traded assets, and hence we lose the “kernel” condition that requires the agents to agree on asset prices, but this also implies that the holdings of that asset at that time are equal to the holdings in the previous date.

In this way we are able to solve the system recursively in a backward fashion, knowing for each set of values of state variables if we are currently in the no-trade region with a smaller number of equations to be solved for consumption only, or the full set of equations to be solved for consumption and the investment portfolio. After we solve the dynamic program recursively

up to time $t = 0$, we undertake a single “forward step” to determine the equilibrium quantities for each state of nature that satisfy the initial conditions.

We now explain more precisely the recursive solution method (dynamic program) described above.¹⁰ Observe that the *unknowns* to be solved for are the choice variables consumption and investments of each agent, and the prices of the available assets. For each choice variable, there is a corresponding first-order condition, and the prices of the assets are determined from the market-clearing conditions. Finally, the Lagrange multiplier is identified by the budget constraint. As explained above, one can substitute out the Lagrange multiplier, so that the system of equations to be solved consists of the budget constraint in (8), the first-order conditions for optimal investment in (9), and the market clearing conditions in (10) and (11). In the recursive formulation, at date t we “time-shift” forward to date $t + 1$ equations (8), (10) and (11), while leaving (9) as it is.

Thus, we start at the terminal date T , where there are no choices to be made: $\theta(n, k, T) = 0$, $S(n, k, T) = 0$, and the consumption of each agent is determined by the portfolio she chose at $T - 1$; that is, $c(k, T) = \sum_{n=0}^N \theta(n, k, T - 1) d(n, T)$.

Stepping back one date, at $T - 1$ we solve for $\theta(n, k, T - 1)$ and $c(k, T)$; and once we have these quantities, we use (9) to identify $S(n, k, T - 1)$. We solve for the number of bonds and stocks to hold, $\theta(n, k, T - 1)$, using the time-shifted market-clearing conditions at T for the bond and stocks, (10) and (11); and, we solve for $c(k, T)$ using the budget constraint in (8), again for date T . The recursive optimization problem of each agent is written as a function of the state variables consisting of the portfolio holdings of Agent k coming into this period, $\theta(n, k, T - 2)$, and the consumptions of both agents at date $T - 1$, $c(1, T - 1)$ and $c(2, T - 1)$.

For any date $t < T$, the recursive system to be solved is the same as that described for date $T - 1$: we solve for $\theta(n, k, t)$, $c(k, t + 1)$, and $S(n, k, d)$ using Equation (9) for date t and the time-shifted Equations (8), (10), and (11) for date $t + 1$. And, at each date t , the recursive optimization problem of each agent is written as a function of the portfolio holdings of Agent k

¹⁰We show in the appendix that the principle of the dynamic programming applies to the problem with transaction costs, that is, the maximization goal of Agent k at time 0 is achieved if and only if the value function of the recursive problem is maximized at all times and states. In particular, we show that the first-order conditions of the dynamic program are equivalent to the first order conditions (8) and (9); moreover, one can also show that the value function is concave, and thus, satisfying the first-order conditions of the dynamic program is necessary and sufficient for optimality.

coming into this period, $\theta(n, k, t-1)$, and the consumptions of both agents at that date, $c(1, t)$ and $c(2, t)$.

At date $t = 0$, we solve for $\theta(n, k, 0)$, $S(n, k, 0)$, and $c(k, 1)$ using Equation (9) for date $t = 0$ and the time-shifted Equations (8), (10) and (11) for date $t = 1$. The recursive optimization problem of each agent is written as a function of the portfolio holdings of Agent k coming into this period (the holdings with which the agent was initially endowed), $\theta(n, k, -1)$, and the consumptions of both agents at that date, $c(1, 0)$ and $c(2, 0)$.

This leaves us with the unknown $c(k, 0)$ to be determined using the equation we have not used so far: the budget constraint in Equation (8) for date $t = 0$. Once we have the optimal $c(k, 0)$, we then identify the solution by walking forward through the entire tree from $t = 0$ to $t = T$ using this optimal $c(k, 0)$ as the starting value. Thus, one obtains the solution at all dates and states using only *one* forward step.

5 Quantitative Analysis of the Model

In this section, we undertake a quantitative analysis of the model described in the previous section. In our analysis, the quantities we study are: the consumption and portfolio choices of the two investors; the turnover (trading volume) of the bond and the two stocks; the prices of the risk-free bond and stocks; the risk-free rate and expected returns of the risky assets; the liquidity and equity risk premia for these assets; and the volatility of stock returns.

We examine how the quantities listed above change with respect to four sets of parameters of the model: (i) the level of proportional transactions costs; (ii) the preference parameters that dictate the level of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) for the two investors; (iii) the parameters driving the labor income processes, which are the initial level of labor income relative to the level of dividends, the growth rate (drift), and the volatility of the labor income processes; and, (iv) the parameters driving the dividend processes, which are the drift and volatility for the two stocks (we assume that the dividend processes are uncorrelated). The benchmark values of these parameters are given in Table 1.

We also investigate how the behavior of agents, and the impact of this behavior on prices in the capital market, are different in our model compared to the models studied earlier in the literature, where: (a) instead of using Epstein-Zin-Weil utility functions, the utility functions are power, log, or exponential functions; (b) instead of allowing asset prices to be determined in general equilibrium, the price processes are specified exogenously.

5.1 Consumption

Aggregate consumption decreases with transaction costs, because some resources are devoted to transaction costs. However, the drop in aggregate consumption is quite small: a transaction cost of 1% reduces consumption by 0.3% for the base case. In cases where the heterogeneity in preferences or labor income is smaller, the effect of transactions costs on aggregate consumption is even smaller because the desire to trade financial assets is smaller.

Transaction costs also have a small effect on the distribution of consumption between the two investors. Note that in the absence of heterogeneity in preference and labor income, each investor's share of aggregate output would be 0.50. When the investors differ in risk aversion, elasticity of intertemporal substitution, or labor income, then there is a desire to trade financial securities in order to achieve optimal consumption sharing. For instance, if the only difference between the investors is in their labor income, and they have identical Epstein-Zin-Weil preferences, then in the absence of transactions costs the deviation of the optimal consumption share of the two investors from 0.5 increases with RRA: it is 0.5151 when RRA is 2, 0.5251 when RRA is 3.5, and 0.5335 when RRA is 5. If, in addition to differing with RRA, they also differ with respect to their EIS, the deviation of consumption from 0.5 increases further with EIS: it is 0.5224 when EIS is 0.5, 0.5280 when EIS is 0.9, and 0.5350 when EIS is 1.5. Of course, this consumption reallocation needs to be financed by trading in financial securities and so it indicates when transactions costs are likely to have the strongest effect. But, the magnitude of the effect of transaction costs on consumption sharing is still small. For the base case of the model, Investor 1 would ideally like to consume 0.5350 of aggregate consumption in the absence of transaction costs. In the presence of transaction costs of 1%, the optimal consumption share of Investor 1 changes by only 0.11%; for Investor 2, who has higher

RRA but the same EIS as the Investor 1 and a labor income process that has a higher growth rate and volatility, the change in the consumption share in response to a 1% transaction cost is 0.74%. The consumption share of Investor 2 changes by more than the transaction cost of 1% only if we increase the heterogeneity in risk aversion between the two investors: for example, if the risk aversion of Investor 2 goes from 5 to 7, while keeping fixed the risk aversion of Investor 1 at its base-case value of 2, then the change in the consumption share of Investor 2 is 1.15%.

Changes in the growth rate and volatility of labor income have only small effects on the results described above. Similarly, changes in the growth rate and volatility have negligible effects on these results.

However, working with simpler utility functions does influence the effect of transaction costs on consumption sharing. The effect of transactions costs on aggregate consumption and also on the share of consumption is much greater with power utility (when the two agents have RRA of 2 and 5) than with Epstein-Zin-Weil utility functions (with the two agents still having a RRA of 2 and 5, and with EIS equal to 1.5). When both agents have log utility, the effect of transactions costs is negligible. For exponential utility, the magnitude of the results depends on the heterogeneity in risk aversion. If the two agents have the same absolute risk aversion (ARA) and differ only with respect to their labor income, then transactions costs have no effect at all on aggregate consumption or consumption sharing; if the ARA of the two agents is 0.75 and 1.25, then a 1% transaction cost has no effect on aggregate consumption and a 0.01% effect on consumption sharing; and if the ARA of the two agents is 0.5 and 2.0, then the effect of a 1% transaction cost is about 0.57% on aggregate consumption and more than 1% on the consumption shares of the two agents (2.47% for the less risk averse agent and 1.32% for the more risk averse agent).

5.2 Portfolio Choice

We start by describing the portfolios of the two investors for the case of zero transactions costs. Note that both agents start out with an initial holding of 0.5 shares of each of the two risky assets and 0.0 bonds. When we introduce heterogeneity in EIS, the two investors

continue to hold 0.5 shares in each of the risky assets, but Investor 1 has a negative position in bonds (sells bonds) while Investor 2 has a positive position in bonds; these bonds are used to smooth consumption over time. When we introduce heterogeneity in labor income, the motive to smooth consumption over time becomes stronger, and so the amount of borrowing/lending between the two agents increases; however, there is no change in the holding of risky assets.

However, when we allow also for heterogeneity in risk aversion, then there is a substantial change in the investment in risky assets. Because Agent 1 is less risk averse than Agent 2, the risk-sharing motive drives Agent 1 to increase her holding of the risky assets from the initial position of 0.5 shares of each of the two risky assets (that is, to increase the 50% of her share of wealth invested in each of the two stocks), while Agent 2 decreases her holding of risky assets. Thus, the total proportion of financial wealth invested in risky assets by Agent 1 exceeds 1; this is financed by borrowing (selling bonds). The total proportion of wealth invested in risky assets by Agent 2, is always less than 1, with the rest of her wealth invested in bonds (this is equal to the amount borrowed by Agent 1).¹¹ For the base case of our model, Investor 1 has -73% of her wealth invested in bonds, and about 86.5% of her wealth invested in each of the two risky assets.

With the introduction of transactions costs, there is only a small change in the bond positions for the case where the only motive for trade is consumption smoothing over time, that is, the cases where the source of heterogeneity between investors is differences in EIS and labor income. However, there is a substantial decrease in the risk-sharing between the two investors, and this is reflected in the holdings of the two risky assets that are now closer to the initial allocation 0.5 shares, with a commensurate reduction in borrowing/lending between the investors (that is, because of transactions costs, investors choose allocations in the risky assets that are closer to their initial endowment of shares). For the base case of our model, a 1% transaction cost on Stock 1 leads to a decrease in the magnitude of the bond position from -0.73% to -47% of Investor 1's financial wealth; a decrease in the investment in Stock 1 from 86.5% to 63% of wealth; and, a much smaller decrease in the investment in Stock 2, from 86.5% to 84% of wealth. Thus, transaction costs lead to a sizable decrease in leverage and

¹¹See Dumas (1989) for a detailed description of the choice of portfolios when the two agents differ in their relative risk aversion.

investment in the risky asset that incurs the transaction cost, but only a small decrease in the investment in the other risky asset (that does not incur the transaction cost).

When there is a decrease in EIS of both agents from the base case value of 1.5 to 0.5, Investor 1's investment in the risky assets is larger: 101% instead 86.5% when there are no transactions costs, and when a 1% transaction cost is introduced, 74% instead of 63% in the base case. The change in investment, upon the introduction of transaction costs, is about the same for both values of EIS. Similarly, an increase in the heterogeneity in risk aversion of the two agents (RRA of 2 and 7 instead of 2 and 5) increases the risk-sharing motive and lead to more leverage and a larger deviation from the 50% investment in the two assets with which the investors started: in this case, in the absence of transactions costs, Investor 1 has 99% instead of 86.5% of wealth invested in Stock 1, which with a 1% transaction cost drops to 74% instead of 63%.

If one worked with power utility functions instead of the Epstein-Zin-Weil utility specification we have adopted, then for the case where the two agents have RRA of 2 and 5, which for power utility implies that their EIS is $1/2$ and $1/5$, we get virtually the same portfolios as the one described in the paragraph above for the case where $EIS = 0.5$. In the case of log utility, the investor invests 50% of wealth in each of the two stocks and do not borrow/lend; and, the introduction of a transaction cost of 1% has no effect on portfolio weights. For the case of exponential utility where the two investors have an absolute risk aversion (ARA) of 0.75 and 1.25, we get results that are similar to the base case: Investor 1 borrows 44% of his wealth, and invests 72% in the two risky assets; with a 1% transaction cost, leverage drops to 17%, investment in Stock 1 decreases to 50% and that in Stock 2 decreases to 67%. An increase in heterogeneity where the ARA of the two agents is 0.5 and 2 instead of 0.75 and 1.25 has only a small effect on the portfolio composition.

5.3 Portfolio Turnover

If the two agents were identical, then there would no trade in securities and turnover would be zero. When the two agents have the same RRA and EIS, but have different labor income processes, then, as we have described above, the two agents engage in consumption smoothing

by trading mostly in bonds. Similarly, when the two agents differ only with respect to EIS, they trade only in bonds. The volume of trading, however, is much higher when agents differ also with respect to their RRA. In the base case of our model, the average turnover per period is 0.828 for the bond, and 0.074 for each of the two stocks.

With the introduction of transaction costs, there is significant decrease in turnover. A 0.50% transaction cost on Stock 1 reduces turnover to 0.688 for the bond, 0.005 for Stock 1, and 0.036 for Stock 2. A 1% transaction cost on Stock 1 reduces turnover to 0.585 for the bond, 0.0001 for Stock 1, and 0.035 for Stock 2.

We can also measure the fraction of nodes (date-states) where we trade the bond and stocks. If there are no transactions costs, then for the base case of our model, we trade the bond and the two stocks at every node. When transaction costs for trading Stock 1 are introduced, the agents continue to trade the bond and Stock 2 at every node. However, there is a decrease in the fraction of nodes where Stock 1 is traded. With a 0.50% transaction cost on Stock 1, the fraction of nodes where Stock 1 is traded drops to 0.124, and with a 1% transaction cost, the fraction of nodes where Stock 1 is traded is only 0.001.

Our analysis suggests that a transaction cost on a stock has a large effect on its turnover, but a much smaller effect on the turnover of other stocks, and an even smaller effect on the turnover of the bond.

5.4 Asset Prices

The introduction of the transaction cost for Stock 1 results in less efficient allocation of aggregate output across the two agents. This has an effect on the price of not just Stock 1, but also on the prices of the bond and Stock 2. In the base case of our model, in the absence of transactions costs, the price of the one period bond is 0.9838, and the price of Stock 1 and Stock 2 is 5.7680. With the introduction of a transaction cost of 0.50% on Stock 1, the price of the bond drops by 0.18%, the price of Stock 1 drops by 0.28%, and the price of Stock 2 drops by 0.09%. With a transaction cost of 1% on Stock 1, relative to the base case without transaction costs, the price of the bond drops by 0.21%, the price of Stock 1 drops by 0.44%, and the price of Stock 2 drops by 0.05% (yes, it increases relative to the case of 0.50% transaction cost).

Note that in the absence of transaction costs, both investors trade the bond and the two stocks at each node, and agree on the prices of these assets, which are essentially the payoffs of the assets appropriately discounted. With transaction costs incurred for trading Stock 1, we have discussed above, that the bond and Stock 2 are still traded at each node; thus, agents will always agree on the price of the one-period discount bond and Stock 2. But, because Stock 1 is not traded at all the nodes, agents can disagree at those nodes about the price of this stock. However, at date 0 the investors always trade both stocks for the level of transaction costs we have considered (from 0% to 2%), and so at date 0 there is agreement about the prices of all assets.

Observe also that if one works with a low value of EIS, the prices of assets are lower: compared to the base case prices of 0.9838 for the bond and 5.7680 for the two stocks in the absence of transactions costs and EIS of 1.5, if we set EIS to 0.5 then the price of the bond is only 0.9374 for the bond and 4.9773 for the stocks. Thus, it is not surprising that in the case of power utility with RRA of 2 and 5 for the two investors, one gets prices that are similar to the case of Epstein-Zin-Weil preferences with an EIS of 0.5: in the absence of transaction costs, with power utility the bond price is 0.9073, and the prices of the two stocks are 4.5183. In the model with power utility, a 1% transaction cost on Stock 1 causes the bond price to drop by 1.42%, the price of Stock 1 to drop by 1.62%, and the price of Stock 2 to drop by 1.29%; these changes, quite clearly, are much larger than for the case of Epstein-Zin-Weil utility functions.

In the case of exponential utility, the difference in results is even more striking. For the case where the absolute risk aversion (ARA) of the two investors is 0.75 and 1.25, the introduction of a 1% transaction cost *increases* the bond price by 0.04%, *increases* the price of Stock 1 by 0.95% from the point of view of Investor 1 and decreases it by 0.79% from the point of view of Investor 2, and *increases* the price of Stock 2.

In the special case where both investors have log utility, there is almost no effect of a 1% transaction cost on asset prices.

The above experiments suggest that using a particular functional form for the utility function can have a strong bearing, not just on the quantitative results, but also about the qualitative inferences about the effect of transaction costs on asset prices.

5.5 Equity Risk Premium and the Risk-free Rate

The equity risk premium on each stock, that is the return of each stock in excess of the risk-free rate, is 3.92% p.a. when both investors have the same preferences (with RRA of 3.5 and EIS of 1.5) and there is no labor income. If we endow the two agents with the same labor income, the equity premium decreases to 2.22% p.a. If we introduce heterogeneity in RRA so that the RRA of the two agents is 2 and 5, (but no labor income), then the equity risk premium changes to 3.24% instead of 3.92% p.a. In our base case, where we have both heterogeneity in RRA and heterogeneous labor income, the equity risk premium is only 1.62% p.a. A transaction cost of 1% on Stock 1 increases the equity risk premium for Stock 1 by about 0.14% and *decreases* it for Stock 2 by about 0.09%; that is, not only is the effect of transaction costs on the equity risk premium for each asset one order of magnitude smaller than the transaction cost, but it also has opposite effects on the two stocks, implying that the net effect on the economy-wide equity risk premium is even smaller.

The one-period risk-free rate is equal to the inverse of the bond price minus 1. We have already discussed the bond price above, so we will not discuss it again. However, it is worth pointing out that only under the Epstein-Zin-Weil specification is the interest rate at a reasonable level compared to its empirical value of about 1% p.a. For our base case, in the absence of transaction costs, the interest rate is 1.65% p.a. and in the presence of a 1% transaction cost, the interest rate is 1.86% p.a. However, for power utility (with agents having the same RRA as in the base case), the interest rate is 10.22% without transactions costs, and it increases to 11.81% with transactions costs. With log utility for both agents, the interest rate is 5.44% for both the case without and with transaction costs. With exponential utility, with the absolute risk aversion of the two investors being 0.75 and 1.25, the interest rate is 9.32% without transaction costs and 9.27% with a 1% transaction cost, which decreases to 8.57% p.a. when the two agents have absolute risk aversion of 0.5 and 2.0.

5.6 Liquidity Risk Premium

In this section, we wish to study the effect of transaction costs on the liquidity risk premium, and compare the magnitude of this effect to that in Lynch and Tan (2011), where there is a

single-agent with power utility, and asset prices and the interest rate are exogenous. We define the liquidity risk premium to be the extra return that has to be paid by Stock 1, which incurs a transaction cost, relative to the return on Stock 2, which can be traded at zero cost. This is the definition used by Lynch and Tan (2011) and other papers in this literature.

Lynch and Tan (2011) find that the liquidity risk premium is of the same order as the transaction cost. The main driving force for this result is the idiosyncratic labor income. Because of the permanent shocks to labor income, the investor desires to smooth consumption over time and across states of nature; but, because labor income shocks are not spanned by the financial assets, these shocks cannot be hedged perfectly, and hence, this forces investors to rebalance their portfolios frequently. Consequently, the transaction cost has a larger bite than it would in the absence of labor income shocks.

In order to make sure that we are comparing apples with apples when comparing the liquidity risk premium in our model with that in Lynch and Tan (2011), we first reproduce the large liquidity risk premium in a *partial-equilibrium* version of our economy (with investors who have power utility). The main difference in the partial equilibrium version of our model and that studied by Lynch and Tan (2011) is in the specification of the transaction cost function. They require the investor to invest the full labor income in financial assets and to finance consumption by selling financial assets, so the transaction cost function is given by:

$$\left| \theta_t - \frac{\theta_{t-1} \frac{W_t}{Y_{t-1}}}{\frac{W_t}{Y_{t-1}} + \exp(g_t)} \right| \times \kappa;$$

we, on the other hand, allow the investor to consume labor income directly, and transaction costs are incurred only for rebalancing the portfolio, which means that the transaction cost function is:

$$|\theta_t - \theta_{t-1}| \times S_t \times \kappa.$$

We find that the liquidity risk premium that we get in the *partial-equilibrium* version of our model is similar to that obtained by Lynch and Tan (2011). For example, if the ratio of financial wealth to labor income is 5, then the liquidity risk premium for a 1% transaction cost in the Lynch and Tan model is 0.55%, and in our model it is 0.58%.

Now, we evaluate the liquidity risk premium in general equilibrium. For the base case of our general-equilibrium model with two heterogeneous investors, when the ratio of financial wealth to labor income is 5, we find that the liquidity risky premium is 0.23% for a transaction cost of 1%. That is, the liquidity premium is about one quarter of the transaction cost, instead of being of about half in the partial-equilibrium setting. When the ratio of financial wealth to labor income is 20, the liquidity risky premium is 0.16% for a transaction cost of 1%. When the ratio of financial wealth to labor income is 3, the liquidity risky premium is 0.24% for a transaction cost of 1%.

In the absence of labor income, the liquidity premium for a 1% transaction cost is only 0.13%; that is, about one-seventh of the transaction cost. The liquidity risk premium is not very sensitive to the growth rate of labor income. However, it does change with the volatility of labor income: in the base case, the volatility of labor income for Investor 2 is 30% p.a., and the liquidity risk premium for this case is 0.23%. If we decrease the volatility of labor income for Investor 2 to 20%, the liquidity premium is 0.20%; and, if we further decrease the volatility of labor income for Investor 2 to 10%, then the liquidity premium is 0.13%.

With a power utility function (where the RRA of the investors is 2 and 5), the liquidity risk premium is a bit smaller: instead of 0.23% in the base case, it is 0.19%. With exponential utility, the liquidity risk premium is 0.04% when the absolute risk aversion of the two investors is 0.75 and 1.25; and, the liquidity risk premium increases to 0.24% if the absolute risk aversion of the two investors is 0.50 and 2. With log utility, the liquidity risk premium is zero.

From the above discussion, we observe that the greater the desire to trade financial securities (for instance, when volatility of dividends is higher), the larger is the liquidity risk premium; but, in the cases we have considered, it is never greater than one-quarter of the transaction cost.

5.7 Stock-return Volatility

We know that if the two investors have the same preferences and they do not have any labor income, then the volatility of stock returns should be equal to “fundamental volatility,” the volatility of dividends, which in our base case is set equal to 15% p.a. In our base case, where

the two investors are heterogeneous and are endowed with labor income, the volatility of the two stocks is 13.42%, slightly lower than 15%. Bhamra and Uppal (2009) explain why volatility can be below fundamental volatility when investors are heterogeneous. Transactions costs have a negligible effect on the volatility of stock returns.

6 Conclusion

In this paper, we develop a method that allows us to obtain asset prices in a general equilibrium economy with multiple agents who are *heterogeneous* when there are proportional costs for trading financial assets. The agents in our model have Epstein-Zin-Weil utility functions and can be heterogeneous with respect to endowments and all three characteristics of their utility functions—time preference, risk aversion, and elasticity of intertemporal substitution. The securities traded in the financial market include a one-period discount bond and *multiple* risky stocks. Our method allows us to identify the equilibrium in a recursive fashion even in the presence of transactions costs, which make markets incomplete. This is in contrast to the usual approach for identifying the equilibrium in a general equilibrium model with incomplete financial markets, where one needs to iterate backwards and forwards in time until the system converges. We use our model to study the effect of transactions costs on the consumption and portfolio choices of investors, and the resulting impact on asset prices and risk premia.

We find that aggregate *consumption* decreases with transaction costs, because some resources are devoted to transaction costs. However, the drop in aggregate consumption is quite small: a transaction cost of 1% reduces consumption by 0.3% for the base case we consider. Transaction costs also have a small effect on the distribution of consumption between the two investors.¹² Transaction costs also lead to a substantial change in leverage and *portfolio* investment in the risky asset that incurs the transaction cost, but only a small change in the investment in the other risky asset. The change in demand for assets is reflected in their prices.

¹²Note that the effect of transactions costs on aggregate consumption and also on the share of consumption is much greater with power utility. On the other hand, when both agents have log utility, the effect of transactions costs is negligible, while for exponential utility the magnitude of the results depends on the heterogeneity in risk aversion.

For the base case of our model, a transaction cost of 1% on Stock 1, the price of the bond drops by 0.21%, the price of Stock 1 drops by 0.44%, and the price of Stock 2 drops by 0.05%.¹³

The effect of a transaction cost on the *equity risk premium* is small. A transaction cost of 1% on Stock 1 increases the equity risk premium for Stock 1 by about 0.14% and *decreases* it for Stock 2 by about 0.09%; that is, not only is the effect of transaction costs on the equity risk premium for each asset one order of magnitude smaller than the transaction cost, but it also has opposite effects on the two stocks, implying that the net effect on the economy-wide equity risk premium is even smaller. Transactions costs also have a negligible effect on the volatility of stock returns. The liquidity risk premium generated by a transaction cost is smaller in general equilibrium than in the partial-equilibrium model considered in Lynch and Tan (2011). In the cases we have considered, the liquidity risk premium is less than one-quarter of the transaction cost instead of being of the same order of magnitude as the transaction cost.

¹³However, if one works with exponential utility, the introduction of transaction costs for Stock 1 leads to an *increase* in the prices of the assets.

A The Proofs for Dynamic Programming

A.1 The Derivations of the First-Order Conditions

Note, for ease of exposition we omit the subscript k for agent k and concentrate on the case for one asset with transaction costs. The case with several assets subject to transaction costs follows accordingly. Below we show that the first-order conditions derived from the recursive utility function and from the indirect utility function in the dynamic programming formulation are the same.

A.1.1 Global Solution

We have to solve at *each time t simultaneously* the following problem:

$$\sup_{c_t, \theta_t} V_t = \sup_{c_t, \theta_t} \left[(1 - \beta) c_t^{1 - \frac{1}{\psi}} + \beta E_t \left[V_{t+1}^{1 - \gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1 - \gamma}} \quad (\text{A1})$$

subject to for all points in time t :

$$c_t + \theta_t \cdot S_t + \tau(\theta_t, \theta_{t-1}) = \theta_{t-1} \cdot (S_t + d_t) \quad (\text{A2})$$

Form the Lagrangian with ξ_t as the multiplier for the budget constraint:

$$\begin{aligned} \mathcal{L}_t = & \left[(1 - \beta) c_t^{1 - \frac{1}{\psi}} + \beta E_t \left[V_{t+1}^{1 - \gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1 - \gamma}} \\ & + \xi_t \cdot (\theta_{t-1} \cdot (S_t + d_t) - c_t - \theta_t \cdot S_t - \tau(\theta_t, \theta_{t-1})), \end{aligned} \quad (\text{A3})$$

and take the first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial c_t} &= \frac{\phi}{1 - \gamma} \cdot \left[(1 - \beta) c_t^{1 - \frac{1}{\psi}} + \beta E_t \left[V_{t+1}^{1 - \gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1 - \gamma} - 1} \\ &\quad \cdot (1 - \beta) \cdot \left(1 - \frac{1}{\psi} \right) \cdot c_t^{-\frac{1}{\psi}} - \xi_t \\ &= V_t^{\frac{1}{\phi}} \cdot c_t^{-\frac{1}{\psi}} \cdot (1 - \beta) - \xi_t = 0 \\ \frac{\partial \mathcal{L}_t}{\partial \xi_t} &= \theta_{t-1} \cdot (S_t + d_t) - c_t - \theta_t \cdot S_t - \tau(\theta_t, \theta_{t-1}) = 0 \end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \theta_t} &= \frac{\phi}{1-\gamma} \cdot \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}-1} \cdot \frac{\beta}{\phi} \cdot E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \\
&\quad \cdot (1-\gamma) \cdot E_t \left[V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial \theta_t} \right] - \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right) \\
&= V_t^{\frac{1}{\psi}} \cdot \beta \cdot E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \cdot E_t \left[V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial \theta_t} \right] - \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right) = 0 \quad (\text{A4})
\end{aligned}$$

Now use that:

$$\begin{aligned}
\frac{\partial V_{t+1}}{\partial \theta_t} &= \frac{\partial V_{t+1}}{\partial c_{t+1}} \cdot \frac{\partial c_{t+1}}{\partial \theta_t} \\
&= \left[V_{t+1}^{\frac{1}{\psi}} \cdot c_{t+1}^{-\frac{1}{\psi}} \cdot (1-\beta) \right] \cdot \left[S_{t+1} + d_{t+1} - \frac{\partial \tau(\theta_{t+1}, \theta_t)}{\partial \theta_t} \right]
\end{aligned}$$

and plug into (A4), we get:

$$\begin{aligned}
&V_t^{\frac{1}{\psi}} \cdot \beta \cdot E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \cdot E_t \left[V_{t+1}^{-\gamma+\frac{1}{\psi}} c_{t+1}^{-\frac{1}{\psi}} \cdot (1-\beta) \cdot \left[S_{t+1} + d_{t+1} - \frac{\partial \tau(\theta_{t+1}, \theta_t)}{\partial \theta_t} \right] \right] \\
&= \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right)
\end{aligned}$$

A.1.2 Dynamic Programming Solution

Define the value function recursively as:

$$\begin{aligned}
J_T &= \sup_{\{c_T\}} (1-\beta) c_T \\
J_t &= \sup_{\{c_t\}} \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}}
\end{aligned}$$

and write the problem as follows

$$J_t = \sup_{c_t, \theta_t} \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}} \quad (\text{A5})$$

subject to (A2). The Lagrangian:

$$\mathcal{L}_t = \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}} + \xi_t \cdot (\theta_{t-1} \cdot (S_t + d_t) - c_t - \theta_t \cdot S_t - \tau(\theta_t, \theta_{t-1}))$$

with first-order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_t} &= \frac{\phi}{1-\gamma} \cdot \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}-1} \\
&\quad \cdot (1-\beta) \cdot \left(1 - \frac{1}{\psi} \right) \cdot c_t^{-\frac{1}{\psi}} - \xi_t \\
&= J_t^{\frac{1}{\psi}} \cdot c_t^{-\frac{1}{\psi}} \cdot (1-\beta) - \xi_t \\
&= 0.
\end{aligned} \tag{A6}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \xi_t} &= \theta_{t-1} \cdot (S_t + d_t) - c_t - \theta_t \cdot S_t \\
&= 0.
\end{aligned} \tag{A7}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \theta_t} &= \frac{\phi}{1-\gamma} \cdot \left[(1-\beta) c_t^{1-\frac{1}{\psi}} + \beta E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}} \right]^{\frac{\phi}{1-\gamma}-1} \cdot \frac{\beta}{\phi} \cdot E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \\
&\quad \cdot (1-\gamma) \cdot E_t \left[J_{t+1}^{-\gamma} \frac{\partial J_{t+1}}{\partial \theta_t} \right] - \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right) \\
&= J_t^{\frac{1}{\psi}} \cdot \beta \cdot E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \cdot E_t \left[J_{t+1}^{-\gamma} \frac{\partial J_{t+1}}{\partial \theta_t} \right] - \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right) \\
&= 0.
\end{aligned} \tag{A8}$$

The envelope theorem gives us:

$$\frac{\partial J_t}{\partial \theta_{t-1}} = \frac{\partial \mathcal{L}_t}{\partial \theta_{t-1}} = \xi_t \cdot \left(S_t + d_t - \frac{\partial \tau(\theta_t, \theta_{t-1})}{\partial \theta_{t-1}} \right),$$

which we can plug into (A8) to get:

$$\begin{aligned}
&J_t^{\frac{1}{\psi}} \cdot \beta \cdot E_t \left[J_{t+1}^{1-\gamma} \right]^{\frac{1}{\phi}-1} \cdot E_t \left[J_{t+1}^{-\gamma+\frac{1}{\psi}} c_{t+1}^{-\frac{1}{\psi}} \cdot (1-\beta) \cdot \left[S_{t+1} + d_{t+1} - \frac{\partial \tau(\theta_{t+1}, \theta_t)}{\partial \theta_t} \right] \right] \\
&= \xi_t \cdot \left(S_t + \frac{\tau(\theta_t, \theta_{t-1})}{\partial \theta_t} \right)
\end{aligned} \tag{A9}$$

The first-order conditions for the global case and the recursive case are the same, except that in the global formulation the utility function V_t shows up, and in the recursive formulation the value function J_t shows up. However, in optimum, when we solve the global case for all t , the utility function and the value function are the same.

Table 1: Base-case Parameter Values

Parameter	Symbol	Base case value	Range considered
Parameter for transaction cost			
Rate of proportional transaction cost	κ	0.00	0.00–0.02
Parameters of the utility functions			
Time discount factor for Agents 1 and 2	β	0.99	—
Elasticity of intertemporal substitution of Agent 1	ψ_1	1.50	0.5–1.5
Elasticity of intertemporal substitution of Agent 2	ψ_2	1.50	0.5–1.5
Relative risk aversion of Agent 1	γ_1	2.00	1.0–5.0
Relative risk aversion of Agent 2	γ_2	5.00	1.0–7.0
Absolute risk aversion (for exponential utility) of Agent 1	α_1	0.75	0.5–1.0
Absolute risk aversion (for exponential utility) of Agent 2	α_2	1.25	1.0–2.0
Parameters for the labor-income processes			
Level of initial labor income for Agents 1 and 2		1.00	0–2.00
Drift of labor income process for Agent 1	$\bar{g}_1$	0.03	0–0.03
Drift of labor income process for Agent 2	$\bar{g}_2$	0.04	0–0.04
Volatility of labor income process for Agent 1	$\sigma_{u,1}$	0.14	0–0.14
Volatility of labor income process for Agent 2	$\sigma_{u,2}$	0.30	0–0.30
Parameters for the dividend processes			
Level of initial dividends for Agents 1 and 2		1.00	—
Drift of dividend process for Stock 1	μ_1	0.08	0.06–0.10
Drift of dividend process for Stock 2	μ_2	0.08	0.06–0.10
Volatility of dividend process for Stock 1	σ_1	0.15	0.13–0.18
Volatility of dividend process for Stock 2	σ_2	0.15	0.13–0.18
Correlation between dividends of Stock 1 and 2	ρ	0.00	—

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