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Abstract

The (additive) generator of an Archimedean copula - as well as the inverse of the generator - is a strictly decreasing and convex function, while Morgenstern utility functions (applying to risk averse decision makers) are nondecreasing and concave. This provides a basis for deriving either a generator of Archimedean copulas, or its inverse, from a Morgenstern utility function. If we derive the generator in this way, dependence properties of an Archimedean copula that are often taken to be desirable, match with generally sought after properties of the corresponding utility function. It is shown how well known copula families are derived from established utility functions. Also, some new copula families are derived, and their properties are discussed. If, on the other hand, we instead derive the inverse of the generator from the utility function, there is a link between the magnitude of measures of risk attitude (like the very common Arrow-Pratt coefficient of absolute risk aversion) and the strength of dependence featured by the corresponding Archimedean copula.

Keywords: copula; Archimedean generator; utility function; risk aversion; dependence

1 Introduction

Archimedean copulas are constructed using a one-dimensional function, the generator, which is strictly decreasing and convex. The same applies to the inverse of the generator. Von Neumann-Morgenstern utility functions, on the other hand, are nondecreasing (decision makers prefer more to less) and concave (decision makers are risk averse). Therefore, an affine transformation of a utility function, with sign changed, could act as a generator for an Archimedean copula or its inverse, subject to some additional conditions. Applying this methodology can lead to copula families that are either new or well known.

This paper examines relationships between (generators of) Archimedean copulas and Von Neumann-Morgenstern utility functions. The contributions in this paper are two-fold. Firstly, following a round of research in relevant literature on economics and decision theory, examples are given of Archimedean copulas generated in this way. Some of the copula families derived from utility functions are new, to the best of our knowledge. These families can be found in Sections 3.2.2 (derived from the SAHARA utility function) and 3.2.4 (based on a utility family that can be found in the seminal paper on risk aversion by Pratt, see [30]). Secondly, relationships are explored between properties and quantities of a utility function on the one hand, and type and strength of dependence induced by the Archimedean copula generated from it on the other hand. Many of these properties and quantities of utility functions are well established in the literature and can help when choosing the most appropriate Archimedean copula family. For

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example, a key measure of risk attitude is the coefficient of absolute risk aversion (defined in [1] and [30]) which is calculated as minus the ratio of the second derivative to the first derivative of the underlying utility function (assuming differentiability). Both the direction and size of the very common Arrow-Pratt coefficient of absolute risk aversion have been studied extensively in the economic literature and properties of this concept can help to choose the most appropriate copula function.

Section 2 gives a brief definition of generators of Archimedean copulas and introduces the two ways of constructing an Archimedean copula from a utility function.

Section 3 deals with the one way of expressing the generator in terms of the utility function. Links between the two functions are then explored in terms of the type of risk attitude as specified above for utility functions on the one hand, and type of dependence of the copula on the other hand. We start with two dimensions, using the findings of [2] who derive relationships between dependence properties (both positive and negative dependence) of Archimedean copulas and aging properties of their generator. Subsequently, we discuss properties of utility functions that are generally considered to be desirable. We also deal with the case of more than two dimensions, consider the additional constraints of the generator (and hence the utility function) and discuss the multivariate dependence notions explored in [27]. Several utility functions that appeared in the literature as examples will be considered. Finally, an alternative approach of using the derivative of a utility function as basis of the generator (rather than the generator itself) is discussed.

In Section 4 on the other hand we consider the alternative approach of expressing the inverse of the generator in terms of the utility function. Unlike the previous section, relationships between the two functions are in terms of the strengths of risk attitude (rather than its direction) and the strength of dependence (rather than its type). Again, some examples are given.

Conclusions are presented in Section 5.

As a final note, a potential link between copulas and utility functions, albeit from a different angle and within a different context when compared to this paper, is suggested in [11]. Durante and Spizzichino, discussing the concept of semi-copula as a mode of description of the set of level curves of a joint survival function of a vector of nonnegative random variables, identify a connection between such sets and families of indifference curves. For more details about semi-copulae, in particular regarding bivariate aging, see [3].

2 Archimedean copulas and utility functions

We define $C(\cdot,\cdot)$ to be a d-dimensional copula with $d \geq 2$. An Archimedean copula can be specified as:

$$C_{\varphi}(v_1,...,v_d) = \varphi\left(\varphi^{[-1]}(v_1)+,...,+\varphi^{[-1]}(v_d)\right). \qquad 0 \leq v_1,...,v_d \leq 1, \tag{1}$$

with φ, the generator, nonincreasing and convex, $\varphi(0) = 1$ and $\varphi(s) = 0$ for $s \geq s^*$ for some nonnegative s^*. The generator is strict if $\lim_{s\to\infty}\varphi(s) = 0$ (so $s^* = \infty$), and non-strict if s^* is finite. The function $\varphi^{[-1]}$ is defined as the generalized inverse of φ:

$$\varphi^{[-1]}(s) = \begin{cases} \varphi^{-1}(s) & \text{for} \quad 0 < s \leq 1 \\ s^* & \text{for} \quad s = 0 \end{cases}.$$

Remark 1 *In some literature about Archimedean copulas, the generator is defined in terms of $\varphi^{[-1]}$ rather than φ ("inverse operator outside, rather than inside, the brackets"). However, for the general ease of exposition, we prefer the notation above, also adopted by e.g. [24] and [27].*

So for instance, the function $s \mapsto \exp[-s]$ is used as generator for the independence copula, rather than $s \mapsto -\log[s]$.

Remark 2 *The generator, as specified in this paper, is invariant to multiplication of the argument by a positive constant. For $\epsilon > 0$, $\varphi(s)$ and $\varphi(\epsilon s)$ lead to the same copula. Likewise, $\varphi^{[-1]}(s)$ and $\epsilon\varphi^{[-1]}(s)$ generate the same copula. This often leads to simplifications, which in the sequel will usually tacitly be performed.*

A utility function $\psi : I_1 \to I_2$, with I_1 and I_2 being subsets of $\mathbb{R}$, is of a von Neumann-Morgenstern type if it is nondecreasing and concave. In this paper we will assume ψ to be strictly increasing on s with $s \in I_1$. Hence, the function $-\psi$ is strictly decreasing and convex. This does not mean that $-\psi$ could serve as a generator of an Archimedean copula, since in general the additional requirements are not satisfied. For the generator this concerns $\psi(0) = -1$ and $\lim_{s\to\infty} \psi(s) \geq 0$, while for the inverse generator it is simply required that $\psi(1) = 0$.

However, generators of Archimedean copulas can be constructed from affine transformations of utility functions. We define

$$u(s) = \alpha + \beta\psi(s),$$

with α and β real, $\beta > 0$. Clearly, if ψ is nondecreasing and concave, then so is u. Assuming that ψ is twice continuously differentiable on I_1 (a property applying to most common utility functions), an important measure for risk perception in utility theory is the Arrow-Pratt coefficient of absolute risk aversion, defined by [1] and [30] as

$$AR_\psi(s) = -\frac{\psi''(s)}{\psi'(s)} \geq 0, \quad s \in I_1. \tag{2}$$

(The subscript ψ in AR_ψ indicates that the degree of absolute risk aversion is related to the utility function). It is easy to verify that $AR_u(s) = AR_\psi(s)$, so risk perception is invariant up to an affine transformation.

In the remainder of this paper, we will distinguish between two approaches:

1. Derive the generator φ from ψ;
2. Derive the inverse generator $\varphi^{[-1]}$ from ψ;

The two approaches lead to different relationships between properties of the utility function (in terms of risk attitude) and properties of the Archimedean copula reflected by the generator or its inverse. We will deal with both in turn in Sections 3 and 4..

3 Deriving the generator from the utility function

For $s \geq 0$, $\max[-u(s), 0]$ could serve as a generator of an Archimedean copula, provided that: a) I_1 includes an interval $[0, J)$ with $J > 0$ and allowed to be infinity; b) $u(0) = -1$; c) $\lim_{s\to J} u(s) \geq 0$. Applying the second condition gives $\alpha = -1 - \beta\psi(0)$. The corresponding generator will then be

$$\varphi_\beta(s) = \max[1 - \beta(\psi(s) - \psi(0)), 0], \quad s \geq 0. \tag{3}$$

Satisfaction of the third condition requires $\beta \geq (\lim_{s\to J} \psi(s) - \psi(0))^{-1}$. Necessary conditions for strictness of the generator are that a) $J = \infty$, and b) $\lim_{s\to\infty} \psi(s) = \psi(\infty) < \infty$. Then a strict generator is obtained for $\beta = (\lim_{s\to\infty} \psi(s) - \psi(0))^{-1}$, reducing (3) to

$$\varphi(s) = \frac{\psi(\infty) - \psi(s)}{\psi(\infty) - \psi(0)}, \quad s \geq 0. \tag{4}$$

Remark 3 *Note that J as above is actually equal to s^* as defined in Section 2.*

In all other cases, and also in all cases with $\lim_{s\to\infty}\psi(s)=\infty$, the generator is not strict. The inverse of the generator is

$$\varphi_\beta^{[-1]}(v)=\psi^{-1}\left(\psi(0)+\frac{1-v}{\beta}\right),$$

in the strict case reducing to

$$\varphi_\beta^{[-1]}(v)=\psi^{-1}\left(v\psi(0)+(1-v)\psi(\infty)\right).$$

Remark 4 *Note the requirement that $\psi(0)$ be well defined and finite. This condition is e.g. not met for the widely applied utility functions $\psi(s)=\log s$, and $\psi(s)=-s^{1-\gamma}$ with $\gamma>1$. These utility functions belong to the class of Constant Relative Risk Aversion (CCRA), i.e. the coefficient of relative risk aversion $sAR_\psi(s)$ is constant and equal to γ. This class is widely applied in economics and decision theory. It is only for the third family of the CRRA class, namely $\psi(s)=s^{1-\gamma}$ with $0<\gamma<1$ that $\psi(0)$ is well defined, leading to a valid Archimedean generator.*

Two observations can be made regarding the role of the parameter β. Both have been derived in [34]:

1. The upper tail dependence coefficient, as defined in [28], denoted by λ_u, is

$$\lambda_u=2-\lim_{s\downarrow 0}\frac{1-\varphi(2s)}{1-\varphi(s)}=2-\lim_{s\downarrow 0}\frac{\psi(2s)-\psi(0)}{\psi(s)-\psi(0)},$$

   implying that λ_u does not depend on β.

2. For $\left(\psi^{-1}\right)'(\psi(0))>0$, (or, equivalently, $\psi'(0)<\infty$) the family includes W (Fréchet-Höffding's lower bound) as a limiting member for $\beta\to\infty$.

The second observation suggests that family (3) may be negatively ordered in terms of β.

From [2] and [6] it is known that an ordering of the copulas, i.e. $C_{\varphi_{\beta_1}}\prec C_{\varphi_{\beta_2}}$, is implied by an ordering of the corresponding Kendall functions, denoted by $C_{\varphi_{\beta_1}}\prec_{PKD}C_{\varphi_{\beta_2}}$ which is short for $K_{\varphi_{\beta_1}}(v)\geq K_{\varphi_{\beta_2}}(v)$ with

$$K_{\varphi_{\beta_i}}(v)=v-\frac{\varphi_{\beta_i}^{[-1]}(v)}{\left(\varphi_{\beta_i}^{[-1]}\right)'(v^+)},\quad i\in\{1,2\},v\in[0,1]. \tag{5}$$

Now suppose that $\beta_1>\beta_2$. According to Proposition 4 of [2], $K_{\varphi_{\beta_1}}(v)\geq K_{\varphi_{\beta_2}}$ if, and only if, $\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}$ is star-shaped, i.e. $s^{-1}\cdot\left(\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}\right)(s)$ is decreasing in s. We have that

$$\begin{aligned}\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}(s) &= \psi^{-1}\left(\psi(0)+\frac{1-\max\left[1-\beta_2\left(\psi(s)-\psi(0)\right),0\right]}{\beta_1}\right)\\ &= \begin{cases}\psi^{-1}\left(\left(1-\frac{\beta_2}{\beta_1}\right)\psi(0)+\frac{\beta_2}{\beta_1}\psi(s)\right) & \text{if}\quad s\leq\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right)\\ \psi^{-1}\left(\psi(0)+\beta_1^{-1}\right) & \text{if}\quad s>\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right)\end{cases}.\end{aligned}$$

It follows that the first derivative of $\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}$ is

$$\frac{\partial}{\partial s}\left(\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}\right)(s)=\begin{matrix}\frac{\beta_2}{\beta_1}\frac{\psi'(s)}{(\psi)'\left(\psi^{-1}\left(\left(1-\frac{\beta_2}{\beta_1}\right)\psi(0)+\frac{\beta_2}{\beta_1}\psi(s)\right)\right)} & \text{if}\quad s\leq\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right)\\ 0 & \text{if}\quad s>\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right),\end{matrix}$$

while the second derivative, after some rewriting gives

$$\begin{aligned}
&\frac{\partial^2}{\partial s^2}\left(\varphi_{\beta_1}^{[-1]}\circ\varphi_{\beta_2}\right)(s)\\
=\quad &\begin{array}{ll}\frac{\beta_2}{\beta_1}\left((\psi)'(t)\right)^{-1}\left(-\psi''(s)\right)\left(\frac{\beta_2}{\beta_1}\left(-\frac{\psi''(t)}{(\psi'(t))^2}\right)\left(-\frac{(\psi'(s))^2}{\psi''(s)}\right)-1\right) & \text{if}\quad s\leq\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right)\\ 0 & \text{if}\quad s>\psi^{-1}\left(\psi(0)+\beta_2^{-1}\right),\end{array}
\end{aligned}$$

defining

$$t=\psi^{-1}\left(\left(1-\frac{\beta_2}{\beta_1}\right)\psi(0)+\frac{\beta_2}{\beta_1}\psi(s)\right).$$

For $s=0$, this second derivative is negative. By continuity of ψ and its first and second derivative, it follows that the second derivative is negative for s in a neighborhood of 0. Also note that the second derivative is negative for $\frac{\beta_2}{\beta_1}\downarrow 0$. We can conclude that $C_{\varphi_{\beta_1}}\prec_{PKD}C_{\varphi_{\beta_2}}$ if either β_2 is small, compared to β_1, or β_2 is large. A sufficient condition for the given order is that

$$\frac{\beta_2}{\beta_1}\left(-\frac{\psi''\left(\psi^{-1}\left(\left(1-\frac{\beta_2}{\beta_1}\right)\psi(0)+\frac{\beta_2}{\beta_1}\psi(s)\right)\right)}{\left(\psi'\left(\psi^{-1}\left(\left(1-\frac{\beta_2}{\beta_1}\right)\psi(0)+\frac{\beta_2}{\beta_1}\psi(s)\right)\right)\right)^2}\right)$$

is increasing in terms of $\frac{\beta_2}{\beta_1}$.

3.1 Relationships between properties of utility function and properties of generator

Using concepts from reliability theory, [2] derive several relationships between type of dependence of a copula, and aging properties of the generator, which is in fact a survival function. Given the expressions (3) and (4), these aging characteristics translate into properties of the corresponding utility function. In this section, links between type of dependence of copulas and behavior of corresponding utility functions will be investigated.

Types of dependence, as appeared in the literature, usually fall in either one of the categories of positive dependence or negative dependence. All notions of positive dependence that appeared in the literature, including the weakest one of Positive Quadrant Dependence (PQD) as defined by [20], require the generator to be strict. As pointed out by [2], copulas obtained from non-strict generators can possess strong properties of negative dependence.

While for two dimensions, (3) generates a valid Archimedean copula for any β satisfying the given lower bound, this is not the case for $d>2$, since the addition of dimensions entails additional constraints to the generator, limiting the possible values of β. For this reason, we will distinguish between a) two dimensions (where any kind of positive or negative dependence can be considered) in Subsection 3.1.1, and b) more than two dimensions (where the discussion about dependence is limited to the notions of positive dependence) in Subsection 3.1.3. Subsection 3.1.2 considers properties of Morgenstern utility functions that are generally accepted as desirable.

3.1.1 Two dimensions

In the sequel, we consider two continuous random variables X_1 and X_2, and either an Archimedean *distribution copula* C_φ with generator φ defined in (1) such that
$\Pr\left[X_1\leq x_1,X_2\leq x_2\right]=C_\varphi\left(\Pr\left[X_1\leq x_1\right],\Pr\left[X_2\leq x_2\right]\right)$, or an Archimedean *survival copula* defined as $\widehat{C}_{\widehat{\varphi}}$ with generator $\widehat{\varphi}$ such that $\Pr\left[X_1>x_1,Y>x_2\right]=\widehat{C}_{\widehat{\varphi}}\left(\Pr\left[X_1>x_1\right],\Pr\left[X_2>x_2\right]\right)$.

The notation $\widehat{\psi}$ refers to the Morgenstern utility function, from which the generator $\widehat{\varphi}$ is constructed, just as in (4). Furthermore, we define $f(x_1, x_2)$ as the joint density function of X_1 and X_2, valued at (x_1, x_2).

We will consider some of the notions discussed in [2]. For positive dependence, we have SI (Stochastically Increasing) (from [20]), as well both LTD (Left Tail Decreasing) and RTI (Right Tail Increasing) (from [12]). On the other hand, concerning negative dependence, we will discuss their respective negative counterparts: SD (Stochastically Decreasing), LTI (Left Tail Increasing) and RTD (Right Tail Decreasing). In addition the type of strong positive dependence induced by TP2 (Total Positivity of Order 2 or Positive Likelihood Ratio Dependence), introduced by [20], will also be considered. We will not elaborate on the concepts of Positive Quadrant Dependence and Negative Quadrant Dependence as in [20], since the matching properties of the utility functions do not look natural or tractable. For the same reason we will not discuss the notions of Positive K-Dependence and Negative K-Dependence from [6] either.

Definition 5 *X_2 is LTD in $X_1 \Leftrightarrow \Pr[X_2 \leq x_2 | X_1 \leq x_1]$ is nonincreasing in x_1 for all x_2.*

Definition 6 *X_2 is RTI in $X_1 \Leftrightarrow \Pr[X_2 > x_2 | X_1 > x_1]$ is nondecreasing in x_1 for all x_2.*

Definition 7 *X_2 is SI in $X_1 \Leftrightarrow \Pr[X_2 \leq x_2 | X_1 = x_1]$ is nonincreasing in x_1 for all x_2.*

Note that the SI concept can also be applied to survival copulas since Definition 7 is equivalent to: $\Pr[X_2 > x_2 | X_1 = x_1]$ is nondecreasing in x_1 for all x_2.

The negative dependence counterparts are defined by reversing inequalities or changing "nonincreasing" in "nondecreasing" or vice versa:

Definition 8 *X_2 is LTI in $X_1 \Leftrightarrow \Pr[X_2 \leq x_2 | X_1 \leq x_1]$ is nondecreasing in x_1 for all x_2.*

Definition 9 *X_2 is RTD in $X_1 \Leftrightarrow \Pr[X_2 > x_2 | X_1 > x_1]$ is nonincreasing in x_1 for all x_2.*

Definition 10 *X_2 is SD in $X_1 \Leftrightarrow \Pr[X_2 \leq x_2 | X_1 = x_1]$ is nondecreasing in x_1 for all x_2.*

The TP2 concept is based on the joint density function, rather than the joint distribution function or joint survival function.

Definition 11 *(X_1, X_2) is TP2 $\Leftrightarrow f(x_1, x_2) f(x_1', x_2') \geq f(x_1, x_2') f(x_1', x_2)$ for all real valued x_1, x_1', x_2, x_2' such that $x_1 \leq x_1'$ and $x_2 \leq x_2'$.*

As pointed out in [2], $SI \Longrightarrow (LTD$ or $RTI)$ and $SD \Longrightarrow (LTI$ or $RTD)$. The implication $TP2 \Rightarrow SI$ has been proven in e.g. [16].

Next, by means of two propositions, we discuss the connection between the dependence properties of either the distribution copula C_φ or the survival copula $\widehat{C}_{\widehat{\varphi}}$, and the risk perception properties of the utility functions ψ and $\widehat{\psi}$, respectively. To this end, we will need two measures of risk attitude that are less common than the Arrow-Pratt coefficient of absolute risk aversion r_ψ as defined in (2). It concerns the *coefficient of asymptotic risk aversion* and the *absolute prudence function.*

The coefficient of asymptotic risk aversion of a utility function ψ in terms of wealth s, defined as $RL_\psi(s)$ is:

$$RL_\psi(s) = \frac{\psi'(s)}{\psi(J) - \psi(s)}. \tag{6}$$

This quantity was defined in [17]. A closely related - and more general - notion called "happiness of win" is discussed in [22].

The absolute prudence function is defined by [18] as

$$AP_{\psi}(s) = -\frac{\psi'''(s)}{\psi''(s)}, \quad s \in I_1.$$

In the next Propositions, The variable s^* is as defined in Section 2.

Proposition 12 *i) C_{φ} is LTD$\Longleftrightarrow$ $RL_{\psi}(s)$ is nonincreasing in s , for all $s \in \mathbb{R}^+$ (with $J = \infty$).*

C_{φ} is LTI$\Longleftrightarrow$ $\beta\left(\psi''(s)(\psi(s) - \psi(0)) - (\psi'(s))^2\right) - \psi''(s) \leq 0$, for all $s \in (0, s^*)$.
ii) $\widehat{C}_{\widehat{\varphi}}$ is RTI$\Longleftrightarrow$ $RL_{\widehat{\psi}}(s)$ is nonincreasing in s , for all $s \in \mathbb{R}^+$ (with $J = \infty$).
$\widehat{C}_{\widehat{\varphi}}$ is RTD$\Longleftrightarrow$ $\beta\left(\widehat{\psi}''(s)\left(\widehat{\psi}(s) - \widehat{\psi}(0)\right) - \left(\widehat{\psi}'(s)\right)^2\right) - \widehat{\psi}''(s) \leq 0$, for all $s \in (0, s^*)$.

Proof. i) Follows from [2] or [9], in connection with Equation (4). See [34] for more details.
ii) In [34] it is noted that RTI is actually the "survival copula" counterpart of LTD. Likewise, LTI can be considered to be the "survival copula" counterpart of RTD. Bearing this in mind, the proofs are exactly the same as in ii), except that the generator of C_{φ} is replaced by the generator of $\widehat{C}_{\widehat{\varphi}}$. ■

The dependence types of SI/SD and TP2 can apply to either C_{φ} or $\widehat{C}_{\widehat{\varphi}}$. Therefore, to avoid tedious repetition of notation, the following Proposition will give the result in terms of the generator ψ, although the same applies to the generator $\widehat{\psi}$.

Proposition 13 *i) C_{φ} is SI $\Longleftrightarrow$ $AR_{\psi}(s)$ is nonincreasing in s, for all $s \in \mathbb{R}^+$.*

ii) C_{φ} is SD $\Longleftrightarrow$ $AR_{\psi}(s)$ is nondecreasing in s for all $s \in (0, s^*)$.
iii) C_{φ} is TP2 $\Longleftrightarrow$ $AP_{\psi}(s)$ is nonincreasing in s for all $s \in \mathbb{R}^+$.

Proof. i) and ii) Follows from [2], in connection with Equation (4). See [34] for more details.
iii) Follows from Proposition 3.3 (part ii)) of [7], in connection with Equation (4). ■

3.1.2 Common properties of Morgenstern utility functions

Morgenstern utility functions are usually considered to be strictly increasing: "the more wealth the better". Strictly increasing utility functions are also required in order to get a valid generator as in (3).

Remark 14 *Strictly speaking, constant utility is a special case of a Morgenstern utility function. Let $\psi(s) \equiv c$ with c real-valued. This gives $\varphi(s) = \max[1 + \beta(c - c), 0] = 1$ for any positive β. According to [28] (see also [15]): a) $K_{\varphi}(v)$ has a geometric interpretation as being the x-intercept of the line tangent to the graph $y = \varphi^{[-1]}(x)$ at the point $\left(v, \varphi^{[-1]}(v)\right)$; b) Define φ_{θ} to be an Archimedean generator with $\theta \in \Theta$ where Θ denotes the parameter interval. Then the Fréchet upper bound is obtained as a limiting case, if and only if $\lim \frac{\varphi^{[-1]}(v)}{\left(\varphi^{[-1]}\right)'(v^+)} = 0$ for all $v \in (0, 1)$, where the limit is taken with respect to θ approaching the end point of the interval. Combining a) and b) leads to the conjecture that the Fréchet upper bound corresponds to $\varphi^{[-1]}$ being a vertical line $x = 1$, or, equivalently φ being a horizontal line $y = 1$. In words, comonotonicity matches a utility function of an individual who is indifferent regarding wealth.*

Furthermore, utility functions are usually taken to be strictly concave. This means that, within an expected utility framework, an individual is risk averse for any level of wealth: he would rather not be exposed to a lottery with zero mean payoff.

Remark 15 *A special case of a concave utility function is a linear one. The corresponding generator of an Archimedean copula is* $\varphi(s) = \max[1 - \beta s, 0]$ *reducing to* $\max[1 - s, 0]$*, (since, as stated above, a generator determines a copula, up to a constant positive factor) being the generator of the Fréchet-Höffding lower bound copula* $C(v_1, v_2) = \max[v_1 + v_2 - 1, 0]$*. In words: countermonotonicity corresponds to risk neutrality.*

The Arrow-Pratt risk aversion coefficient as defined by [1] and [30] in (2) plays a key role in micro-economics. As we saw above, nondecreasing (nonincreasing) risk aversion corresponds to SI (SD) of the corresponding two-dimensional copula.

Remark 16 *A standard example of a utility function featuring nondecreasing risk aversion, discussed by [30], is* $\psi(s) = -(b - s)^c$ *for* $s \leq b$ *and* $c > 1$*. This leads to generator* $\varphi(s) = \max[1 - \beta(1 - (1 - s)^c), 0]$ *with* $\beta \geq 1$*. For* $\beta = 1$ *we get* $\varphi(s) = (\max[(1 - s), 0])^c$ *which is the generator of a subclass of the Clayton family featuring negative dependence and SD.*

However, it is generally accepted in economics that risk aversion should be nondecreasing. One interpretation is that the premium that an individual is prepared to pay to get rid of a risk is decreasing in terms of wealth, see [30]. It is also known that under DARA, all undesirable risks (any risk which if taken on would lead to a lower expected utility than without that risk) are known to be loss aggravating (the loss in utility due to a nonrandom decrease in wealth is more painful if combined with a loss aggravating risk). As we saw before, DARA is equivalent to SI in the two-dimensional case, providing the generator is strict.

Decreasing Absolute Risk Aversion (DARA) requires the utility function to have a convex first derivative. Individuals to whom such utility function applies (though not necessarily being DARA) are called *prudent*, see [18]. This property leads to an Archimedean generator whose first derivative is concave, which, as pointed out in [24], is a necessary and sufficient condition for generating a valid Archimedean copula in three dimensions. The importance of prudence in determining a precautionary savings motive was recognized in [21] and [32].

A stronger requirement than DARA, also quite widely accepted as desirable in economics and decision theory, is that of Decreasing Absolute Prudence (DAP), which means that $AP_\psi(s)$ is nonincreasing in s. This is the necessary and sufficient condition for an Archimedean copula being TP2 in the two-dimensional space, as seen above. Kimball, see [19], defines a utility function to feature standard risk aversion if an undesirable risk and a loss aggravating risk aggravate each other. [19] shows that standard risk aversion is equivalent to DAP. Note that a necessary condition of DAP is concavity of the second derivative of the utility function.

3.1.3 Extension to three or more dimensions

McNeil and Nešlehová, see [24], specify the additional constraints to be imposed on a generator in order to get a valid Archimedean copula in d dimensions with $d \geq 3$. Basically, such a generator needs to be d-monotone, i.e. differentiable up to the order $d - 2$ with all derivatives alternating in sign, and $(-1)^{d-2}\varphi^{(d-2)}$ being nonincreasing and convex. Here $\varphi^{(d)}$ denotes the d-th derivative of φ. We will now see how these additional constraints affect (3). For $d = 3$, (3) needs to be once differentiable with the first derivative being nondecreasing and concave. Differentiability requires that

$$\lim_{s \to s^* = \psi^{-1}(\psi(0) + \beta^{-1})} \frac{\partial}{\partial s}[1 - \beta(\psi(s) - \psi(0))] = 0,$$

which, given $\beta > 0$, implies that $\psi(s)$ is differentiable on I_1 and $\psi'(s^*) = 0$. Given that ψ needs to be strictly increasing on I_1, $\psi'(s)$ may only vanish for $s = J$. It follows that (3) may only

generate a valid three-dimensional copula for β equal to the minimum value of $(\psi(J) - \psi(0))^{-1}$. In addition, $\psi(J)$ must be finite (since $\beta = 0$ never yields a valid Archimedean generator) and ψ' needs to be nonincreasing and convex.

Let $\psi^{(k)}$ denote the derivative of ψ of order k. For any $d \geq 3$ the necessary and sufficient properties of ψ on I_1 that are to be met in order to generate a valid Archimedean copula can be spelled out in the same vein as [24]: a) ψ is differentiable up to order $d-2$; b) $(-1)^{d-2}\psi^{(d-2)}$ is nondecreasing and concave; c) $\psi(J) < \infty$; d) $\psi^{(k)}(J) = 0$ for $k \in \{1, ..., d-2\}$. Note that for $J = \infty$ and $\psi(\infty) < \infty$, only strict generators are allowed for higher dimensions. It is for this reason that in the examples in Subsection 3.2, we study in particular the case of $\psi(J) < \infty$ and $\beta = (\psi(J) - \psi(0))^{-1}$.

Müller and Scarsini, see [27], discuss the dependence notions of Conditionally Increasing (CI), Conditionally Increasing in Sequence (CIS) and Multivariate Totally Positive of order 2 (MTP2). CI and CIS are multivariate analogs of SI, while MTP2 is a multivariate analogue of TP2. A copula C is CIS if for any random vector $\mathbf{X} = (X_1, ..., X_d)$ having a distribution with copula C, X_i is stochastically increasing in $(X_1, ..., X_{i-1})$ for all $i \in \{2, ..., d\}$, which means that

$$P[X_i > s \,|X_1 = x_1, ..., X_{i-1} = x_{i-1}]$$

is increasing in $x_1, ..., x_{i-1}$. A copula C is CI if for any random vector $\mathbf{X} = (X_1, ..., X_d)$ having a distribution with copula C, X_i is stochastically increasing in $\mathbf{X}_M$ for all $i \notin M$ and all $M \subset \{1, ..., d\}$ where $\mathbf{X}_M = (X_j, j \in M)$. As pointed out by [27], the notions of CI and CIS coincide for exchangeable copulas (like any Archimedean copula). Müller and Scarsini show that the concepts of MTP2 in $d-1$ dimensions and CI or CIS in d dimensions are identical in the sense that they entail exactly the same property of the generator, i.e. $(-1)^{d-1}\varphi^{(d-1)}$ is log-convex. Assuming the generator is strict and ψ being differentiable up to order $d-1$, this is equivalent to $(-1)^{d}\psi^{(d-1)}$ being log-convex. For $d = 3$, this is equivalent to DAP as seen above.

3.2 Examples

In this Section, we will discuss several utility functions, derive the corresponding Archimedean copulas, and give a thorough analysis of the dependence properties concerned. For the reader's convenience, first an overview of the main characteristics of the several copula families satisfying the necessary conditions of strictness of the generator - i.e. where the generator is of type 4 - is presented in Table 1. For each utility function we give a) an expression for the generator; b) the name of the corresponding copula (and indicate "New" if, to the best of our knowledge, it has not appeared in the literature yet); c) an expression for Kendall's tau; and d) the range of dependence (where "Comprehensive" means that the entire spectrum of positive and negative dependence is covered, including independence, comonotonicity and countermonotonicity; idem for "No independence", except that independence is not included; "Positive only" means that, unlike "Comprehensive", negative dependence is not covered).

3.2.1 The HARA family

This family (Hyperbolic Absolute Risk Aversion), which contains several utility functions discussed above as special cases, has been introduced in [25]. It is specified as:

$$\psi(s) = \frac{1-\gamma}{\gamma}\left(\frac{\delta s}{1-\gamma} + \epsilon\right)^{\gamma}; \quad \delta > 0; \gamma \notin \{0, 1\}; \frac{\delta s}{1-\gamma} + \epsilon > 0; \ \epsilon = 1 \text{ if } \gamma = -\infty.$$

Utility	**Generator** $\varphi(s)$	**New?**	**Tau**	**Range of dependence**
HARA	$\left(\max\left[1-\frac{s}{\gamma},0\right]\right)^{\gamma}$, $\gamma>1 \vee \gamma<0$.	Clayton	$(1+2\gamma)^{-1}$	Comprehensive
SAHARA	$\left(s+\sqrt{1+s^2}\right)^{-(1+\gamma)} \cdot\left(\frac{s}{1+\gamma}+\sqrt{1+s^2}\right)$, $\gamma>0$.	New	$(1+\gamma)^{-1}$	Positive only
Xie/Saha	$\exp\left(-s^{\delta}\right), \quad 0 \leq \delta \leq 1$.	Gumbel-Hougaard	$1-\delta$	Positive only
Conniffe	$\left(\max\left[1+\gamma s^{\delta},0\right]\right)^{-\frac{1}{\gamma}}$, $\gamma \geq -1; 0 \leq \delta \leq 1$.	Clayton's exterior power extension	$1-\frac{2\delta}{2+\gamma}$	Comprehensive
Pratt (Ex. 2)	$1-\left(\frac{s^{\delta}}{1+s^{\delta}}\right)^{\frac{1}{\delta}}, \quad \delta>0$.	New	$\frac{2-\delta}{2+\delta}$	No independence

Table 1: Derived Archimedean copula families and their main properties.

This utility function has risk aversion coefficient $r(s)=\left(\frac{\delta s}{1-\gamma}+\epsilon\right)^{-1}$. Given that the utility function must be well-defined for $s=0$, $\epsilon \geq 0$ is required, with strict inequality for $\gamma<0$ or $\gamma>1$. For $\epsilon=0$ (requiring $0<\gamma<1$) we get the CRRA case indicated in Remark 4. This gives $\varphi(s)=\max\left[1-s^{\gamma},0\right]$, which is Family (4.2.2) of Table 4.1 in [28]. This generator is not strict, since $\varphi(1)=0$. Otherwise, the generator $\varphi(s)=\max\left[1-\beta\frac{1-\gamma}{\gamma}\left(\left(\frac{s}{1-\gamma}+1\right)^{\gamma}-1\right),0\right]$ is obtained. The case $\gamma \rightarrow 1$ gives linear utility (and hence W), while $\gamma \rightarrow \pm\infty$ leads to $\varphi(s)=\max\left[1-\beta\left(1-\exp\left[-s\right]\right),0\right]$ which is Family 4.2.7 of Table 4.1 in [28]. For $\gamma>1$, the generator reduces to $\varphi(s)=\max\left[1-\beta\left(1-(1-s)^{\gamma}\right),0\right]$. We have that $J=1$ while the minimum value of β is also equal to 1. In that case, we end up with $\varphi(s)=\left(\max\left[1-s,0\right]\right)^{\gamma}$ which is the generator of the non-strict part of the Clayton copula for negative dependence (see also Remark 16). We know from [24] that, to get an Archimedean copula for d dimensions it is required that $\gamma \geq d-1$. For $0<\gamma<1$, the generator reduces to $\varphi(s)=\max\left[1-\beta\left((s+1)^{\gamma}-1\right),0\right]$. Given that $\lim_{s\rightarrow\infty}(s+1)^{\gamma}=\infty$, a valid generator is only obtained for two dimensions (and is never strict). Strictness of the generator requires $\gamma<0$, yielding $\varphi(s)=\max\left[1-\beta\left(1-(s+1)^{\gamma}\right),0\right]$ which for $\beta=1$ leads to $\varphi(s)=(s+1)^{\gamma}$ being the generator of the strict part of the Clayton copula, which is a standard example of an SI copula.

3.2.2 SAHARA utility

SAHARA stands for Symmetric Asymptotic Hyperbolic Absolute Risk Aversion. This class was introduced in [8]. The family is based on a risk aversion coefficient $r(s)=\frac{1+\gamma}{\sqrt{s^2+\delta^2}}$, with $\gamma>-1$

and $\delta > 0$. The special case $\delta \downarrow 0$ leads to CRRA, requiring $\gamma > 0$. The utility function is:

$$\psi(s) = \begin{cases} -\frac{1}{(1+\gamma)^2-1}\left(s+\sqrt{\delta^2+s^2}\right)^{-(1+\gamma)}\left(s+(1+\gamma)\sqrt{\delta^2+s^2}\right) & \gamma \neq 0 \\ \frac{1}{2}\ln\left(s+\sqrt{\delta^2+s^2}\right)+\frac{1}{2}\delta^{-2}s\left(\sqrt{\delta^2+s^2}-s\right) & \gamma = 0. \end{cases} \tag{7}$$

The "SA"-part of the name SAHARA refers to the fact that risk aversion attains a maximum for $s = 0$. In other words, risk aversion increases for negative s, while it decreases for positive s. Since only positive values of the argument are of interest to us, it is clear that this utility function should lead to a generator of an SI copula. For $\gamma \leq 0$, $\lim_{s\to\infty}\psi(s) = \infty$ while for $\gamma > 0$, $\lim_{s\to\infty}\psi(s) = 0$. For this reason we will confine this analysis to $\gamma > 0$ and $\beta = (\psi(\infty) - \psi(0))^{-1} = \frac{(1+\gamma)\delta^{-\gamma}}{(1+\gamma)^2-1}$, leading to the strict generator

$$\varphi(s) = \left(s+\sqrt{1+s^2}\right)^{-(1+\gamma)}\left(\frac{s}{1+\gamma}+\sqrt{1+s^2}\right).$$

The limiting case $\gamma \downarrow 0$ leads to Fréchet's upper bound, while for $\gamma \to \infty$ we get $\varphi(s) = \exp(-s)$ which corresponds to independence. Kendall's tau of this copula is equal to $(1+\gamma)^{-1}$. The family therefore covers the entire range of positive dependence. There is no upper tail dependence while the lower tail dependence coefficient is $2^{-\gamma}$. The function φ has a negative third derivative and a positive fourth derivative, so that an Archimedean copula for up to at least four dimensions can be constructed. Additional constraints apply for more than four dimensions; for instance $d = 5$ and $d = 6$ require $\gamma \geq 2$ and $\gamma \geq 3$. Furthermore, $(-1)^k\,\varphi^{(k)}$ is log-convex for $k \in \{1, 2\}$, implying that the copula is MTP_2 for two dimensions and SI for up to three dimensions. Since a closed form expression of the inverse of φ is unavailable, we are unable to state whether the family is negatively ordered in γ (although Kendall's tau suggests that this could well be the case).

3.2.3 The utility family by Xie and Saha and Conniffe's generalization

Xie, see [35] discusses the utility family

$$\psi(s) = \frac{1}{\epsilon}\left(1-\exp\left[-\epsilon\left(\frac{s^\delta-1}{\delta}\right)\right]\right), \quad \epsilon \geq 0, \delta \leq 1.$$

Actually, the class introduced by Saha in [31] is essentially the same. For notational convenience, however, we will stick to Xie's format. To construct a valid Archimedean generator, we need to impose the additional constraint $\delta \geq 0$. A special case results from $\lim_{\epsilon\downarrow 0}$ or $\lim_{\delta\downarrow 0}$, giving the CRRA family already considered before. From now on we assume $\epsilon > 0$. The family is generalized in [10], defined as the Flexible Three Parameter (FTP) utility function through an additional real-valued parameter γ:

$$\psi(s) = \frac{1}{\epsilon}\left\{1-\left(1+\gamma\,\epsilon\frac{s^\delta-1}{\delta}\right)^{-\frac{1}{\gamma}}\right\}.$$

The Saha/Xie family comes as a special case by taking $\lim_{\gamma\to 0}$. To avoid non-real numbers for non-integer values of $-\frac{1}{\gamma}$, it is required that

$$1+\gamma\,\epsilon\left(\frac{s^\delta-1}{\delta}\right) > 0.$$

Now assume that this condition is satisfied. Then, for $\gamma < 0$ this implies that s has an upper bound equal to $\left(1 - \frac{\delta}{\gamma\,\epsilon}\right)^{\frac{1}{\delta}}$ with corresponding upper bound of the utility function being $\frac{1}{\epsilon}$ while for nonnegative γ, s is unbounded from below with $\psi(\infty) = \frac{1}{\epsilon}$. We get as generator

$$\varphi(s) = \max\left[1 - \beta\left(1 - \left[1 + s^{\delta}\right]^{-\frac{1}{\gamma}}\right), 0\right].$$

The minimum value for β is 1, leading, after simplifications, to

$$\varphi(s) = \left(\max\left[1 + \gamma s^{\delta}, 0\right]\right)^{-\frac{1}{\gamma}}.$$

The basic requirement of convexity of this generator is satisfied for $\gamma \geq -1$, while it is strict for $\gamma \geq 0$. Actually, this family can be considered as an exterior power family extension (due to δ) of the Clayton family. The case $\gamma \geq 0$ is defined as "BB1 family" in [16], and is briefly discussed in Example 4.22, p. 144/145 of [28], while the portion $\gamma \geq 0$ is considered in Example 4.22, p. 145 of [28]. The family is positively ordered in γ and negatively ordered in δ. Comonotonicity is obtained for $\delta \to 0$ or $\gamma \to \infty$. Note the six special cases of one-parameter families: $\delta = 1$ is Clayton, while $\gamma = -1$ and $\gamma = 1$ lead to Families 4.2.2 and 4.2.12, respectively, of [28]. Furthermore, $\gamma = \delta$ and $\gamma = -\delta$ corresponds to Families 4.2.14 and 4.2.15 of [28]. Finally, $\gamma \to 0$ is Gumbel-Hougaard, showing that the utility function by Saha and Xie provides the basis of the Gumbel-Hougaard family. Kendall's tau is equal to $1 - 2\delta(2+\gamma)^{-1}$. Given that Clayton has no upper tail dependence, and lower tail dependence parameter, $I_{\theta \geq 0} 2^{-\frac{1}{\theta}}$, applying Theorem 5.4.4. of [28] leads to an upper tail dependence parameter of $2 - 2^{\delta}$ and lower tail dependence parameter of $2^{-\frac{\delta}{\gamma}}$ if $\theta \geq 0$ (no lower tail dependence for negative values of γ). For d dimensions, the minimum value of γ - in order to get a valid Archimedean generator - is $-\frac{1}{d}$. This is in accordance with the findings by [24]. The cross-ratio function, which originates from [29] and counts as a key measure of time-dependent association, is

$$CR[v] = \left(\frac{\varphi''(s)\,\varphi(s)}{(\varphi'(s))^2}\right)_{s=\varphi^{-1}(v)} = 1 + \frac{\gamma}{\delta}\left(1 + \frac{1-\delta}{v^{-\gamma} - 1}\right), \tag{8}$$

which is increasing in v. Association starts at ∞ (when time is at 0 for $v = 1$) (unless $\delta = 1$ when it is constant at $1 + \gamma$) and has a limiting value (for time at infinity when $v = 0$) of $1 + \gamma\delta^{-1}$.

3.2.4 Example 2 of Pratt (1964)

Pratt (1964) considers several utility functions with possible decreasing risk aversion. Here we will discuss a family with given derivative

$$\psi'(s) = \left(s^{\delta} + \epsilon\right)^{-\frac{1}{\gamma}}, \quad \delta > 0, \gamma > 0, s^{\delta} + \epsilon > 0. \tag{9}$$

Clearly, $\epsilon > 0$ is required to accomplish a valid generator. In general, ψ is a hypergeometric function and therefore mathematically not tractable. For this reason, we only consider some special cases. $\delta = 1$ leads to $\psi(s) = \frac{\gamma}{\gamma-1}(s+1)^{\frac{\gamma-1}{\gamma}}$ (disregarding constants) and therefore

$$\varphi(s) = \max\left[1 - \beta\frac{\gamma}{\gamma - 1}\left((s+1)^{\frac{\gamma-1}{\gamma}} - 1\right), 0\right],$$

which basically covers some cases of the HARA family as above. Furthermore, $\gamma = \delta = 1$ and $\gamma = \delta = 2$ yield $\psi(s) = \ln[s + \epsilon]$ and $\psi(s) = \ln\left[s + \sqrt{s + \epsilon^2}\right]$ which are defined for any

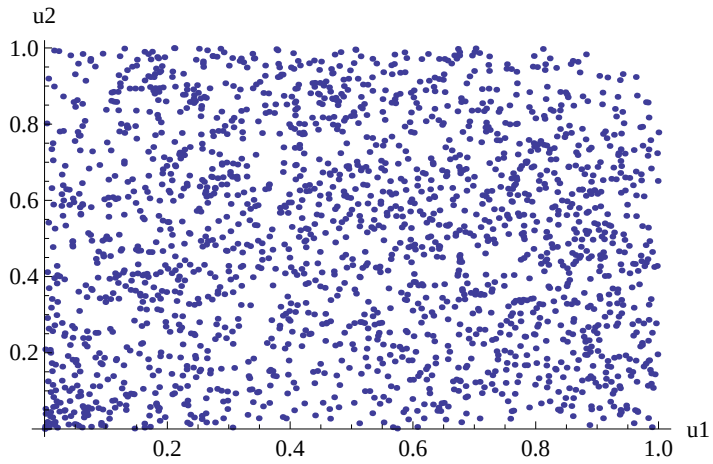

Figure 1: Scatterplot copula with generator as in Pratt (1964), with $\gamma = 2$ and 2,000 simulations.

nonnegative s, but do not attain a finite limit for $s \to \infty$. An interesting case arises when $\gamma = \delta(\delta+1)^{-1}$. Then we get

$$\psi(s) = \frac{1}{\epsilon}\left(\frac{s^{\delta}}{\epsilon + s^{\delta}}\right)^{\frac{1}{\delta}},$$

giving the generator

$$\varphi(s) = \max\left[1 - \beta\left(\frac{s^{\delta}}{1+s^{\delta}}\right)^{\frac{1}{\delta}}\right].$$

The strict generator, resulting for $\beta = 1$, is

$$\varphi(s) = 1 - \left(\frac{s^{\delta}}{1+s^{\delta}}\right)^{\frac{1}{\delta}}.$$

This family is negatively ordered in δ. The limits $\delta \to 0$ and $\delta \to \infty$ correspond to Fréchet's upper and lower bound, respectively, while for $\delta = 1$ we get the copula $C_{\varphi}(v_1, ..., v_d) = \left(\sum_{i=1}^{d} v_i^{-1} - (d-1)\right)^{-1}$ which arises as a limiting case in several copula families (see [28] for the two-dimensional case). This family covers the entire range of positive and negative dependence, but not the special case of independence. Kendall's tau is $(2-\delta)(2+\delta)^{-1}$. There is no upper tail dependence whereas the lower tail dependence parameter is $2^{-\delta}$. To have a copula with more than two dimensions, it is required that $\delta < 1$. The cross-ratio function is

$$CR[v] = (1+\delta)\frac{(1-v)^{-1} - 1}{(1-v)^{-\delta} - 1},$$

which is increasing in v for $\delta < 1$ and decreasing in v for $\delta > 1$. The limiting association as time approaches infinity is $1 + \delta^{-1}$. The initial value of the cross ratio function is at time 0 is ∞ for $\delta < 1$ and 0 for $\delta > 1$.

For two dimensions, a scatterplot for $\delta = 0$ (that is, Kendall's tau equal to zero) is displayed in Figure 1. Note the spike in the bottom left corner, showing the lower tail dependence which the independence copula does not have.

3.3 Derivative of utility function as basis for generator

In the bulk of literature about expected utility based decision models, utility families are discussed where a functional form for the utility function is given. Although probably most people would consider this property as a definite asset, a recent contribution by Meyer, see [26], suggests that knowledge of the marginal utility function itself is sufficient to represent risk preferences under the axiom of individuals maximizing expected utility. In other words: it is the availability of a closed form for the derivative of a utility function that matters, rather than a closed form for the utility function itself.

In this section, we consider Archimedean generators derived from the first derivative of a utility function. Indeed, one possible reason to rely on the derivative is that the utility function itself has no closed form expression, and therefore the Archimedean generator does not have it either. And even if the utility function does have a functional form, it may not have a closed form for the inverse of the utility function, and therefore not for the inverse of the Archimedean generator either. In the latter case, it would be hard to work out if the resulting Archimedean copula is positively or negatively ordered in terms of its parameters.

We give some examples to illustrate these points. A drawback of using the first derivative as basis is that an additional constraint needs to be met. Whereas any strictly increasing Morgenstern utility function is concave, we only know that the first derivative is strictly positive and decreasing. It is only convex if the third derivative is nonnegative. If this condition is satisfied, alongside all the basic conditions spelled out in Section 2 (i.e. the domain includes an interval $[0, J)$ with $J > 0$ and allowed to be infinity; b) $\psi'(0)$ well defined and finite) the generator is

$$\varphi_\beta(s) = \max\left[1 - \beta\left(\psi'(0) - \psi'(s)\right), 0\right], \quad s \geq 0,$$

and if the requirements of strictness are satisfied:

$$\varphi(s) = \frac{\psi'(s) - \psi'(\infty)}{\psi'(0) - \psi'(\infty)}, \quad s \geq 0.$$

Example 17 *Example 2 of Pratt (1964) gives the derivative as in (9), leading to a an Archimedean copula that is essentially the same as the one derived from Conniffe's family as discussed above. To ensure DARA, Pratt imposes the constraint $\gamma > 0$ which is stronger than $\gamma > -d^{-1}$.*

Example 18 *[4] and [5] consider a class of utility functions where the marginal utility is a Laplace transform of a distribution function:*

$$\psi'(s) = \int_{x=0}^{\infty} e^{-sx} dF(x), \quad s > 0.$$

where F is a distribution function defined on $[0, \infty)$. It is well known that Laplace transforms have values running from 1 (for $s = 0$) to 0 (for $s \to \infty$). Moreover, they are completely monotone, i.e. they have derivatives alternating in sign. For β equal to the minimum value of 1, we get

$$\varphi(s) = \psi'(s) = \int_{x=0}^{\infty} e^{-sx} dF(x).$$

In words, the generator is equal to a Laplace transform, which is the class that leads to a valid Archimedean generator for any dimension d. Assuming $\psi(0) = 0$, [4] and [5], also show that

the utility has a functional form

$$\psi(s) = \int_{x=0}^{\infty} \frac{1-e^{-sx}}{x} dF(x),$$

which is less tractable than the form of the derivative.

Example 19 *As shown in [8], the derivative of a SAHARA utility function as in (7) - is*

$$\psi'(s) = \left(s + \sqrt{\delta^2 + s^2}\right)^{-\theta}, \quad \theta > 0.$$

defining $\theta = 1-\gamma$, where γ is as in (7). This derivative, contrary to the SAHARA function, has a functional form for the inverse. For $s \to \infty$, we have that $\psi'(s) \to 0$. The strict generator is

$$\varphi(s) = \left(s + \sqrt{1 + s^2}\right)^{-\theta}.$$

Some properties of the copula generated are set out in [33], whereas the family is also involved in a study on mortality of coupled lives in [23], who named it the "Special Copula". Both these contributions give a representation in terms of the inverse of the derivative which is

$$\varphi^{-1}(s) = \frac{1}{2}\left(s^{-\frac{1}{\theta}} - s^{\frac{1}{\theta}}\right).$$

This family is negatively ordered in θ, and covers the entire spectrum of positive dependence, from independence ($\theta \to \infty$) to comonotonicity ($\theta \to 0$). Kendall's tau, however, has an awkward expression (in terms of PolyGamma functions). This family has no upper tail dependence whereas the lower tail dependence coefficient is equal to $2^{-\theta}$. The parameter space gets more constrained for higher dimensions. For instance $\theta \geq 1$ is required for $d = 3$, while the condition $\theta \geq 2$ is required to generate and Archimedean copula in four dimensions. Note that for $\theta = 1$, Nelsen points out that φ is 3-monotonic, but not 4-monotonic (see [28], Exercise 4.25, p. 155). For two dimensions, the copula features LTD/RTI, but no SI for $\theta \leq 1$ and no TP2 for $\theta \leq 2$. Families that are LTD/RTI without being SI could be useful when modelling dependence of remaining lifetimes of members of a couple. The "broken heart syndrome", experienced in some empirical studies, indicates that bereaved lives whose partner died recently have a higher mortality than those who lost their partner years ago.

4 Deriving the inverse of the generator from the utility function

For $v \in [0,1]$, $-u(s)$ could serve as a(n) (inverse) generator of an Archimedean copula, provided that: a) I_1 includes the interval $[0,1]$ b) $u(1) = 0$. Applying the second condition gives

$$\varphi^{[-1]}(v) = \psi(1) - \psi(v), \quad 0 \leq s \leq 1. \tag{10}$$

Note that the factor $\beta > 0$ has been omitted since $\varphi^{[-1]}$ and $\beta\varphi^{[-1]}$ produce the same copula. The generator is then

$$\varphi(s) = \max\left[\psi^{-1}(\psi(1) - s), 0\right], \tag{11}$$

being strict if $\lim_{s\downarrow 0} \psi(s) = -\infty$.

In one respect, expressing the inverse of the generator (rather than the generator itself) in terms of the utility function seems to be a natural approach. Consider an individual with initial

wealth equal to zero, who considers participating in a lottery without stake which would net him an amount $x_i \in [0,1]$ with probability $\pi_i > 0, i \in \{1, ..., d\}$. Then the expected utility after the lottery would be

$$\sum_{i=1}^{d} \pi_i \psi(x_i).$$

Define y to be the *certainty equivalent* of this lottery. The individual would be indifferent between receiving that amount y with certainty and participating in the lottery. This satisfies the equation

$$\psi(y) = \sum_{i=1}^{d} \pi_i \psi(x_i), \tag{12}$$

which, solving for y, gives

$$y = \psi^{-1}\left(\sum_{i=1}^{d} \pi_i \psi(x_i)\right),$$

Due to Jensen's inequality, people who are risk averse (i.e. have a concave utility function) would require a certainty equivalent to be smaller than the expected payoff of $\sum_{i=1}^{d} \pi_i x_i$. In loose terms, the "more risk averse" the individual, the smaller the certainty equivalent y.

Note that $y \in [0,1]$. From (12), we have

$$\psi(1) - \psi(y) = \sum_{i=1}^{d} \pi_i (\psi(1) - \psi(x_i)),$$

which is equivalent to

$$\varphi^{[-1]}(y) = \sum_{i=1}^{d} \pi_i \varphi^{[-1]}(x_i) = \sum_{i=1}^{d} \varphi^{[-1]}(v_i),$$

with v_i such that $\pi_i \varphi(x_i) = \varphi(v_i)$, $i \in \{1, ..., d\}$. This leads to

$$y = \varphi\left(\sum_{i=1}^{d} \varphi^{[-1]}(v_i)\right),$$

which looks like a typical expression for an Archimedean copula, on the face of it very similar to (1). Note, however, that for a strict generator (i.e. $\lim_{s \downarrow 0} \psi(s) = -\infty$), $x_i = 0 \Longrightarrow v_i = 0$, $i \in \{1, ..., d\}$, whereas for a non-strict generator, all v_i are strictly positive. Obviously, the domain of a copula function is $[0,1]^d$, rather than $(0,1]^d$. The copula function obtained from (10) and (11) is

$$C_\varphi(v_1, ..., v_d) = \max\left[\psi^{-1}\left(\sum_{i=1}^{d} \psi(v_i) - (d-1)\psi(1)\right), 0\right]. \qquad 0 \le v_1, ..., v_d \le 1.$$

The above suggests that the greater the risk aversion represented by the underlying utility, the more positive the dependence of the Archimedean copula. We next show that this is indeed the case.

4.1 Relationships between properties of utility function and properties of inverse generator

While the generator φ gives information about the type of dependence of the copula (LTD/RTI, SI, etc., as above, as well as tail dependence), it is its inverse $\varphi^{[-1]}$ that permits to judge on a possible positive or negative order between two copulas. For the sake of brevity, we will confine the analysis to two dimensions only. Now let $\varphi_1^{[-1]}$ and $\varphi_2^{[-1]}$ be inverse generators concerned. Using the PKD-order from Section 4, from (5) we observe that $C_{\varphi_1} \prec_{PKD} C_{\varphi_2}$ if and only if

$$-\frac{\varphi_1^{[-1]}(v)}{\left(\varphi_1^{[-1]}\right)'(v^+)} \geq -\frac{\varphi_2^{[-1]}(v)}{\left(\varphi_2^{[-1]}\right)'(v^+)} \quad \forall v \in [0,1],$$

which is equivalent to

$$RL_{\psi_1}(v) = \frac{\psi_1'(v)}{\psi_1(1) - \psi_1(v)} \leq \frac{\psi_2'(v)}{\psi_2(1) - \psi_2(v)} = RL_{\psi_2}(v) \quad \forall v \in [0,1].$$

In words: C_{φ_2} exhibiting stronger dependence in PKD-sense than C_{φ_1} is equivalent to the underlying utility function having greater asymptotic risk aversion. Note that Kendall's tau, denoted by τ_ψ can be expressed as

$$\tau_\psi = 1 + 4\int_{v=0}^{1} \frac{\varphi^{[-1]}(v)}{\left(\varphi^{[-1]}\right)'(v^+)} dv = 1 - 4\int_{v=0}^{1} \left\{RL_\psi(v)\right\}^{-1} dv,$$

clearly showing that this measure of concordance is increasing as a function of the coefficient of asymptotic risk aversion.

Remark 20 *As pointed out in Remark 14, the Fréchet upper bound arises as a limiting case when* $\varphi^{[-1]}(v) \Big/ \left(\varphi^{[-1]}\right)'(v^+)$ *tends to zero for each* $v \in [0,1]$ *as the copula parameter attains the value concerned. This implies that comonotonicity is equivalent to an asymptotic risk aversion coefficient of infinity.*

As demonstrated in [6], $C_{\varphi_1} \prec_{PKD} C_{\varphi_2}$ is equivalent to $\varphi_1^{[-1]} \Big/ \varphi_2^{[-1]}$ nondecreasing, which, as shown in [28] and [14] is implied by $\left(\varphi_1^{[-1]}\right)' \Big/ \left(\varphi_2^{[-1]}\right)'$ nondecreasing, which is equivalent to

$$AR_{\psi_1}(v) \leq AR_{\psi_2}(v) \quad \forall v \in [0,1].$$

Therefore, an order between two Archimedean copulas is implied by an order between the Arrow-Pratt risk aversion coefficients.

Remark 21 *It follows from above that an order between Arrow-Pratt risk aversion coefficients implies an order between coefficients of asymptotic risk aversion. This is formally proved in [13].*

A key measure of time-dependent association is given by Oakes' cross-ratio function, which was already involved in the discussion of some examples above. From (8) we see that $CR[v]$ is uniformly greater than 1 if and only if $\varphi''(s)\varphi(s) - (\varphi'(s))^2 \geq 0$ for all $s \geq 0$, which is

equivalent to LTD or RTI of the copula. Oakes shows in [29] that the cross ratio function can also be expressed in terms of the inverse generator.

$$CR[v] = -v\frac{\left(\varphi^{[-1]}(v)\right)''}{\left(\varphi^{[-1]}(v)\right)'}, \quad v \in [0,1].$$

Substituting (10) in the expression above gives

$$CR[v] = RR_{\psi}(v), \quad v \in [0,1],$$

where

$$RR_{\psi}(v) = -v\frac{\psi(v)''}{\psi(v)'} = vAR_{\psi}(v),$$

is the Arrow-Pratt coefficient of *relative* risk aversion (again, see [1] and [30] for further details). It follows that LTD/RTI (LTI/RTD) is equivalent to relative risk aversion uniformly being greater (smaller) than one. Also note that in biostatistical applications the argument v of the cross-ratio function is usually a multivariate survival probability which is decreasing as a function of duration. This means that $v = 1$ corresponds to time zero and $CR[v]$ is usually plotted as a function of $1 - v$ rather than v. It follows that a cross-ratio function decreasing in time is equivalent to relative risk aversion increasing in wealth and vice versa.

We could display several examples of Archimedean copulas that can be generated in the way as above. For instance, [30] lists several utility functions. Unfortunately, however, almost all utility families take a finite value at 0 and therefore fail to lead to strict generators, therefore limiting the value in applications. Therefore we will confine ourselves to the special cases of CARA (Constant Absolute Risk Aversion) and CRRA (see Remark 4).

Example 22 (CARA) *Utility functions with Constant Absolute Risk Aversion are specified as* $\psi(v) = -\exp[-\gamma v], \quad \gamma > 0$. *They derive their name from* $AR_{\psi}(v) \equiv \gamma$, *being independent of* v. *This gives* $\varphi^{[-1]}(v) = \exp[-\gamma v] - \exp[-\gamma]$, *being a non-strict generator, and* $\varphi(s) = \max[-\ln[\exp[-\gamma] + s]/\gamma, 0]$. *Clearly, this family is positively ordered in* γ. *It covers the entire range of dependence from countermonotonicity (for* $\gamma \downarrow 0$*) to comonotonicity (for* $\gamma \to \infty$*). Kendall's tau is equal to* $\left((\gamma-2)^2 + 4\exp[-\gamma]\right)\gamma^{-2}$. *The family has no tail dependence (except for the obvious case of* $\gamma \to \infty$*). Given that the generator is not strict, the copula cannot feature LTD or RTI. This also follows from the fact that* $CR[v] = \gamma v$ *which will be smaller than* 1 *for* v *large enough.*

Example 23 (CRRA) *Utility functions featuring CRRA are given in summarized format as* $\psi(v) = v^{1-\gamma}/(1-\gamma)$ *for* $\gamma \geq 0$ *with the special case* $\psi(v) = \ln v$ *for* $\gamma \to 1$. *we get* $\varphi^{[-1]}(v) = \left(1 - v^{1-\gamma}\right)/(1-\gamma)$ *which is the Clayton family. This is no surprise since Clayton is the only copula family with constant cross-ratio function (see [29]).*

5 Conclusions

In this paper, we have derived an Archimedean copula from a Morgenstern utility function in two different ways, which have also led to different ways of extracting information about dependence featured in the copula from the risk attitude exhibited by the corresponding utility function.

The first method - expressing the generator directly in terms of the utility function - leads to a copula family with an additional parameter, which, if set at its minimum value, often gives a copula with strict generator, therefore more likely to be suitable for applications in more than

two dimensions. It also allows to judge on the type of dependence of the copula. Several features of a utility function that are generally considered as desirable in wide areas as economics and decision theory include Decreasing Absolute Risk Aversion and Decreasing Absolute Prudence, as well as a nonnegative third derivative and a nonpositive fourth derivative. It is striking to see how such properties are translated into characteristics of the corresponding Archimedean generator - like feasibility in higher dimensions, SI and MTP2 - that are taken to be desirable in fields like statistics, finance and actuarial science. This is perhaps best illustrated by the class due to [4] and [5], that contains the most commonly applied utility functions. The Archimedean "counterpart" is the class of completely monotone functions yielding Archimedean copulas for any dimension. Other examples include the HARA and Xie/Saha utility families, from which we derive the Clayton and Gumbel-Hougaard copula, respectively.

While some copulas derived as above are already well established, there are some others that are not. An example is the SAHARA utility function that was only recently introduced in the literature. The corresponding copula family is new, to the best of our knowledge, while the family derived from the first derivative of SAHARA utility has not been widely applied so far.

On the other hand, the second method - expressing the inverse of the generator directly in terms of the utility function - permits to judge on the magnitude of dependence featured by the copula. Ordering copulas by means of the asymptotic risk aversion coefficient and the very common Arrow-Pratt measures of absolute and relative risk aversion is straightforward. The method also looks natural as it allows (to some extent) for an interpretation of the Archimedean copula in terms of a certainty equivalent under the expected utility hypothesis. A slight drawback - compared to the first method - is that most utility families in the literature do not start at minus infinity and therefore do not give rise to a strict generator.

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