1. INTRODUCTION

Asset pricing models generally express expected returns on financial assets as linear functions of covariances of returns with some systematic risk factors. Several formulations of this general paradigm have been proposed in the literature (Sharpe 1964;Lintner 1965; Black 1972; Merton 1973; Rubinstein 1973; Kraus and Litzenberger 1976; Ross 1976; Breeden 1979; Barone Adesi and Talwar 1983; Barone Adesi 1985; Jagannathan and Wang 1996; Harvey and Siddique 2000; Dittmar 2002). However, most of the empirical tests suggested to date have produced negative or ambiguous results. These findings have spurred renewed interest in the statistical properties of the currently available testing methodologies. Among recent studies, Shanken (1992) and Kan and Zhang (1999a,b) analyzed the commonly used statistical methodologies and highlighted the sources of ambiguity in their findings.

Although a full specification of the return-generating process is not needed for the formulation of most asset pricing models, it appears that only its a priori knowledge may lead to the design of reliable tests. Because this condition is never met in practice, researchers are forced to make unpalatable choices between two alternative approaches. On the one hand, powerful tests can be designed in the context of a (fully) specified return-generating process, but they are misleading in the presence of possible model misspecifications. On the other hand, more tolerant tests may be considered, but they may not be powerful, as noted by Kan and Zhou (1999) and Jagannathan and Wang (2001). Note that the first choice may lead not only to the rejection of the correct model, but also to the acceptance of irrelevant factors as sources of systematic risk, as noted by Kan and Zhang (1999a,b).

To complicate the picture, a number of empirical regularities have been detected that are not consistent with standard asset pricing models, such as the mean-variance capital asset pricing model (CAPM). Among other studies, Banz (1981) related expected returns to firm size, and Fama and French (1995) linked expected returns also to the ratio of book value to market value. Although the persistence of these anomalies over time is still subject to debate, the evidence suggests that the mean-variance CAPM is not a satisfactory description of market equilibrium. These pricing anomalies may be related to the possibility that useless factors appear to be priced. Of course, it is also possible that pricing anomalies are due to omitted factors. Although statistical tests do not allow us to choose between these two possible explanations of pricing anomalies, Kan and Zhang (1999a,b) suggested that perhaps a large increment in $R^2$ and the persistence of sign and size of coefficients over time are most likely to be associated with truly priced factors.

In the light of the foregoing, the aim of this article is to consider market coskewness and to investigate its role in testing asset pricing models. A dataset of monthly returns on 10 stock portfolios is used. Following Harvey and Siddique (2000), an asset is defined as having "positive coskewness" with the market when the residuals of the regression of its returns on a constant and the market returns are positively correlated with squared market returns. Therefore, an asset with positive (negative) coskewness reduces (increases) the risk of the portfolio to large absolute market returns, and should command a lower (higher) expected return in equilibrium.

Rubinstein (1973), Kraus and Litzenberger (1976), Barone Adesi (1985), and Harvey and Siddique (2000) studied nonnormal asset pricing models related to coskewness. Kraus and Litzenberger (1976) and Harvey and Siddique (2000) formulated expected returns as a function of covariance and coskewness with the market portfolio. In particular, Harvey and Siddique (2000) assessed the importance of coskewness in explaining expected returns by the increment of $R^2$ in cross-sectional regressions. More recently, Dittmar (2002) presented a framework in which agents are also adverse to kurtosis, implying that asset returns are influenced by both coskewness and cokurtosis with the return on aggregate wealth. The author tests...
an extended asset pricing model within a generalized method of moments (GMM) framework (see Hansen 1982).

Most of the foregoing formulations are very general, because the specification of an underlying return-generating process is not required. However, we are concerned about their possible lack of power, made worse in this context by the fact that covariance and coskewness with the market are almost perfectly collinear across portfolios. Of course, in the extreme case, in which market covariance is proportional to market coskewness, it will be impossible to identify covariance and coskewness premia separately. Therefore, to identify and accurately measure the contribution of coskewness, in this article we propose an approach (see also Barone Adesi 1985) based on the prior specification of an appropriate return-generating process, the quadratic market model. The quadratic market model is an extension of the traditional market model (Sharpe 1964;Lintner 1965), including the square of the market returns as an additional factor. The coefficients of the quadratic factor measure the marginal contribution of coskewness to expected excess returns. Because market returns and the square of the market returns are almost orthogonal regressors, we obtain a precise test of the significance of quadratic coefficients. In addition, this framework allows us to test an asset pricing model with coskewness by checking the restrictions that it imposes on the coefficients of the quadratic market model. The specification of a return-generating process provides more powerful tests, as confirmed in a series of Monte Carlo simulations (see Sec. 5).

In addition to evaluating asset pricing models that include coskewness, it is also important to investigate the consequences on asset pricing tests when coskewness is erroneously omitted. We consider the possibility that portfolio characteristics, such as size, are empirically found to explain expected excess returns because of the omission of a truly priced factor, namely coskewness. To explain this problem, let us assume that coskewness is truly priced but is omitted in an asset pricing model. Then, if market coskewness is correlated with a variable such as size, this variable will have spurious explanatory power for the cross-section of expected returns, because it proxies for omitted coskewness. In our empirical application (see Sec. 4), we actually find that coskewness and firm size are correlated. This finding suggests that the empirically observed relation between size and assets excess returns may be explained by the omission of a systematic risk factor, namely market coskewness (see also Harvey and Siddique 2000, p. 128).

The article is organized as follows. Section 2 introduces the quadratic market model. An asset pricing model including coskewness is derived using arbitrage pricing, and the testing of various related statistical hypotheses is discussed. Section 3 reports estimators and test statistics used in the empirical part of the article. Section 4 describes the data and reports empirical results. Section 5 provides Monte Carlo simulations for investigating the finite sample properties of our test statistics, and Section 6 concludes.

2. ASSET PRICING MODELS WITH COSKEWNESS

In this section we introduce the econometric specifications considered in the article. We describe the return-generating process, derive the corresponding restricted equilibrium models, and finally compare our approach with a GMM framework.

2.1 The Quadratic Market Model

Factor models are among the most widely used return-generating processes in financial econometrics. They explain movements in asset returns as arising from the common effect of a (small) number of underlying variables, called factors (see, e.g., Campbell, Lo, and MacKinlay 1987; Gourieroux and Jasiak 2001). In this article we use a linear two-factor model, the quadratic market model, as a return-generating process. Market returns and the square of the market returns are the two factors. Specifically, we denote by \( R_t \) the \( N \times 1 \) vector of returns in period \( t \) of \( N \) portfolios and by \( R_{M,t} \) the return of the market. If \( R_{F,t} \) is the return in period \( t \) of a (conditionally) risk-free asset, then portfolio and market excess returns are defined by \( r_t = R_t - R_{F,t} \) and \( r_{M,t} = R_{M,t} - R_{F,t} \), where \( t \) is a \( N \times 1 \) vector of 1's. Similarly, the excess squared market return is defined by \( q_{M,t} = R_{M,t}^2 - R_{F,t}^2 \).

The quadratic market model is specified by

\[
\begin{align*}
  r_t &= \alpha + \beta R_{M,t} + \gamma q_{M,t} + \epsilon_t, \\
  \epsilon_t &= \epsilon_{t-1} + \vdots + \epsilon_1 + \epsilon_{t+1} + \cdots + \epsilon_T,
\end{align*}
\]

where \( \alpha, \beta, \gamma \) are known constants (e.g., \( \alpha = 0 \) in (1)), \( \beta, \gamma \) are unknown constants, \( \epsilon_t \) is an \( N \times 1 \) vector of errors satisfying

\[
E[\epsilon_t | R_{M,t}, R_{F,t}] = 0
\]

with \( R_{M,t} \) and \( R_{F,t} \) denoting all present and past values of \( R_{M,t} \) and \( R_{F,t} \).

The quadratic market model is a direct extension of the well-known market model (Sharpe 1964; Lintner 1965), which corresponds to restriction \( \gamma = 0 \) in (1),

\[
\begin{align*}
  r_t &= \alpha + \beta R_{M,t} + \epsilon_t, \\
  \epsilon_t &= \epsilon_{t-1} + \vdots + \epsilon_1 + \epsilon_{t+1} + \cdots + \epsilon_T,
\end{align*}
\]

The motivation for including the square of the market returns is to fully account for coskewness with the market portfolio. In fact, deviations from the linear relation between asset returns and market returns implied by (2) are empirically observed. More specifically, for some classes of assets, residuals from the regression of returns on a constant and market returns tend to be positively (negatively) correlated with squared market returns. These assets therefore show a tendency to have relatively higher (lower) returns when the market experiences high absolute returns, and are said to have positive (negative) coskewness with the market. This finding is supported by our empirical investigations in Section 4, where, in accordance with the results of Harvey and Siddique (2000), we find that portfolios formed by assets of small firms tend to have negative coskewness with the market, whereas portfolios formed by assets of large firms have positive market coskewness. In addition to classical beta, market coskewness is therefore another important risk characteristic; an asset that has positive coskewness with the market diminishes the sensitivity of a portfolio to large absolute market returns. Therefore, everything else being equal, investors should prefer assets with positive market coskewness to those with negative coskewness. The quadratic market model (1) is a specification that provides us with a very simple way to take into account market coskewness. Indeed, we have

\[
y = \frac{1}{V[\epsilon_{q,t}]} \text{cov}[\epsilon_t, R_{M,t}^2].
\]
where \( \epsilon_t \) (resp. \( \epsilon_q t \)) are the residuals from a theoretical regression of portfolio returns \( R_t \) (market square returns \( R_{M,t}^2 \), resp.) on a constant and market return \( R_{M,t} \). Because coefficients \( y \) are proportional to \( \text{cov}[\epsilon_t, R_{M,t}] \), we can use the estimate of \( y \) in model (1) to investigate the coskewness properties of the 
N portfolios in the sample. Moreover, although \( y \) does not correspond exactly to the usual probabilistic definition of market coskewness, coefficient \( y \) is a very good proxy for \( \text{cov}(r_t, R_{M,t}^2) / V(R_{M,t}^2) \), as pointed out by Kraus and Litzenberger (1976). Within our sample, the approximation error is smaller than 1% (see App. A). Finally, the statistical (joint) significance of coskewness coefficient \( y \) can be assessed by testing the null hypothesis \( H_F^y \) against the alternative \( H_F \).

### 2.2 Restricted Equilibrium Models

From the standpoint of financial economics, a linear factor model is only a return-generating process, which is not necessarily consistent with notions of economic equilibrium. Constraints on its coefficients are imposed for example by arbitrage pricing (Ross 1976; Chamberlain and Rothschild 1983). The arbitrage pricing theory implies that expected excess returns of assets following the factor model (1) satisfy the restriction (Barone-Adesi 1985)

\[
E(r_t) = \beta \lambda_1 + y \lambda_2, \tag{4}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are expected excess returns on portfolios whose excess returns are perfectly correlated with factors \( R_{M,t} \) and \( q_{M,t} \). Equation (4) is in the form of a typical linear asset pricing model, which relates expected excess returns to covariances and coskewnesses with the market. In this article we test the asset pricing model with coskewness (4) through the restrictions that it imposes on the coefficients of the return-generating process (1). We derive these restrictions. Because the excess market return \( R_{M,t} \) satisfies (4), it must be that

\[
\lambda_1 = E(R_{M,t}). \tag{5}
\]

A similar restriction does not hold for the second factor \( q_{M,t} \) because it is not a traded asset. However, we expect \( \lambda_2 < 0 \), because assets with positive coskewness decrease the risk of a portfolio with respect to large absolute market returns and thus should command a lower risk premium in an arbitrage equilibrium. By taking expectations on both sides of (1) and substituting (4) and (5), we deduce that the asset pricing model (4) implies the cross-equation restriction \( \alpha = \theta y \), where \( \theta \) is the scalar parameter \( \theta = \lambda_2 / E(q_{M,t}) \). Thus arbitrage pricing is consistent with the restricted model

\[
r_t = \beta R_{M,t} + y q_{M,t} + \theta \tilde{y} + \epsilon_t, \quad t = 1, \ldots, T. \tag{6}
\]

\[
H_1: \quad \exists \; \tilde{\theta}: \alpha = \tilde{\theta} y \quad \text{in} \; (1). \tag{7}
\]

Therefore, the asset pricing model with coskewness (4) is tested by testing \( H_1 \) against \( H_F \).

When model (4) is not supported by data, there exists an additional component \( \bar{\alpha} \) (a \( N \times 1 \) vector) in expected excess returns that cannot be fully related to market risk and coskewness risk, \( E(r_t) = \beta \lambda_1 + y \lambda_2 + \bar{\alpha} \). In this case, intercepts \( \alpha \) of model (1) satisfy \( \alpha = \tilde{\theta} y + \bar{\alpha} \). It is crucial to investigate how the additional component \( \bar{\alpha} \) varies across assets. Indeed, if this component arises from an omitted factor, then it will provide us with information about the sensitivities of our portfolios to this factor. Furthermore, variables representing portfolio characteristics, which are correlated with \( \bar{\alpha} \) across portfolios, will have spurious explanatory power for expected excess returns, because they are a proxy for the sensitivities to the omitted factor. A case of particular interest is when \( \bar{\alpha} \) is homogeneous across assets, \( \bar{\alpha} = \lambda_0 \tilde{t} \), where \( \lambda_0 \) is a scalar, that is,

\[
E(r_t) = \lambda_0 + \beta \lambda_1 + y \lambda_2, \tag{7}
\]

corresponding to the specification

\[
r_t = \beta R_{M,t} + y q_{M,t} + \theta \tilde{y} + \lambda_0 \tilde{t} + \epsilon_t, \quad t = 1, \ldots, T. \tag{8}
\]

Specification (8) corresponds to the case where the factor omitted in model (4) has homogeneous sensitivities across portfolios. From (7), \( \lambda_0 \) may be interpreted as the expected excess returns of a portfolio with covariance and coskewness with the market both equal to 0. Such a portfolio may correspond to the analogous of the zero-beta portfolio in the Black version of the capital asset pricing model (Black 1972). Alternatively, \( \lambda_0 > 0 \) \( (\lambda_0 < 0) \) may be due to the use of a risk-free rate lower (higher) than the actual rate faced by investors. With reference to the observed empirical regularities and model misspecifications mentioned in Section 1, the importance of model (8) is that if hypothesis \( H_2 \) is not rejected against \( H_F \), then we expect portfolio characteristics such as size to have additional explanatory power for expected excess returns, once coskewness is taken into account. In addition, a more powerful evaluation of the validity of the asset pricing model (4) should be provided by a test of \( H_1 \) against the alternative \( H_F \).

### 2.3 The Generalized Method of Moments Framework

Asset pricing models of the type (4) have been considered by Kraus and Litzenberger (1976) and Harvey and Siddique (2000). Harvey and Siddique (2000) introduced their specification as a model in which the stochastic discount factor is quadratic in market returns. Specifically, in our notation, the asset pricing model with coskewness (4) is equivalent to the orthogonality condition

\[
E[r_{M,t}(\delta)] = 0, \tag{9}
\]

where the stochastic discount factor \( m_t(\delta) \) is given by \( m_t(\delta) = 1 - R_{M,t} \delta_1 - q_{M,t} \delta_2 \) and \( \delta = (\delta_1, \delta_2) \) is a two-dimensional parameter. A quadratic stochastic discount factor \( m_t(\delta) \) can be justified as a (formal) second-order Taylor expansion of a stochastic discount factor, which is nonlinear in the market returns. Thus in the GMM approach, the derivation and testing of the orthogonality condition (9) do not require a prior specification of a data-generating process.

More recently, in a conditional GMM framework, Dittmar (2002) used a stochastic discount factor model embodying both quadratic and cubic terms. The validity of the model is tested by a GMM statistic using the weighting matrix proposed by Jagannathan and Wang (1996) and Hansen and Jagannathan (1997). As explained earlier, the main contribution of our article, beyond the results obtained by Harvey and Siddique (2000) and Dittmar (2002), is that we focus on testing the asset pricing model with coskewness (4) through the restrictions that it imposes on the return-generating process (1), instead of adopting a methodology using an unspecified alternative (e.g., a GMM test).
3. ESTIMATORS AND TEST STATISTICS

In this section we derive the estimators and test statistics used in our empirical applications. Following an approach widely adopted in the literature (see, e.g., Campbell et al. 1987; Gourieroux and Jasiak 2001), we consider the general framework of pseudo-maximum likelihood (PML) methods. We derive the statistical properties of the estimators and test statistics within the different coskewness asset pricing models presented in Sections 2.1 and 2.2. For completeness, we provide full derivations in the Appendixes.

3.1 The Pseudo-Maximum Likelihood Estimator

We assume that the error term \( e_t \) in model (1) with \( t = 1, \ldots, T \) is a homoscedastic martingale difference sequence satisfying:

\[
E[e_t | e_{t-1}, R_{M,t}, R_{F,t}] = 0,
\]
\[
E[e_t e'_t | e_{t-1}, R_{M,t}, R_{F,t}] = \Sigma,
\]

where \( \Sigma \) is a positive-definite \( N \times N \) matrix. The factor \( \mathbf{f}_t = (r_{M,t}, q_{M,t})' \) is supposed to be exogenous in the sense of Engle, Hendry, and Richard (1988). The expectation and the variance-covariance matrix of factor \( \mathbf{f}_t \) are denoted by \( \mu \) and \( \Sigma_f \). Statistical inference in the asset pricing models presented in Section 2 is conveniently cast in the general framework of PML methods (White 1981; Gourieroux, Monfort, and Trognon 1984; Bollerslev and Wooldridge 1992). If \( \hat{\theta} \) denotes the parameter of interest in the model under consideration, then the PML estimator is defined by the maximization

\[
\hat{\theta} = \arg \max_\theta L_T(\theta),
\]

where the criterion \( L_T(\theta) \) is a (conditional) pseudo-log-likelihood. More specifically, \( L_T(\theta) \) is the (conditional) log-likelihood of the model when we adopt a given conditional distribution for error \( e_t \) that satisfies (10) and is such that the resulting pseudo-true density of the model is exponential. Under standard regularity assumptions, the PML estimator \( \hat{\theta} \) is consistent for any chosen conditional distribution of error \( e_t \) satisfying the foregoing conditions (see the aforementioned references). Estimator \( \hat{\theta} \) is efficient when the pseudo-conditional distribution of \( e_t \) coincides with the true one, being then the PML estimator identical with the maximum likelihood (ML) estimator. Finally, because the PML estimator is based on the maximization of a statistical criterion, hypothesis testing can be conducted by the usual general asymptotic tests.

In what follows, we systematically analyze in the PML framework the alternative specifications introduced in Section 2.

3.2 The Return-Generating Process

The quadratic market model (1) [and the market model (2)] are seemingly unrelated regressions (SUR) systems (Zellner 1962), with the same regressors in each equation. Let \( \theta \) denote the parameters of interest in model (1),

\[
\theta = (\alpha', \beta', \gamma', \text{vech}(\Sigma)'),
\]

where \( \text{vech}(\Sigma) \) is a \((N+1)N/2 \times 1\) vector representation of \( \Sigma \) containing only elements on and above the main diagonal. The PML estimator of \( \theta \) based on the normal family is obtained by maximizing

\[
L_T(\theta) = -\frac{T}{2} \log \det \Sigma - \frac{1}{2} \sum_{t=1}^{T} e_t(\theta)' \Sigma^{-1} e_t(\theta),
\]

where

\[
e_t(\theta) = r_t - \alpha - \beta r_{M,t} - \gamma q_{M,t}, \quad t = 1, \ldots, T.
\]

As is well known, the PML estimator for \((\alpha', \beta', \gamma')'\) is equivalent to the generalized least squares (GLS) estimator on the SUR system and also to the ordinary least squares (OLS) estimator performed equation by equation in model (1). Let \( \mathbf{B} \) denote the \( N \times 3 \) matrix defined by \( \mathbf{B} = [\alpha \beta \gamma]' \). The PML estimator \( \hat{\mathbf{B}} = [\hat{\alpha} \hat{\beta} \hat{\gamma}] \) is consistent when \( T \to \infty \), and its asymptotic distribution is given by

\[
\sqrt{T}(\hat{\mathbf{B}} - \mathbf{B}) \xrightarrow{d} N(0, \Sigma \otimes \mathbf{F} \mathbf{F}'\mathbf{I}^{-1}).
\]

(Upper indices in a matrix denote elements of the inverse.) The statistic \( \xi_{F*}^T \) is asymptotically \( \chi^2(p) \)-distributed, with \( p = N \), when \( T \to \infty \).

3.3 Restricted Equilibrium Models

We now consider the constrained models (6) and (8) derived by arbitrage equilibrium. The estimation of these models is less simple, because they entail cross-equation restrictions. We let

\[
\theta = (\beta', \gamma', \hat{\theta}, \lambda_0, \text{vech}(\Sigma)'),
\]

denote the vector of parameters of model (8). The PML estimator of \( \theta \) based on a normal pseudo-conditional log-likelihood is defined by maximization of

\[
L_T(\theta) = -\frac{T}{2} \log \det \Sigma - \frac{1}{2} \sum_{t=1}^{T} e_t(\theta)' \Sigma^{-1} e_t(\theta),
\]

where

\[
e_t(\theta) = r_t - \beta r_{M,t} - \gamma q_{M,t} - \gamma \hat{\theta} - \lambda_0, \quad t = 1, \ldots, T.
\]

The PML estimator \( \hat{\theta} \) is given by the following system of implicit equations (see App. B):

\[
(\hat{\beta}', \hat{\gamma}') = \left( \sum_{t=1}^{T} (r_t - \lambda_0) \hat{H}_t \right) \left( \sum_{t=1}^{T} \hat{H}_t \hat{H}_t \right)^{-1},
\]

\[
(\hat{\hat{\theta}}, \hat{\lambda}_0) = (\hat{Z}' \hat{\Sigma}^{-1} \hat{Z})^{-1} \hat{Z}' \hat{\Sigma}^{-1} (\hat{r} - \hat{\beta} \hat{r}_M - \hat{\gamma} \hat{q}_M),
\]

and

\[
\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} \hat{e}_t \hat{e}_t'.
\]
Estimator \((\hat{\beta}', \hat{\gamma}')\) is obtained by (time series) OLS regressions of \(\tilde{r}_t - \tilde{\lambda}_{0t}\) on \(\tilde{H}_t\) in a SUR system, performed by equation, whereas \((\hat{\tilde{\beta}}, \hat{\tilde{\lambda}}_0)'\) is obtained by (cross-sectional) GLS regression of \(\tilde{r}_t - \tilde{\beta} R_{M,t} - \tilde{\gamma} q_{M,t} - \tilde{\lambda}_0 t\) on \(\tilde{Z}\). A step of a feasible algorithm consists of (1) starting from old estimates; (2) computing \((\hat{\beta}', \hat{\gamma}')\) from (16); (3) computing \((\hat{\tilde{\beta}}, \hat{\tilde{\lambda}}_0)'\) from (17) using new estimates for \(\hat{\beta}, \hat{\gamma}, \text{ and } \hat{\Sigma}\); and (4) computing \(\hat{\Sigma}\) from (18), using new estimates. The procedure is iterated until a convergence criterion is met. The starting values for \(\hat{\beta}, \hat{\gamma}, \text{ and } \hat{\Sigma}\) are provided by the unrestricted estimates on model (1), whereas the parameters \(\lambda_0\) and \(\theta\) they are provided by (17), where estimates from (1) are used.

The asymptotic distributions of the PML estimator are reported in Appendix B. In particular, it is shown that the asymptotic variance of the estimator of \((\beta', \gamma', \lambda_0, \lambda_1, \lambda_2)\) is independent of the true distribution of the error term \(e_t\), as long as it satisfies the conditions for PML estimation. The results for the constrained PML estimation of model (6) follow by setting \(\lambda_0 = 0\) and \(\tilde{Z} = \tilde{\gamma}\) and deleting the vector \(t\).

We now consider the problem of testing hypotheses \(H_1\) and \(H_{12}\), corresponding to models (6) and (8), against the alternative \(H_F\). If \(\theta\) denotes the parameter of model (1), then each of these two hypotheses can be written in mixed form,

\[
\{\theta : \exists a \in A \subset \mathbb{R}^d : g(\theta, a) = 0\},
\]

for an appropriate vector function \(g\) with values in \(\mathbb{R}\) and suitable dimensions \(q\) and \(r\). Let us assume that the rank conditions

\[
\text{rank}(\frac{\partial g}{\partial \theta}) = r \quad \text{and} \quad \text{rank}(\frac{\partial g}{\partial a}) = q
\]

are satisfied at the true values \(\theta^0, a^0\). The test of hypothesis (20) based on asymptotic least squares (ALS) consists of verifying whether the constraints \(g(\theta, a) = 0\) are satisfied, where \(\theta\) is an unconstrained estimator of \(\theta\), the PML estimator in our case (Gouriéroux et al. 1985). More specifically, the test is based on the statistic

\[
\xi_T = \arg \min_a T g(\tilde{\theta}, a)' S g(\tilde{\theta}, a),
\]

where \(S\) is a consistent estimator for

\[
S_0 = \left(\frac{\partial g}{\partial \theta} \Omega_0 \frac{\partial g}{\partial \theta}\right)^{-1},
\]

evaluated at the true values \(\theta^0\) and \(a^0\), where \(\Omega_0 = V_{aa} [\sqrt{T} \times (\tilde{\theta} - \theta^0)]\). Under regularity conditions, \(\xi_T\) is asymptotically \(\chi^2(r - q)\)-distributed and is asymptotically equivalent to the other asymptotic tests.

By applying these general results, we derive the ALS statistic for testing the hypotheses \(H_1\) and \(H_{12}\) against the alternative \(H_F\) (see App. C for a full derivation). The hypothesis \(H_1\) against \(H_F\) is tested by the statistic

\[
\xi_T = T \frac{(\hat{\alpha} - \hat{\gamma} \hat{\beta})' \Sigma^{-1} (\hat{\alpha} - \hat{\gamma} \hat{\beta})}{1 + \Sigma_f^{-1} \hat{\lambda}} \rightarrow \chi^2(p),
\]

with \(p = N - 1\), where \(\hat{\lambda} = \hat{\mu} + (0, \hat{\gamma})'\) and

\[
\hat{\gamma} = \arg \min_\theta (\hat{\alpha} - \hat{\gamma} \hat{\beta})' \Sigma^{-1} (\hat{\alpha} - \hat{\gamma} \hat{\beta})
\]

\[
= (\hat{\gamma} \hat{\Sigma}^{-1} \hat{\beta})' \hat{\Sigma}^{-1} \hat{\alpha}.
\]

The ALS statistic for testing hypothesis \(H_2\) against \(H_F\) is given by

\[
\xi_T = T \frac{(\hat{\alpha} - \hat{\gamma} \hat{\beta} - \hat{\lambda}_{0t})' \Sigma^{-1} (\hat{\alpha} - \hat{\gamma} \hat{\beta} - \hat{\lambda}_{0t})}{1 + \Sigma_f^{-1} \hat{\lambda}} \rightarrow \chi^2(p),
\]

with \(p = N - 2\), where \(\hat{\lambda} = \hat{\mu} + (0, \hat{\gamma})'\) and

\[
\hat{\lambda}_{0t} = \arg \min_{\lambda, \theta} (\hat{\alpha} - \hat{\gamma} \hat{\beta} - \lambda_{0t})' \Sigma^{-1} (\hat{\alpha} - \hat{\gamma} \hat{\beta} - \lambda_{0t})
\]

\[
= (\hat{\lambda} \hat{\Sigma}^{-1} \hat{\beta})' \hat{\Sigma}^{-1} \hat{\alpha}, \quad \hat{\lambda} = (\hat{\gamma}, \hat{\beta}).
\]

Finally, a test of hypothesis \(H_1\) against \(H_2\) is simply performed as a \(t\)-test for the parameter \(\lambda_0\).

4. EMPIRICAL RESULTS

In this section we report the results of our empirical application, performed on monthly returns of stock portfolios. We first estimate the quadratic market model, then test the different associated asset pricing models with coskewness. Finally, we investigate the consequences of erroneously neglecting coskewness when testing asset pricing models. The section begins with a brief description of the data.

4.1 Data Description

Our dataset comprises 450 (percentage) monthly returns of the 10 stock portfolios formed according to size by French, for the period July 1963–December 2000. Data are available at http://web.mit.edu/kfrench/www/data\_library.html, in the file “Portfolios formed on size.” The portfolios are constructed at the end of June each year, using June market equity data and NYSE breakpoints. The portfolios from July of year \(t\) to June of \(t + 1\) include all NYSE, AMEX, and NASDAQ stocks for which we have market equity data for June of year \(t\). Portfolios are ranked by firm size, with portfolio 1 being the smallest and portfolio 10 the largest.

The market return is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks. The risk-free rate is the 1-month Treasury Bill rate from Ibbotson Associates. Market returns and risk-free returns are available at http://web.mit.edu/kfrench/www/data\_library.html, in the files “Fama-French benchmark factors” and “Fama-French factors.” We use the T-Bill rate, because other money-market series are not available for the whole period of our tests.
4.2 Results

4.2.1 Quadratic Market Model. We begin with the estimation of the quadratic market model (1). PML-SUR estimates of the coefficients $\alpha$, $\beta$, and $\gamma$ and of the variance $\Sigma$ in model (1) are reported in Tables 1 and 2. As explained in Section 3.2, these estimates are obtained by OLS regressions, performed equation by equation on system (1). As expected, the beta coefficients are strongly significant for all portfolios, with smaller portfolios having higher betas in general. From the estimates of the $\gamma$ parameter, we see that small portfolios have significantly negative coefficients of market coskewness (e.g., $\gamma = -0.17$ for the smallest portfolio). Coskewness coefficients are significantly positive for the two largest portfolios ($\gamma = 0.03$ for the largest portfolio). In particular, we observe that the $\beta$ and $\gamma$ coefficients are strongly correlated across portfolios. We can test for joint significance of the coskewness parameter $\gamma$ by using the Wald statistic $\xi_T^\gamma$ in (14). The statistic $\xi_T^\gamma$

<table>
<thead>
<tr>
<th>Portfolio $i$</th>
<th>$\hat{\alpha}_i$</th>
<th>$\hat{\beta}_i$</th>
<th>$\hat{\gamma}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.18</td>
<td>1.01</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(0.84)</td>
<td>(24.23)</td>
<td>(3.32)</td>
</tr>
<tr>
<td>2</td>
<td>0.22</td>
<td>1.18</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(1.65)</td>
<td>(32.62)</td>
<td>(3.05)</td>
</tr>
<tr>
<td>3</td>
<td>0.18</td>
<td>1.12</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(1.88)</td>
<td>(38.37)</td>
<td>(3.84)</td>
</tr>
<tr>
<td>4</td>
<td>0.23</td>
<td>1.16</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(1.96)</td>
<td>(39.99)</td>
<td>(3.00)</td>
</tr>
<tr>
<td>5</td>
<td>0.28</td>
<td>1.13</td>
<td>-0.09</td>
</tr>
<tr>
<td></td>
<td>(2.73)</td>
<td>(46.94)</td>
<td>(3.34)</td>
</tr>
<tr>
<td>6</td>
<td>0.16</td>
<td>1.10</td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td>(1.59)</td>
<td>(54.02)</td>
<td>(2.58)</td>
</tr>
<tr>
<td>7</td>
<td>0.10</td>
<td>1.05</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(1.29)</td>
<td>(64.37)</td>
<td>(8.88)</td>
</tr>
<tr>
<td>8</td>
<td>0.07</td>
<td>1.08</td>
<td>-0.00</td>
</tr>
<tr>
<td></td>
<td>(1.02)</td>
<td>(72.59)</td>
<td>(1.16)</td>
</tr>
<tr>
<td>9</td>
<td>-0.16</td>
<td>1.07</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(-3.30)</td>
<td>(92.76)</td>
<td>(2.06)</td>
</tr>
<tr>
<td>10</td>
<td>-0.07</td>
<td>0.93</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(-1.10)</td>
<td>(88.77)</td>
<td>(2.04)</td>
</tr>
</tbody>
</table>

The variance estimates of Model 1 are reported in Table 2. The variance $\Sigma$ is the vector of parameters $\Sigma$ in the quadratic market model

$\Sigma = E[\varepsilon_1\varepsilon_1'] / \varepsilon_{\Sigma, i} / \varepsilon_{\Sigma, j}, \quad i, j = 1, \ldots, N,$

where $\varepsilon_i = R_t - R_{F, t}, \varepsilon_{\Sigma, i} = R_{M, t} - R_{F, t}$, and $\varepsilon_{\Sigma, j} = R_{M, t} - R_{F, t}$. $R_t$ is the return of portfolio $i$ in month $t$, and $R_{M, t}$ (or $R_{F, t}$) denotes the market return (the risk-free return). In parentheses we report t-statistics computed under the assumption

$E[\varepsilon_i \varepsilon_{i, t}'] = 0,$

$E[\varepsilon_i \varepsilon_{j, t}'] = \Sigma,$

and

$\varepsilon_i = (\varepsilon_{i, t}, \ldots, \varepsilon_{i, T}).$


<table>
<thead>
<tr>
<th>Portfolio $i$</th>
<th>$\Sigma_{i1}$</th>
<th>$\Sigma_{i2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7945</td>
<td>13.47</td>
</tr>
<tr>
<td>2</td>
<td>11.50</td>
<td>9.02</td>
</tr>
<tr>
<td>3</td>
<td>8.24</td>
<td>7.18</td>
</tr>
<tr>
<td>4</td>
<td>7.39</td>
<td>5.56</td>
</tr>
<tr>
<td>5</td>
<td>5.07</td>
<td>3.71</td>
</tr>
<tr>
<td>6</td>
<td>3.67</td>
<td>2.42</td>
</tr>
<tr>
<td>7</td>
<td>2.56</td>
<td>1.68</td>
</tr>
<tr>
<td>8</td>
<td>1.93</td>
<td>1.04</td>
</tr>
<tr>
<td>9</td>
<td>1.04</td>
<td>-1.00</td>
</tr>
<tr>
<td>10</td>
<td>-0.06</td>
<td>0.96</td>
</tr>
</tbody>
</table>


TABLE 2. Variance Estimates of Model (1)

Note: This table reports the estimate of the variance $\Sigma = E[\varepsilon_i \varepsilon_i'] / \varepsilon_{\Sigma, i} / \varepsilon_{\Sigma, j}, \quad i, j = 1, \ldots, N,$

where $\varepsilon_i = R_t - R_{F, t}, \varepsilon_{\Sigma, i} = R_{M, t} - R_{F, t}$, and $\varepsilon_{\Sigma, j} = R_{M, t} - R_{F, t}$. $R_t$ is the N-vector of portfolio returns, $R_{M, t}$ (or $R_{F, t}$) denotes the market return (the risk-free return), and $\varepsilon$ is a N-vector of 1's.

Assumes the value $\xi_T^\gamma = 35.34$, which is strongly significant at the 5% level, because its critical value is $X^2_{0.05}(10) = 18.31$. Finally, from Table 2, we also see that smaller portfolios are characterized by larger variances of the residual error terms.

We performed several specification tests of the functional form of the mean portfolio returns in (1). First, we estimated a factor SUR model including also a cubic power of market returns, $R_{M, t}^3 - R_{F, t}$, as a factor in addition to the constant, market excess returns, and market squared excess returns. The cubic factor was found to be not significant for all portfolios. Furthermore, to test for more general forms of misspecifications in the mean, we performed the Ramsey (1969) reset test on each portfolio, including quadratic and cubic fitted values of (1) among the regressors. In this case, too, the null of correct specification of the quadratic market model was accepted for all portfolios in our tests.

From the standpoint of our analysis, one central result from Table 1 is that the coskewness coefficients are (significantly) different from 0 for all portfolios in our sample, except for two of moderate size. Furthermore, coskewness coefficients tend to be correlated with size, with small portfolios having negative coskewness with the market and the largest portfolios having positive market coskewness. This result is consistent with the findings of Harvey and Siddique (2000). It is worth noting that the dependence between portfolio returns and market returns deviates from that of a linear specification (as assumed in the market model), generating smaller (larger) returns for small (large) portfolios when the market has a large absolute return. This finding has important consequences for the assessment of risk in various portfolio classes; small firm portfolios, having negative market coskewness, are exposed to a source of risk additional to market risk, related to the occurrence of large absolute market returns. In addition, as we have already seen, market model (2), when tested against quadratic market model (1), is rejected with a largely significant Wald statistic. In the light of our findings, we conclude that the extension of the return-generating process to include the squared market return is valuable.

4.2.2 Restricted Equilibrium Models. We now investigate market coskewness in the context of models that are consistent with arbitrage pricing. To this end, we consider constrained
PML estimation of models (6) and (8). Specification (6) is obtained from the quadratic market model after imposing restrictions from the asset pricing model (4). Specification (8) instead allows for a homogeneous additional constant in expected excess returns. The corresponding PML estimators are obtained from the algorithm based on (16)-(18), as reported in Section 3.3. The results for model (6) are reported in Table 3; those for model (8), in Table 4.

The point estimates and standard errors of parameters \( \beta \) and \( \gamma \) are similar in the two models. Their values are close to those obtained from quadratic market model (1). In particular, the estimates of parameter \( \gamma \) confirm that small (large) portfolios have significantly negative (positive) coskewness coefficients. Parameter \( \delta \) is significantly negative in both models, as expected, but the implied estimate for the risk premium for coskewness, \( \hat{\lambda}_2 \), is not statistically significant in either model. However, the estimate in model (8), \( \hat{\lambda}_2 = -7.439 \), has at least the expected negative sign. Using this estimate, we deduce that for a portfolio with coskewness \( \gamma = .01 \) (a moderate-sized portfolio, such as portfolio 3 or 4), the coskewness contribution to the expected excess return on an annual percentage basis is approximately .9. This value increases to 1.5 for the smallest portfolio in our dataset.

We test the empirical validity of asset pricing model (4) in our sample by testing hypothesis \( H_1 \) against the alternative, \( H_F \). The ALS test statistic \( \xi^2_1 \) given in (21) assumes the value \( \xi^2_1 = 16.27 \), which is not significant at the 5% level, even though very close to the critical value \( \chi^2_{0.95}(9) = 16.90 \). Thus there is some evidence that asset pricing model (4) may not be satisfied in our sample. In other words, an additional component other than covariance and coskewness to market may be present in expected excess returns. To test for the homogeneity of this component across assets, we test hypothesis \( H_2 \) against \( H_F \). The test statistic \( \xi^2_2 \) in (22) assumes the value \( \xi^2_2 = 5.32 \), well below the critical value \( \chi^2_{0.95}(8) = 15.51 \). A more powerful test of asset pricing model (4) should be provided by testing hypothesis \( H_1 \) against the alternative, \( H_2 \). This test is performed by the simple \( t \)-test of significance of \( \lambda_0 \). From Table 4, we see that \( H_2 \) is quite clearly rejected. This confirms that asset pricing model (4) may not be supported by our data. However, because \( H_2 \) is not rejected, it follows that, if the additional component unexplained by model (4) comes from an omitted factor, then its sensitivities should be homogeneous across portfolios in our sample. We conclude that size is unlikely to have explanatory power for expected excess returns when coskewness is taken into account. Moreover, the contribution to expected excess returns of the unexplained component, deduced from the estimate of parameter \( \lambda_0 \), is quite modest, approximately .4 on an annual percentage basis. Note in particular that this is less than half the contribution due to coskewness for portfolios of modest size. As explained in Section 2.2, \( \lambda_0 > 0 \) may be due to the use of a risk-free rate lower than the actual rate faced by investors.

4.2.3 Misspecification From Neglected Coskewness. As already mentioned in Section 2, we are also interested in investigating the consequences on asset pricing tests of erroneously neglecting coskewness. The results presented so far suggest that market model (2) is misspecified, because it does not take into account the quadratic market return. Indeed, when tested against quadratic market model (1), it is strongly rejected. For comparison, we report the estimates of parameters \( \alpha \) and \( \beta \) in market model (2) in Table 5. The \( \beta \) coefficients in Table 5 are close to those obtained in the quadratic market model reported in Table 1. Therefore, neglecting the quadratic market returns
does not seem to have dramatic consequences for the estimation of parameter $\beta$. However, we expect the consequences of this misspecification to be serious for inference. Indeed, we have seen earlier that the skewness coefficients are correlated with size, with small portfolios having negative market skewness and large portfolios having positive market skewness. This feature suggests that size can have a spurious explanatory power in the cross-section of expected excess returns because it is a proxy for omitted skewness. Therefore, as anticipated in Section 2, the ability of size to explain expected excess returns could be due to misspecification of models neglecting skewness risk.

Finally, it is interesting to compare our findings with those reported by Barone Adesi (1985), whose investigation covers the period 1931–1975. We see that the sign of the premium for skewness has not changed over time, with assets having negative skewness commanding, not surprisingly, higher expected returns. In contrast, both the sign of the premium for size and, consequently, the link between skewness and size are inverted. Although it appears to be difficult to discriminate statistically between a structural size effect and reward for skewness, according to the criterion of Kan and Zhang (1999a,b), the size effect is more likely explained by omitted skewness.

5. MONTE CARLO SIMULATIONS

In this section we report the results of a series of Monte Carlo simulations undertaken to assess the importance of specifying the return-generating process to obtain reliably powerful statistical tests. We compare the finite-sample properties (size and power) of two statistics for testing the asset pricing model with skewness (4): the ALS statistic $\xi^A_{T}$ in (21), which tests model (4) by the restrictions imposed on the return-generating process (1), and a GMM test statistic $\xi^G_{T}$, which tests model (4) through the orthogonality conditions (9). In addition, we investigate the effects on the ALS statistic $\xi^A_{T}$ induced by the nonnormality of errors $\varepsilon$, or by the misspecification of the return-generating process (1).

5.1 Experiment 1

The data-generating process used in experiment 1 is given by

$$r_i = \alpha + \beta R_{i,t} + \gamma q_{M,i} + \varepsilon_i, \quad i = 1, \ldots, 450,$$

where $R_{i,t} = R_{M,i} - R_{F,i}$, and $q_{M,i} = R_{M,i}^2 - R_{F,i}$, with

$$R_{M,i} \sim \text{iidN}(\mu_M, \sigma_M^2),$$

$$\varepsilon_i \sim \text{iidN}(0, \Sigma), \quad (\varepsilon_i) \text{ independent of } (R_{M,i}),$$

and

$$\alpha = \beta \gamma + \lambda_0 t.$$

The values of the parameters are chosen to be equal to the estimates obtained in the empirical analysis reported in Section 4. Specifically, $\beta$ and $\gamma$ are the third and fourth columns in Table 1, matrix $\Sigma$ is taken from Table 2, $\beta = -14.995$ from Table 3, $\mu_M = 5.2$, $\sigma_M = 4.41$, and $R_F = 0.4$, corresponding to the average of the risk-free return in our dataset. Different values of parameter $\lambda_0$ are used in the simulations. We refer to this data-generating process as DGPI. Under DGPI, when $\lambda_0 = 0$, quadratic equilibrium model (4) is satisfied. When $\lambda_0 \neq 0$, equilibrium model (4) is not correctly specified, and the misspecification is in the form of an additional component, which is homogeneous across portfolios, corresponding to model (8). However, for any value of $\lambda_0$, quadratic market model (1) is well specified.

We perform a Monte Carlo simulation (10,000 replications) for different values of $\lambda_0$ and report the rejection frequencies of the two test statistics, $\xi^A_{T}$ and $\xi^G_{T}$, at the nominal size of .05 in Table 6. The second row, $\lambda_0 = 0$, reports the empirical test sizes. Both statistics control size quite well in finite samples, at least for sample size $T = 450$. The subsequent rows, corresponding to $\lambda_0 = 0$, report the power of the two test statistics against alternatives corresponding to unexplained components in expected excess returns, which are homogeneous across portfolios. Note that such additional components, with $\lambda_0 = .033$, were found in the empirical analysis reported in Section 4. Table 6 shows that the power of the ALS statistic $\xi^A_{T}$ is considerably higher than that of the GMM statistic $\xi^G_{T}$. This is due to the fact that the ALS statistic $\xi^A_{T}$ use a well-specified alternative given by (1), whereas the alternative for the GMM statistic $\xi^G_{T}$ is left unspecified.

5.2 Experiment 2

Under DGPI, residuals $\varepsilon_i$ are normal. When residuals $\varepsilon_i$ are not normal, the alternative used by the ALS statistics $\xi^A_{T}$ [i.e., model (1)] is still correctly specified, because PML estimators are used to construct $\xi^A_{T}$. However, these estimators are not efficient. In experiment 2 we investigate the effect of nonnormality of residuals $\varepsilon_i$ on the ALS test statistic. The data-generating process used in this experiment, called DGPI, is equal to DGPI.
but residuals $\epsilon_t$ follow a multivariate $t$-distribution with 5 degrees of freedom. The correlation matrix is chosen so that the resulting variance of residuals $e_t$ is the same as under DGP1. The rejection frequencies of this Monte Carlo simulation (10,000 replications) for the ALS statistic $\xi_t$ are reported in Table 7. The ALS statistic appears to be only slightly oversized. As expected, power is reduced compared with the case of normality. However, the loss of power caused by nonnormality is limited. These results suggest that the ALS statistic does not suffer unduly from nonnormality of the residuals.

5.3 Experiment 3

In the Monte Carlo experiments conducted so far, the alternative used by the ALS statistic was well specified. In this last experiment, we investigate the effect of a misspecification in the alternative hypothesis in the form of conditional heteroscedasticity of errors $\epsilon_t$. We thus consider two data-generating processes having the same unconditional variance of the residuals $\epsilon_t$, but so that the residuals $\epsilon_t$ are conditionally heteroscedastic in one case and homoscedastic in the other case. Specifically, DGP3 is the same as DGP1 (see Table 6), but the innovations $\epsilon_t$ follow a conditionally normal, multivariate ARCH(1) process without cross-effects.

### Table 6. Rejection Frequencies in Experiment 1

<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>$\xi^{GMM}_t$</th>
<th>$\xi^{GMM}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.0404</td>
<td>.0559</td>
</tr>
<tr>
<td>.03</td>
<td>.0505</td>
<td>.4641</td>
</tr>
<tr>
<td>.06</td>
<td>.0712</td>
<td>.9746</td>
</tr>
<tr>
<td>.10</td>
<td>.1217</td>
<td>.9924</td>
</tr>
<tr>
<td>.15</td>
<td>.2307</td>
<td>.9945</td>
</tr>
</tbody>
</table>

**NOTE:** This table reports the rejection frequencies of the GMM statistics $\xi^{GMM}_t$ (derived from (9)) and the ALS statistics $\xi_t$ (in (21)) for testing the asset pricing model with coskewness (4).

### Table 7. Rejection Frequencies in Experiment 2

<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>$\xi_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.0617</td>
</tr>
<tr>
<td>.03</td>
<td>.3791</td>
</tr>
<tr>
<td>.06</td>
<td>.9368</td>
</tr>
<tr>
<td>.10</td>
<td>.9876</td>
</tr>
<tr>
<td>.15</td>
<td>.9910</td>
</tr>
</tbody>
</table>

**NOTE:** This table reports the rejection frequencies of the ALS statistics $\xi_t$ (in (21)) for testing (4).

### Table 8. Rejection Frequencies in Experiment 3

<table>
<thead>
<tr>
<th>$\lambda_0$</th>
<th>$\xi_t$ under DGP4 (homoscedasticity)</th>
<th>$\xi_t$ under DGP3 (conditional heteroscedasticity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.0587</td>
<td>.0539</td>
</tr>
<tr>
<td>.03</td>
<td>.3683</td>
<td>.1720</td>
</tr>
<tr>
<td>.06</td>
<td>.9333</td>
<td>.5791</td>
</tr>
<tr>
<td>.10</td>
<td>.9855</td>
<td>.9373</td>
</tr>
</tbody>
</table>

**NOTE:** This table reports the rejection frequencies of the ALS statistics $\xi_t$ (in (21)) for testing (4).
rewarded factors. To obtain testing methodologies that are more powerful than a GMM approach, we tested asset pricing models including coskewness through the restrictions that they impose on the quadratic market model. We used asymptotic test statistics whose finite-sample properties are validated by means of a series of Monte Carlo simulations. We found evidence of a component in expected excess returns that is not explained by either covariance or coskewness with the market. However, this unexplained component is relatively small and is consistent for instance with a minor misspecification of the risk-free rate. More important, the unexplained component is homogeneous across portfolios. This finding implies that additional variables, representing portfolios characteristics such as size, have no explanatory power for expected excess returns once coskewness is taken into account.

Finally, the homogeneity property of the unexplained component in expected excess returns is not satisfied when coskewness is neglected. Therefore, our results have important implications for testing methodologies, showing that neglecting coskewness risk can cause misleading inference. Indeed, because coskewness is positively correlated with size, a possible justification for the anomalous explanatory power of size in the cross-section of expected returns is that it is a proxy for omitted coskewness risk. This view is supported by the fact that the sign of the premium for coskewness, contrary to that of size, has not changed in a very long time.

ACKNOWLEDGMENTS

The authors thank the participants in the 9th Conference on Panel Data, the 10th Annual Conference of the European Financial Management Association, the Conference on Multimoment Capital Asset Pricing Models and Related Topics, the INQUIRE UK 14th Annual Seminar on Beyond Mean-Variance: Do Higher Moments Matter?, the 2004 ASSA Meeting, and P. Balestra, S. Cain-Polli, J. Chen, C. R. Harvey, R. Jagannathan, R. Morek, M. Rockinger, T. Wansbeek, C. Zhang, two anonymous referees, and the editor, Eric Ghysels. All of them have contributed, through discussion, very helpful comments and suggestions that improved this article. The usual disclaimer applies. Thanks are also due to K. French, R. Jagannathan, and R. Kan for providing their dataset. Swiss NCCR Finrisk is gratefully acknowledged by the first two authors.

APPENDIX A: RELATIONSHIP BETWEEN CROSS-MOMENT COSKEWNESS AND THE \( \gamma_1 \) PARAMETER

In this appendix we show how the parameter \( \gamma_1 \) relates to the coskewness term of our quadratic market model. We also report error estimates when the cross-moment coskewness is approximated by the parameter \( \gamma_1 \).

Our basic model is [see (1)]

\[
R_{i,t} = \alpha_i + \beta_i (R_{M,t} - R_{F,t}) + \gamma_i (R_{M,t}^2 - R_{F,t}^2) + \epsilon_{i,t},
\]

where

\[
R_{i,t} = R_{i,t} - R_{F,t}.
\]

The probabilistic measure of coskewness is defined by

\[
\text{cov}(r_{i,t}, R_{M,t}^2) = E[r_{i,t}R_{M,t}^2] - E[r_{i,t}]E[R_{M,t}^2],
\]

which can be rewritten as

\[
\text{cov}(r_{i,t}, R_{M,t}^2) = \beta_i E[r_{i,t}^3] - E[r_{i,t}]E[R_{M,t}^2]E[R_{F,t}^2] + \gamma_i \text{var}[R_{M,t}^2] + E[R_{M,t}^2]E[\epsilon_{i,t}].
\]

In the final equation, the first term is a measure of the market asymmetry, the second term is essentially our measure of coskewness, and the final term is equal to 0 by assumption (10). Evidently, our approach considers the second term only. However, our claim is motivated by the negligible effects of the first term. In fact, for values \( \beta_i = 1 \) and \( \gamma_i = -0.1 \), which are representative for small firm portfolios, the first term is .1, whereas the second is -15. If \( \gamma_i \) is equal to .003, as in large firm portfolios, then the terms are equal to .1 and 4. Finally, if the portfolio has a \( \gamma_i = 0 \), then the second term is also 0.

We are greatly indebted to one of the referees, whose comments highlighted this point.

APPENDIX B: PSEUDO-MAXIMUM LIKELIHOOD IN MODEL (8)

In this appendix we consider the PML estimator of model (8), defined by maximization of (15). We first derive the PML equations. The score vector is given by

\[
\frac{\partial L_T}{\partial (\beta', \gamma')} = \sum_{t=1}^{T} H_t \otimes \Sigma^{-1} \epsilon_t,
\]

and

\[
\frac{\partial L_T}{\partial (\varphi, \lambda)} = \sum_{t=1}^{T} Z' \Sigma^{-1} \epsilon_t,
\]

where

\[
\frac{\partial L_T}{\partial \text{vec}(\Sigma)} = \frac{1}{2} P^t \Sigma^{-1} \otimes \Sigma^{-1} P \text{vec} \left[ \sum_{t=1}^{T} (\epsilon_t \epsilon_t' - \Sigma) \right],
\]

and

where \( H_t = (\eta_{M,t}, q_{M,t} + \vartheta)' \), \( \epsilon_t = r_t - \beta_t R_{M,t} - \gamma q_{M,t} - \vartheta - \lambda \eta_t \), \( Z = (\gamma, t) \), and \( P \) is such that \( \text{vec}(\Sigma) = P \text{vec}(\Sigma) \).

By equating the score to 0, we immediately find (16)–(18).

We now derive the asymptotic distribution of the PML estimator in model (8). Under usual regularity conditions (see the references in the main text), the asymptotic distribution of the general PML estimator \( \hat{\theta} \) defined in (11) is given by

\[
\sqrt{T} (\hat{\theta} - \theta^0) \overset{d}{\to} N(0, J_0^{-1} I_0 J_0^{-1}),
\]

where \( J_0 \) (the so-called “information matrix”) and \( I_0 \) are symmetric, positive-definite matrices defined by

\[
J_0 = \lim_{T \to \infty} E \left[ \frac{1}{T} \frac{\partial^2 L_T}{\partial \theta \partial \theta'} (\theta^0) \right]
\]

and

\[
I_0 = \lim_{T \to \infty} E \left[ \frac{1}{T} \frac{\partial L_T}{\partial \theta} (\theta^0) \frac{\partial L_T}{\partial \theta'} (\theta^0) \right].
\]
We now compute matrices \( J_0 \) and \( I_0 \) in model (8). The second derivatives of the pseudo-log-likelihood are given by 

\[
\frac{\partial^2 L_T}{\partial (\beta', y') \partial (\beta', y')} = -\sum_{t=1}^{T} H_t H_t' \otimes \Sigma^{-1},
\]

\[
\frac{\partial^2 L_T}{\partial (\beta', y') \partial (\theta, \lambda_0)} = -\sum_{t=1}^{T} H_t \otimes \Sigma^{-1} Z,
\]

\[
\frac{\partial^2 L_T}{\partial (\theta, \lambda_0) \partial (\theta, \lambda_0)} = -TZ' \Sigma^{-1} Z.
\]

and

\[
\frac{\partial L_T}{\partial \text{vech}(\Sigma) \partial \text{vech}(\Sigma)'} = \frac{T}{2} \mathbf{P}' \Sigma^{-1} \otimes \Sigma^{-1} \mathbf{P} - \frac{T}{2} \mathbf{P}' \Sigma^{-1} \left( \sum_{i=1}^{T} \varepsilon_i \varepsilon_i' \right) \Sigma^{-1} \mathbf{P} - \frac{T}{2} \mathbf{P}' \Sigma^{-1} \left( \sum_{i=1}^{T} \varepsilon_i \varepsilon_i' \right) \Sigma^{-1} \otimes \Sigma^{-1} \mathbf{P},
\]

with the other ones vanishing in expectation. It follows that matrices \( J_0 \) and \( I_0 \) are given by [in the block representation corresponding to \((\beta', y', \theta, \lambda_0)\)' and \(\text{vech}(\Sigma)\)]

\[
J_0 = \begin{cases} J_0^* \hspace{1cm} \tilde{J}_0 \\ \tilde{J}_0^* \hspace{1cm} J_0 \end{cases} 
I_0 = \begin{cases} J_0^* \hspace{1cm} \tilde{J}_0 \hspace{1cm} \eta \hspace{1cm} \bar{J}_0 \hspace{1cm} \bar{K} \\ J_0 \hspace{1cm} J_0^* \hspace{1cm} S \hspace{1cm} \eta' \hspace{1cm} \bar{J}_0 \hspace{1cm} \bar{K} \end{cases},
\]

where

\[
\tilde{J}_0 = \frac{1}{2} (\mathbf{P}' \Sigma^{-1} \otimes \Sigma^{-1} \mathbf{P}),
\]

\[
S = \operatorname{cov} (\varepsilon_i, \text{vech}(\varepsilon_i \varepsilon_i')),
\]

\[
K = \operatorname{var} (\text{vech}(\varepsilon_i \varepsilon_i')).
\]

and [in the block form corresponding to \((\beta', y', \theta, \lambda_0)\)]

\[
J_0^* = \begin{cases} E [H_t H_t] \otimes \Sigma^{-1} \hspace{1cm} \lambda \otimes \Sigma^{-1} Z \\ \lambda' Z' \otimes \Sigma^{-1} \hspace{1cm} Z' \Sigma^{-1} Z \end{cases},
\]

and

\[
\eta = \begin{cases} \lambda \otimes \Sigma^{-1} \hspace{1cm} Z' \Sigma^{-1} \end{cases}.
\]

(All parameters are evaluated at true value). Therefore, the asymptotic variance–covariance matrix of the PML estimator \( \hat{\theta} \) in model (8) is given by

\[
V_{\text{as}} [\sqrt{T} (\hat{\theta} - \theta_0)] = J_0^{-1} I_0 J_0^{-1} = \begin{cases} J_0^{-1} \eta S \eta' J_0^{-1} \hspace{1cm} \mathbf{K} \\ \mathbf{S} \eta' \eta J_0^{-1} \hspace{1cm} \mathbf{K} \end{cases}.
\]

Note that the asymptotic variance–covariance of \((\hat{\beta}', \hat{\gamma}', \hat{\theta}, \hat{\lambda}_0)\)' (i.e., \( J_0^{-1} \)) does not depend on the distribution of error term \( \varepsilon_i \), and in particular it coincides with the asymptotic variance–covariance matrix of the ML estimator of \((\hat{\beta}', \hat{\gamma}', \hat{\theta}, \hat{\lambda}_0)\)' when \( \varepsilon_i \) is normal. In contrast, asymmetries and kurtosis of the distribution of \( \varepsilon_i \) influence the asymptotic variance–covariance matrix of \( \text{vech}(\Sigma) \) and the asymptotic covariance of \((\hat{\beta}', \hat{\gamma}', \hat{\theta}, \hat{\lambda}_0)\)' and \( \text{vech}(\Sigma) \), through matrices \( S \) and \( K \).

The asymptotic variance–covariance of \((\hat{\beta}', \hat{\gamma}', \hat{\theta})\) and \((\hat{\theta}, \hat{\lambda}_0)\)' is given explicitly in block form by

\[
J_0^{-2} = \begin{cases} J_0^{-11} \hspace{1cm} J_0^{-12} \\ J_0^{-21} \hspace{1cm} J_0^{-22} \end{cases},
\]

where

\[
J_0^{-11} = (\Sigma + \lambda \lambda')^{-1} \otimes \Sigma' + \left( \Sigma^{-1} \lambda \lambda' (\Sigma + \lambda \lambda')^{-1} \right) \otimes (Z' \Sigma^{-1} Z)^{-1} Z' \Sigma^{-1} Z',
\]

\[
J_0^{-12} = -\Sigma^{-1} \lambda \otimes Z (Z' \Sigma^{-1} Z)^{-1},
\]

\[
J_0^{-21} = J_0^{-12}',
\]

and

\[
J_0^{-22} = (1 + \lambda' \Sigma^{-1} \lambda (Z' \Sigma^{-1} Z)^{-1}).
\]

Finally, we consider the asymptotic distribution of estimator \( \hat{\lambda} \) defined in (19). The estimator

\[
\hat{\lambda} = \frac{1}{T} \sum_{t=1}^{T} \hat{\mu}_t,
\]

where \( \hat{\mu}_t = (r_{M,t} q_{M,t})' \), can be seen as a component of the PML estimator on the extended pseudolikelihood

\[
L_T (\theta, \mu, \Sigma) = L_T (\theta) - \frac{T}{2} \log \det \Sigma_T - \frac{T}{2} \sum_{t=1}^{T} (\hat{\mu}_t - \mu)' \Sigma_T^{-1} (\hat{\mu}_t - \mu),
\]

where \( L_T (\theta) \) is given in (15). It is easily seen that \( \theta \) and \((\mu, \Sigma)\) are asymptotically independent. It follows that

\[
V_{\text{as}} (\sqrt{T} (\hat{\lambda}_2 - \lambda_2, 0)) = \Sigma_{T,2} + V_{\text{as}} (\sqrt{T} (\hat{\theta} - \theta_0)).
\]

APPENDIX C: ASYMPTOTIC LEAST SQUARES

In this appendix we derive the ALS statistic \( \xi^2_1 \) in (21) and \( \xi^2_2 \) in (22). In both cases the restrictions [see (20)] are of the form

\[
g (\theta, a) = A_1 (a) \text{vec} (B) + A_2 (a),
\]

where \( B \) is the \( N \times 3 \) matrix defined by \( B = [\alpha, \beta, \gamma] \) and \( A_1 (a) \) is such that

\[
A_1 (a) = (1, 0, -\theta) \otimes I_N = A_1^* (a) \otimes I_N.
\]

We derive the weighting matrix \( S_0 = (\partial g / \partial \theta)^{-1} \cdot \omega_0 = V_{\text{as}} (\sqrt{T} (\hat{\theta} - \theta_0)). \) From (13), we get

\[
\frac{\partial g}{\partial \theta} \omega_0 \frac{\partial g'}{\partial \theta} = A_1^* E [F' F]^{-1} A_1^* \otimes \Sigma
\]

\[
= (1 + \lambda' \Sigma^{-1} \lambda) \Sigma.
\]

The test statistics follow.

It should be noted that exact tests (under normality) can be constructed for testing hypotheses \( H_1 \) and \( H_2 \) against \( H_F \) (see, e.g., Zhou 1995; Velu and Zhou 1999). These tests are asymptotically equivalent to the ALS tests used in this article for their computational simplicity. An evaluation of the finite-sample properties of the ALS test statistics is presented in Section 5.
REFERENCES


