Funding, Collateral and Hedging
Uncovering the Mechanics and the Subtleties of Funding Valuation Adjustments

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Talk Outline

1. Collateralized Credit and Funding Valuation Adjusted Pricing
2. Derivative Pricing and Bank’s Structure
3. Conclusions and Future Developments
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“Counterparty Credit Risk, Collateral and Funding with Pricing Cases for All Asset Classes”
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“Counterparty Risk under Correlation between Default and Interest-Rates".
“Funding, Collateral and Hedging: consistent costs of funding”.  
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Working papers available at SSRN  
→ http://ssrn.com/abstract=1969114  
→ http://ssrn.com/abstract=2161528

Brigo, D., Capponi, A., Pallavicini, A. (2011)  
“Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swap”.  
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Talk Outline

1. Collateralized Credit and Funding Valuation Adjusted Pricing
   - Funding, Collateral and Hedging
   - Continuous Time Approximation
   - Treasury Funding Operations
   - Funding Costs and DVA

2. Derivative Pricing and Bank’s Structure

3. Conclusions and Future Developments
Global Pricing Framework

- A pricing framework inclusive of counterparty risk, collaterals, and funding costs does not require a new pricing theory.
  → We adopt a risk neutral approach.

- We only need a detailed description of all cash flows occurring when the trading position is entered.
  → derivative’s cash flows (e.g. coupons, dividends, etc.);
  → cash flows required by the collateral margining procedure;
  → cash flows required by the funding and hedging procedures;
  → cash flows occurring on default events.

- Prices can be calculated under risk-neutral measure $\mathbb{Q}$ by discounting with the risk-free discount factor $D(t, T)$.

- We will see that whenever cash and risky assets are bought via treasury or market contracts, the risk-free rate disappears from the pricing equations.
Collateralized Credit and Funding Valuation Adjusted Pricing

Funding, Collateral and Hedging

Derivative’s Cash Flows

- We calculate prices by discounting cash-flows under risk-neutral measure.
- We start from derivative’s cash flows.

$$\bar{V}_t(C; F) := \mathbb{E}_t[\Pi(t, T \land \tau) + \ldots]$$

where
- $\tau := \tau_C \land \tau_I$ is the first default time, and
- $\Pi(t, u)$ is the sum of all discounted payoff terms up to time $u$,

- In particular, if the derivative’s contract has a stream of coupon $\pi$ payed on the time grid $\{t_i\}$, we have

$$\Pi(t, u) := \sum_{i=1}^{p} 1_{\{t < t_i \leq u\}} \pi_{t_i} D(t, t_i)$$
Collateral Procedure – I

As second contribution we consider the collateralization procedure and we add its cash flows.

\[
\tilde{V}_t(C; F) := \mathbb{E}_t[\Pi(t, T \wedge \tau)] + \mathbb{E}_t[\gamma(t, T \wedge \tau; C) + 1_{\{t < \tau < T\}}D(t, \tau)C_{\tau-} + \ldots]
\]

where

- \(\rightarrow C_t\) is the collateral account defined by the CSA,
- \(\rightarrow C_{\tau-}\) is the pre-default value of the collateral account, and
- \(\rightarrow \gamma(t, u; C)\) are the collateral margining costs up to time \(u\).

Notice that when applying close-out netting rules, first we will net the exposure against \(C_{\tau-}\), then we will treat any remaining collateral as an unsecured claimed.
Collateral Procedure – II

- The cash flows due to the margining procedure on the time grid \( \{ t_k \} \) are equal to

\[
\gamma(t, u; C) := \sum_{k=1}^{n-1} \mathbf{1}_{t \leq t_k < u} D(t, t_k) C_{t_k} \left( 1 - \frac{P_{t_k}(t_{k+1})}{P_{t_k}^\tilde{c}(t_{k+1})} \right)
\]

where the collateral accrual rates and zero-coupon bonds are given by

\[
\tilde{c}_t := c_t^+ \mathbf{1}_{c_t > 0} + c_t^- \mathbf{1}_{c_t < 0}, \quad P_{t}^\tilde{c}(T) := \frac{1}{1 + (T - t)\tilde{c}_t(T)}
\]

- Then, according to CSA, we introduce the pre-default value of the collateral account \( C_{\tau^-} \) as

\[
C_{\tau^-} := \sum_{k=1}^{n-1} \mathbf{1}_{t_k \leq \tau < t_{k+1}} C_{t_k} \frac{P_{\tau}(t_{k+1})}{P_{t_k}^\tilde{c}(t_{k+1})}
\]
As third contribution we consider the cash flow occurring on default event, and we have

\[
\bar{V}_t(C; F) := \mathbb{E}_t[\Pi(t, T \wedge \tau)] \\
+ \mathbb{E}_t[\gamma(t, T \wedge \tau; C) + 1_{\{t<\tau<T\}}D(t, \tau)C_{\tau-}] \\
+ \mathbb{E}_t[1_{\{t<\tau<T\}}D(t, \tau)(\theta_{\tau}(C, \varepsilon) - C_{\tau-}) + \ldots]
\]

where

\(\theta_{\tau}(C, \varepsilon)\) is the on-default cash flow, and

\(\varepsilon_{\tau}\) is the amount of losses or costs the surviving party would incur on default event (close-out amount).

We define \(\theta_{\tau}\) including the pre-default value of the collateral account since it is used by the close-out netting rule to reduce exposure.
Close-Out Netting Rules – II

- The close-out amount is not a symmetric quantity w.r.t. the exchange of the role of two parties, since it is valued by one party after the default of the other one. More details in the examples and in Brigo, Capponi, Pallavicini and Papatheodorou (2011).

\[ \varepsilon_\tau := 1_{\{\tau = \tau_C\}} \varepsilon_{I,\tau} + 1_{\{\tau = \tau_I\}} \varepsilon_{C,\tau} \]

- The on-default cash flow \( \theta_\tau(C, \varepsilon) \) can be calculated by following ISDA documentation.

- In particular, if collaterals can be re-hypothecated, we obtain

\[
\theta_\tau(C, \varepsilon) := \varepsilon_\tau - L_{GD_C}(\varepsilon_{I,\tau} - C_{\tau-})^+ - L_{GD_I}(\varepsilon_{C,\tau} - C_{\tau-})^-
\]

where loss-given-defaults are defined as \( L_{GD_C} := 1 - R_C \), and so on.
Funding and Hedging – I

As fourth and last contribution we consider the funding and hedging procedures and we add their cash flows.

\[ \bar{V}_t(C; F) := E_t[\Pi(t, T \land \tau)] + E_t[\gamma(t, T \land \tau; C) + 1_{\{t<\tau<T\}}D(t, \tau)\theta_\tau(C, \varepsilon)] + E_t[\varphi(t, T \land \tau; F, H)] \]

where

\[ \rightarrow F_t \text{ is the cash account needed for trading,} \]
\[ \rightarrow H_t \text{ is the risky-asset account implementing the hedging strategy, and} \]
\[ \rightarrow \varphi(t, u; F, H) \text{ are the funding and hedging costs up to time } u. \]
The cash flows due to the funding and hedging strategy on the time grid \( \{ t_j \} \) are equal to

\[
\varphi(t, u; F, H) := \sum_{j=1}^{m-1} 1\{ t \leq t_j < u \} D(t, t_j) F_{t_j} \left( 1 - \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{F}}(t_{j+1})} \right) \\
- \sum_{j=1}^{m-1} 1\{ t \leq t_j < u \} D(t, t_j) H_{t_j} \left( \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{F}}(t_{j+1})} - \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{H}}(t_{j+1})} \right)
\]

where the funding and lending rates are given by

\[
\tilde{f}_t := f_t^+ 1\{ F_t > 0 \} + f_t^- 1\{ F_t < 0 \}, \quad \tilde{h}_t := h_t^+ 1\{ H_t > 0 \} + h_t^- 1\{ H_t < 0 \}
\]

and zero-coupon bonds are defined as in the collateral case.
The amount of cash needed for trading is equal to the amount of cash needed to establish the hedging strategy. Thus, if collateral re-hypothecation is allowed, we have

\[ F_t = \tilde{V}_t(C, F) - C_t - H_t \]

Since the value of the cash \( F_t \) depends on the price \( \tilde{V}_t \) of the derivative, which, in turn, depends on such process, we obtain a recursive pricing equation.

- Numerical solutions based on BSDE techniques can be used to solve the general problem.
- See Burgard and Kjaer (2010,2011) and Crépey et al. (2011,2012) for further examples.
We can simplify the pricing equation, for the examples in the following section, by considering that collateralization, funding and hedging is done in continuous time.

Moreover, we assume that re-hypothecation is holding.

When we consider all time-grids in continuous time we have

\[
\Pi(t, u) = \int_t^u d\pi_v D(t, v), \quad \gamma(t, u; C) = \int_t^u dv \left( r_v - \tilde{c}_v \right) C_v D(t, v)
\]

\[
\varphi(t, u; F, H) = \int_t^u dv \left( r_v - \tilde{f}_v \right) \left( \bar{V}_v(C, F) - C_v \right) D(t, v)
\]

\[
- \int_t^u dv \left( r_v - \tilde{h}_v \right) H_v D(t, v)
\]
Then, if we switch to market filtration $\mathcal{F}$, we obtain

\[
\bar{V}_t(C; F) = 1\{\tau>t\} \int_t^T du \mathbb{E}_t \left[ (\partial_u \pi_u + \lambda_u \theta_u(C, \varepsilon)) D(t, u; r + \lambda) | \mathcal{F}_t \right] \\
+ 1\{\tau>t\} \int_t^T du \mathbb{E}_t \left[ (\tilde{f}_u - \tilde{c}_u) C_u D(t, u; r + \lambda) | \mathcal{F}_t \right] \\
+ 1\{\tau>t\} \int_t^T du \mathbb{E}_t \left[ (r_u - \tilde{f}_u) \bar{V}_u(C, F) D(t, u; r + \lambda) | \mathcal{F}_t \right] \\
- 1\{\tau>t\} \int_t^T du \mathbb{E}_t \left[ (r_u - \tilde{h}_u) H_u D(t, u; r + \lambda) | \mathcal{F}_t \right]
\]

where $\lambda_t$ is the first-default intensity, and

\[
D(t, T; x) := \exp \left\{ - \int_t^T du x_u \right\}
\]
Continuous Time Approximation – III

- We can write the corresponding PDE if we assume that the hypotheses of the Feynman-Kac theorem are holding.
- In particular, we assume that the underlying risk factors are Markov with infinitesimal generator $\mathcal{L}_t$.
- In such case we get for $\tau > t$

$$
\left( \partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}_t \right) \tilde{V}_t(C; F) - (r_t - \tilde{h}_t)H_t + (\tilde{f}_t - \tilde{c}_t)C_t + \partial_t \pi_t + \lambda_t \theta_t(C, \varepsilon) = 0
$$

with boundary condition

$$
\tilde{V}_\tau(C; F) = 0
$$

- Notice we consider all cash flows within the coupon stream $\pi$, so that the boundary condition is zero.
If we consider a diffusive dynamics, and we assume delta-hedging, we can expand the generator $\mathcal{L}_t$ in term of first and second order operators, and we get

$$\mathcal{L}_t \tilde{V}_t(C; F) \equiv (\mathcal{L}^1_t + \mathcal{L}^2_t) \tilde{V}_t(C; F) \equiv r_t H_t + \mathcal{L}^2_t \tilde{V}_t(C; F)$$

Hence, the price PDE becomes for $\tau > t$

$$\left( \partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}^h_t \right) \tilde{V}_t(C; F) + (\tilde{f}_t - \tilde{c}_t) C_t + \partial_t \pi_t + \lambda_t \theta_t(C, \varepsilon) = 0$$

where

$$\mathcal{L}^h_t \tilde{V}_t(C; F) := \tilde{h}_t H_t + \mathcal{L}^2_t \tilde{V}_t(C; F)$$

Notice that the above equation does not depend any more on the risk-free rate.
Continuous Time Approximation – V

- We can apply again Feynman-Kac theorem and we get the result of Pallavicini, Perini, Brigo (2011)

\[
\tilde{V}_t(C; F) := \int_t^T \mathbb{E}_t^{\tilde{h}} \left[ \left( 1_{\{u<\tau\}} d\pi_u + 1_{\{\tau \in du\}} \theta_u(C, \varepsilon) \right) D(t, u; \tilde{f}) \right] \\
+ \int_t^T du \mathbb{E}_t^{\tilde{h}} \left[ 1_{\{u<\tau\}} (\tilde{f}_u - \tilde{c}_u) C_u D(t, u; \tilde{f}) \right]
\]

where the expectations are taken under a pricing measure \( \mathbb{Q}^{\tilde{h}} \) under which the underlying risk factors growth at rate \( \tilde{h} \).

- An explicit specification of the close-out amount \( \varepsilon_{\tau} \) and the collateral process \( C_t \), along with the definition of market and treasury rates \( \{\tilde{f}_t, \tilde{c}_t, \tilde{h}_t\} \), is required to solve the above pricing equation.
The funding rate $\tilde{f}_t$ is determined by the party managing the funding account for the investor, usually the bank’s treasury according to its liquidity policy:

- trading positions may be netted before funding on the market;
- a Funds Transfer Pricing (FTP) process may be implemented to gauge the performances of different business units;
- a maturity transformation rule can be used to link portfolios to effective maturity dates;
- many source of funding can be mixed.

In the literature the role of the treasury is usually neglected, leading to some controversial results.

- The false claim “funding costs are the DVA”, or even “there are no funding costs at all”, are often cited in the practitioners’ literature.
- See the querelle following Hull and White (2012).
Yet, the role of the treasury is crucial within a Bank.

→ Treasury could be designed as a “performance” center, separated from internal business, to implement bank’s business policy.

→ See, for instance, Kratky and Choudhry (2012).

Here, we follow Pallavicini, Perini, Brigo (2011) to investigate the problem.
In general, we assume that on a time grid \( \{ t_j \} \) the treasury opens new positions to fund collateral procedures and traders’ needs of cash:

\[ \rightarrow \] at \( t_j \) the trader asks the funder for a cash amount equal to \( F_{t_j} \);

\[ \rightarrow \] at \( t_{j+1} \) the trader has to reimburse the funder for the cash amount previously obtained and has to pay for funding costs.

In case of default of the investor we have a DVA adjustment.

\[ \rightarrow \] Sometimes in the literature the funding position’s DVA adjustment is known as funding benefit.

In this example we assume \( L_{GD,i} = 1 \), but it is straightforward extending the example to the general case.
Thus, the payoff of the $j$-th funding position is given by

$$\bar{\Phi}_j(t_j, t_{j+1}; F) := -1_{\{\tau_l > t_{j+1}\}} N_{t_j} D(t_j, t_{j+1}) + 1_{\{t_j < \tau_l < t_{j+1}\}} \varepsilon_{F, \tau_l}^+ D(t_j, \tau_l)$$

where $\varepsilon_{F,t}$ is the close-out amount calculated by the (risk-free) funder on investor’s default event, which we assume to be

$$\varepsilon_{F, \tau_l} := -N_{t_j} P_{\tau_l}(t_{j+1})$$

with

$$N_{t_j} := \frac{F^-_{t_j}}{P_{t_j}^{r+\ell^-}(t_{j+1})} + \frac{F^+_{t_j}}{P_{t_j}^{r+\lambda_t^l+\ell^+}(t_{j+1})}$$

where $\ell^{\pm}$ are the market liquidity bases over the risk-free rate $r_t$, and $\lambda_{t}^l$ is the investor’s default intensity.
False Claims: FVA=\text{DVA} and FVA=0 – 1

- We can simplify the funding position’s payoff, if we consider the very stylized case of a treasury
  \rightarrow avoiding any transformation (no netting, no FTP, no maturity rule, etc...), so that a different funding position is opened for each trading position, and
  \rightarrow using only investor’s bond issues to implement the funding strategy.

- Indeed, in such case for lending money ($F_t > 0$) we get

$$
\mathbb{E}_{t_j} [ \Phi_j(t_j, t_{j+1}; F)] = -1_{\{\tau_I > t_j\}} F^+_{t_j} \frac{P_{t_j}^{r+\lambda,c}(t_{j+1})}{P_{t_j}^{r+\lambda,c+\ell,+}(t_{j+1})}
$$

- Furthermore, if we assume that market liquidity bases are independent of other market risks, we get

$$
\mathbb{E}_{t_j} [ \Phi_j(t_j, t_{j+1}; F)] \approx -1_{\{\tau_I > t_j\}} F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{r+\ell,+}(t_{j+1})}
$$
False Claims: FVA = DVA and FVA = 0 – II

- Hence, since the treasury is avoiding any transformation, we can compare the result with the expression for derivative contract’s funding costs, and we can identify

\[ f_t^+ \approx r_t + \ell_t^+ \]

- The resulting picture is a treasury which has no active role any longer in the funding process: the traders are funding directly on the market.
False Claims: $\text{FVA} = \text{DVA}$ and $\text{FVA} = 0$ – III

- The above result is due to a cancellation of default intensities appearing in bond yields and in funding position’s DVA term.
  - We obtain the cancellation only if the treasury is entitled to consider the DVA of funding positions in its balance sheets.
  - We remind the treasury must avoid transformations (no netting, no FTP, no maturity rule, etc...).
  - Moreover, in case of defaultable funders the cancellation is no longer valid.

- This is the source of the false claim: “funding costs are the DVA”.
  - More precisely Hull and White (2012) considers funding costs as the treasury’s DVA.

- If also market liquidity bases are vanishing, then we have not funding costs any longer: $f_t^+ \approx r_t$.
  - This is the source of the false claim: “there are no funding costs at all”.
As a concluding remark, we can cite BIS documentation where the importance of pricing funding liquidity risk is heavily stressed.

“Probably the most striking example of poor practice [...] was that some banks [...] came to view funding liquidity as essentially free, and funding liquidity risk as essentially zero. [...] This approach resulted in the hoarding of long-term highly illiquid assets, and very few long-term stable liabilities to meet funding demands as they became due.”

[“Liquidity Transfer Pricing: a guide to better practice” – BIS (2011)]
Talk Outline

1. Collateralized Credit and Funding Valuation Adjusted Pricing

2. Derivative Pricing and Bank’s Structure
   - The CVA Desk as a Treasury Department
   - Pricing Collateralized Interest-Rate Derivatives
   - The CVA Desk as a Trading Desk
   - Pricing Collateralized Contracts with Gap Risk

3. Conclusions and Future Developments
Effective Discount Factors – I

- We wish to approximate further the pricing equation in continuous time to write the price as an expectation over the derivative’s discounted cash flows.
- In order to achieve such goal, we assume that gap risk is not present, and we consider a particular form for collateral and close-out prices.

\[ C_t \doteq \alpha_t \tilde{V}_t(C, F), \quad \varepsilon_\tau \doteq \tilde{V}_\tau(C, F) \]

with \( 0 \leq \alpha_t \leq 1 \).
- This approach is considered in Brigo, Morini, Pallavicini (2013) and in Biffis et al. (2012).
Effective Discount Factors – II

- We obtain by switching to market filtration $\mathcal{F}$

$$\tilde{V}_t(C; F) = 1_{\{\tau>t\}} \int_t^T \tilde{E}_t^h \left[ d\pi_u D(t, u; \tilde{f} + \lambda) \middle| \mathcal{F}_t \right]$$

$$- 1_{\{\tau>t\}} \int_t^T du \tilde{E}_t^h \left[ (\tilde{\zeta}_u - \lambda_u) \tilde{V}_u(C, F) D(t, u; \tilde{f} + \lambda) \middle| \mathcal{F}_t \right]$$

where we define

$$\tilde{\zeta}_t := (1 - \alpha_t) \left( \lambda_t^{C<\langle LGD} 1_{\{\tilde{V}_t(C; F)>0\}} + \lambda_t^{I<\langle LGD} 1_{\{\tilde{V}_t(C; F)<0\}} \right) - \alpha_t (\tilde{f}_t - \tilde{c}_t)$$

- Now, it is possible to apply again the Feynman-Kac theorem to obtain a differential equation, which can be solved to obtain

$$\tilde{V}_t(C; F) = 1_{\{\tau>t\}} \int_t^T \tilde{E}_t^h \left[ d\pi_u D(t, u; \tilde{f} + \tilde{\zeta}) \middle| \mathcal{F}_t \right]$$
The CVA Desk as a Treasury Department

Such result has implications for the treasury and the CVA desk within the Bank’s structure.

→ The collateral office, calculating $\alpha_t$, is embedded within the treasury and not visible by trading desks.

→ The CVA desk may lock the “all-inclusive” spread $\tilde{\zeta}_t$ for traders with a role similar to the treasury fixing the funding rate $\tilde{f}_t$. 
Perfect Collateralization

Since the rates $\tilde{\zeta}$, $\tilde{f}$ and $\tilde{h}$ depend on the derivative’s price we must resort to numerical simulations to calculate the prices.

Longevity swaps’ case can be found in Biffis et al. (2012).

We can specialize further the pricing equation to two extreme cases:

- perfect collateralization, namely $\alpha_t = 1$, and

$$\bar{V}_t(C; F) = 1\{\tau > t\} \int_t^T \mathbb{E}_t^{\tilde{h}} \left[ d\pi_u D(t, u; \tilde{c}) \middle| \mathcal{F}_t \right]$$

- no collateralization, namely $\alpha_t = 0$.

$$\bar{V}_t(C; F) = 1\{\tau > t\} \int_t^T \mathbb{E}_t^{\tilde{h}} \left[ d\pi_u D(t, u; \tilde{f} + \tilde{\lambda}) \middle| \mathcal{F}_t \right]$$

with

$$\tilde{\lambda}_t = \lambda_t^C < I_{LGD;C} 1\{\bar{V}_t(C; F) > 0\} + \lambda_t^L < C_{LGD} I_{1\{\bar{V}_t(C; F) < 0\}}$$
Pricing Collateralized Interest-Rate Derivatives – I

- All liquid market quotes on the money market are daily collateralized at overnight rate \( (c_t) \), namely
  \[
  \tilde{c}_t = c_t
  \]

- Thus, if we use such instruments to implement hedging strategies for exotic deals we have to consider the collateral accrual rate as the effective lending rate for the underlying assets, namely
  \[
  \tilde{h}_t = c_t
  \]

- We assume that daily collateralization can be considered as a perfect collateralization, and, in particular, we disregard gap risk.
  → See Brigo, Capponi, Pallavicini, Papatheodorou (2011) for a discussion on the impact of partial collateralization on interest-rate derivatives.
Collateralized bilateral CVA for an IRS with ten year maturity and one year coupon tenor for different collateral update intervals with (and without) collateral re-hypothecation. See Brigo, Capponi, Pallavicini and Papatheodorou (2011).
Under the previous assumptions we can write the pricing equation as

$$\bar{V}_t(C; F) = 1_{\{\tau > t\}} \int_t^T \mathbb{E}_t^c \left[ d\pi_u D(t, u; c) | \mathcal{F}_t \right]$$

In the following, we omit to explicitly write the indicator and the conditioning from pricing equations to lighten notation.

The above equation is the same used by Piterbarg (2012) starting from a reformulation of the Black and Scholes theory.

In Fujii and Takahashi (2011a) a similar expression is derived but under risk-neutral measure.
In the previous section we considered the collateral and close-out prices proportional to the derivative price.

In order to accommodate gap risk, it is easier to calculate such prices independently from derivative price.

In particular, by following the suggestion of ISDA documentation, we include funding (and margining) costs when evaluating the close-out amount.

Furthermore, for sake of simplicity, we assume that the close-out amount is symmetric.

Thus, we are implicitly assuming that the investor estimates the funding costs of the counterparty to be equal to his own.

Thus, we write

\[ \varepsilon_{\tau} = \varepsilon_{I,\tau} = \varepsilon_{C,\tau} \]
We introduce a process \( \varepsilon_t \), whose value in \( \tau \) is the close-out amount.

We define \( \varepsilon_t \) as the price process when all costs are considered, but investor’s and counterparty’s default events are not, namely

\[
\varepsilon_t := \int_t^T \mathbb{E}_t^{\tilde{h}} \left[ \left( d\pi_u + (\tilde{f}_u - \tilde{c}_u)M_u \right) D(t, u; \tilde{f}) \right] , \quad C_t := 1_{\{\tau > t\}} M_t
\]

where we introduce the mark-to-market price process of the collateral account \( M_t \).

We make a distinction between \( M_t \) and \( C_t \), since the former is the price process used to evaluate the collateral account, while the latter is the price of the account, which is set to zero at investor’s or counterparty’s default event.

We can explicitly calculate the price processes if we assume also

\[
M_t \doteq \alpha_t \varepsilon_t
\]
Gap Risk as an Additive Correction – III

In particular, we can calculate the derivative price, and we find that it can be expressed as $\varepsilon_t$ plus an additive correction.

$$\tilde{V}_t(C; F) = 1_{\{\tau > t\}}\varepsilon_t$$

$$- \mathbb{E}^h_t\left[ 1_{\{t < \tau < T\}} \left( 1_{\{\tau = \tau_C\}} L_{GD} C \delta^+ + 1_{\{\tau = \tau_I\}} L_{GD} I \delta^- \right) D(t, \tau; \tilde{f}) \right]$$

where

$$\delta_t := \varepsilon_\tau - M_{\tau^-}, \quad \varepsilon_t := \int_t^T \mathbb{E}^h_t\left[ \left( d\pi_u + (\tilde{f}_u - \tilde{c}_u) M_u \right) D(t, u; \tilde{f}) \right]$$

Notice that without any further approximation the right-hand side is still depending on the derivative price via market and treasury rates.
Also in this case we can show the implications for the Bank’s structure.

Collateral informations are explicitly given by the calculation agent or by the collateral desk, when evaluating the mark-to-market price $M_t$.

The trading desk books the first term of the pricing equation, while the CVA desk the second term with a role very similar to any other trading desk.
Pricing Collateralized Contracts with Gap Risk

- In case of perfect collateralization, namely $\alpha_t \triangleq 1$, we can still have a residual exposure.
  - Such exposure is also known as gap risk.

- Gap risk may arise for different causes:
  - evaluation of close-out amount is non tightly defined by CSA;
  - in illiquid markets prices used for the collateral procedure may be different from replacement close-out prices (see also Alavian et al. (2008));
  - market risks may jump at default event due to contagion effects (see also Bielecki et al. (2011) and Fujii and Takahashi (2011)).

- In the following we present the case of a CDS contract as described in Brigo, Capponi and Pallavicini (2011).
  - Accordingly we assume a single rate for CSA and borrowing/lending rates, namely:
    $$\tilde{h} \triangleq \tilde{c} \triangleq c_t$$
Gap Risk for CDS – I

- The price process \( C_{DS_t} \) for a CDS selling protection \( L_{GDU} \) at time \( t \) for default of the reference entity \( U \) between times \( T_a \) and \( T_b \), in exchange of a periodic premium rate \( S_t \) is given by

\[
C_{DS_t} := 1_{\{\tau_U > t\}} S_t \int_{T_a}^{T_b} du \mathbb{Q}^c \{ \tau_U > u \mid \mathcal{G}_t \} D(t, u; c) \\
+ 1_{\{\tau_U > t\}} L_{GDU} \int_{T_a}^{T_b} d\mathbb{Q}^c \{ \tau_U > u \mid \mathcal{G}_t \} D(t, u; c)
\]

where, for simplicity, we assume that the premium leg pays continually, and interest and recovery rates are deterministic.

- We consider the following collateral and close-out processes (\( \alpha_t = 1 \)):

\[
M_t := C_{DS_t} , \quad C_t = 1_{\{\tau > t\}} M_t , \quad \varepsilon_T = M_T
\]
We need to evaluate the on-default survival probabilities, namely

\[ 1\{\tau = \tau_C < T\}1\{\tau_U > \tau_C\}Q^c\{\tau_U > t \mid G_{\tau_C}\} \]

\[ \left. \frac{\partial v}{\partial v} Q^c\{\tau_I > u, \tau_U > t, \tau_C > v \mid F_u\} \right|_{v=\tau_C} \]

\[ = \lim_{u \downarrow \tau_C} 1\{u \leq T\}1\{\tau_U > u\} \frac{\partial v}{\partial v} Q^c\{\tau_I > u, \tau_U > u, \tau_C > v \mid F_u\} \]

\[ \left. \left| \frac{\partial v}{\partial v} Q^c\{\tau_I > u, \tau_U > u, \tau_C > v \mid F_u\} \right|_{v=\tau_C} \right|_{v=\tau_C} \]

and similarly for probabilities conditioned on \( \tau_I \).

In the case of a credit default swap contract, continuous collateralization does not fully eliminate counterparty risk.

\[ \rightarrow \text{ It does so only if the default times of the counterparties and of the reference entity are conditionally independent.} \]
Bilateral CVA for a collateralized payer CDS with five year maturity for different margining practices by varying the Gaussian copula parameter between the reference name’s and the counterparty’s default time. See Brigo, Capponi and Pallavicini (2011).
Talk Outline

1. Collateralized Credit and Funding Valuation Adjusted Pricing
2. Derivative Pricing and Bank’s Structure
3. Conclusions and Future Developments
Conclusions and Future Developments

We describe a risk-neutral pricing equation inclusive of CVA/DVA, collaterals, and costs of margining and funding.

We do not wish to approximate the payout but rather we add collateral, treasury and hedgers fees as explicit cash flows.

There is no need to change the pricing theory, but just the payout!

We keep a sharp distinction between market and theoretical quantities.

For instance, we never claim the (false) fact that overnight rates are risk free.

The risk-free rate is just an instrumental variable that vanishes if funding and hedging are accomplished through concrete market instruments.
We propose two different approximations of pricing equation by properly defining collateral and close-out processes by focusing on:

- perfect vs. partial collateralization and the impact of gap risk for interest-rate and credit derivatives;
- the implications for the treasury and the CVA desk within the Bank’s structure.

We are working on:

- better approximations for the close-out price, which could incorporate an estimate of the funding costs of the counterparties; and
- numerical examples for relevant market cases.

Basel Committee on Banking Supervision
“International Convergence of Capital Measurement and Capital Standards A Revised Framework Comprehensive
Available at www.bis.org.

Counterparty Risk and the Impact of Collateralization in CDS Contracts Available at www.defaultrisk.com


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Arbitrage-Free Bilateral Counterparty Risk Valuation under Collateralization and Application to Credit Default Swaps. Accepted by Mathematical Finance.

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Available at http://www.isda.org.

Funding Valuation Adjustment consistent with CVA, DVA, Wrong-Way Risk, Collateral, Netting and Re-Hypothecation