Expansion Formulas for bivariate payoffs with application to best-Of Option on Equity and Inflation

J. Hok joint work with E. Gobet

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Introduction

Inflation: Increase or decrease in the general level of prices of goods and services in an economy over a period of time.
Examples of reference index: CPI in France and US, RPI in UK.

See graph below:

- Average of annual inflation: 4% (resp. 5.3% and 1.94%) for UK (resp. US and Eurozone).
- Some pikes during the 80s: annual inflation reaches 20% (resp. 14%) for UK (resp. US).

⇒ Potential risk for inflation.
**Fig.**: Period 1976-2011 for UK and US and period 2001-2011 for the Eurozone.
A growing market for inflation derivatives since 1980. Most liquid ones are:

- **Zero coupon swap**: at the single maturity date $T$, Party B pays Party A the fixed amount:

  $$N[(1 + K)^T - 1]$$

  $K$ fixed rate, $N$ the nominal value. In exchange, Party A pays to Party B at the final $T$ the floating amount

  $$N\left[\frac{I_T}{I_0} - 1\right]$$

  $I_T$ the value of the inflation index at time $T$. 

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- YoY swap: \( T_1, \ldots, T_M \) a set dates for the YoY swap. At each \( T_i \), Party B pays Party A the fixed amount

\[
N \varphi_i K
\]

\( \varphi_i \): fixed-leg year fraction for \( [T_{i-1}, T_i] \). In exchange, Party A pays Party B the floating amount

\[
N \psi_i \left[ \frac{I_{T_i}}{I_{T_{i-1}}} - 1 \right]
\]

\( \psi_i \): floating leg year fraction for \( [T_{i-1}, T_i] \) with \( T_0 := 0 \).

ZC swap and YoY swap: core for the trading on inflation derivatives and building blocks for more complex structures

Ex: LPI in UK, asian option on inflation, hybrid products linked to inflation.
Introduction

Our study: Best-Of option whose payoff is given by

\[(\max(S_T, I_T) - K)_+\]

- Analytical formula under log-normal assumptions.

- To take into account the skew effect observed on equity implied volatilities, natural to incorporate the local volatility model for S (Dupire 94, Rubinstein 94, Derman, Kani 94).

Inconvenient: No closed formula in general. Analytical formulas in a few cases (Albanese and al 01).

⇒ PDE or Monte Carlo methods ⇒ Time consuming.
Approximation formulas with some restrictions:

- Geodesic expansion for short maturity options by Berestycki and al 04 or Labordere 05
- Ergodic approach for long maturity options by Fouque and al 00
- Extreme strikes options by Lee 04
- Singular perturbation techniques for homogeneous volatility by Hagan and al 99
- Recent PDE approach for local volatility model by Pascussi and al (12)

Objective: approximation formula of Black and Scholes type by applying the perturbation method using a proxy introduced by Gobet and al 09.
Plan:

1. Literature review for inflation models
2. Best-Of equity inflation payoff and modeling
3. Pricing and the approximation formulas
4. Proofs
5. Numerical experiments
6. Extension to stochastic interest rate
7. Conclusions
Multi foreign currency analogy (Turnbull-Jarrow 98) : the inflation Index $I$ as an exchange rate between nominal and real economies (Jarrow-Yildirim 03).

Contribution : no arbitrage condition between these 3 components :

\[
\begin{align*}
\frac{dI_t}{I_t} &= (r^n_t - r^r_t) dt + \sigma^I_t dW^I_t, \\
\frac{dr^k_t}{r_t} &= a^k (\theta^k_t - r^k_t) dt + \sigma^k_t dW^k_t, \quad k \in \{n, r\},
\end{align*}
\]

Market model (Koehler, Belgrade, Benhamou 04 and Mercurio 05) : It consists to model all the forward prices of the inflation index. Given a discrete set of times relevant ($T_1, T_2...$) to the product (YoY swap), we have a separate dynamics for the forward $I$ for each of these times : for a given maturity $T_i$, the forward price $I$ at time $t$ is defined by

\[
\mathcal{I}(t, T_i) = I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)}
\]

Under the forward measure $Q_n^{T_i}$ with $P_n(t, T_i)$ as a numeraire, it becomes a martingale. → Tractability.

But problem for path-dependent product. Drift freezing method (Mercurio 05). Extension to stochastic volatility model in (Moreni, Mercurio 06).
Direct modelling for the short term inflation rate (Kainth, Dodgson 06):

\[
\begin{align*}
\frac{dI_t}{I_t} &= i_t \, dt, \\
di_t &= a_i(\theta^i_t - i_t) \, dt + \sigma_i dW^i_t, \\
dr_t &= a_r(\theta^r_t - r_t) \, dt + \sigma_r dW^r_t, \\
d\langle W^i, W^r \rangle_t &= \rho_{ir} \, dt.
\end{align*}
\]
Payoff at maturity $T$:

$$\max \left[ \max (S_T, I_T) - K, 0 \right]$$

\[
\begin{align*}
\frac{dI_t}{I_t} &= i_t \, dt, \\
\frac{di_t}{S_t} &= a(\theta_t - i_t) \, dt + \sigma_I \, dW^I_t, \\
\frac{dS_t}{S_t} &= (r_t - q_t) \, dt + \sigma_S(t, S_t) \, dW^S_t
\end{align*}
\]

- $a > 0$ the mean reverting parameter,
- $(\theta_t)_{t \geq 0}$ the time-dependent long term average inflation rate,
- $\sigma_I$ the inflation rate volatility.
- $\sigma_S(t, S)$ the local volatility function for $S$.
- $r_t$ and $q_t$ the short-term interest rate and the repo or dividend rate.
- $d < W^S, W^I >_t = \rho_{S,I} \, dt$
Pricing and the approximation formula

\[ \text{price} = B(0, T) E^Q(\max[S_T, I_T, K]) - KB(0, T) \]

\(B(0, T)\) zero coupon price.

Closed formula under log-normal assumptions

\((\sigma_S(t, x) = \sigma_S(t) := \sigma_{S,t})_{0 \leq t \leq T}\) deterministic time-dependent volatility for \(S\).

\(\Rightarrow (S_T, I_T) \sim \text{bivariate log-normal distribution}\)

\[ \text{price} = S_0 e^{-\int_0^T q_t dt} A_1 + I_0 e^{b_{I, T}} B(0, T) A_2 - KB(0, T)(1 - A_3) \]

with

\[ A_1 = \mathcal{N}_2(d_1, d_3, \rho_1), \quad A_2 = \mathcal{N}_2(d_2, d_4, \rho_2), \quad A_3 = \mathcal{N}_2(\tilde{C}_{11} - d_1, \tilde{C}_{22} - d_2, \rho), \]

\[ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + b_{S, T} + \frac{1}{2} \tilde{C}_{11}^2}{\tilde{C}_{11}}, \quad d_2 = \frac{\ln\left(\frac{I_0}{K}\right) + b_{I, T} + \frac{1}{2} \tilde{C}_{22}^2}{\tilde{C}_{22}}, \]

\[ d_3 = \frac{\ln\left(\frac{S_0}{I_0}\right) + b_{S, T} - b_{I, T} + \frac{1}{2} \tilde{C}'^2}{\tilde{C}} , \quad d_4 = \frac{\ln\left(\frac{I_0}{S_0}\right) + b_{I, T} - b_{S, T} + \frac{1}{2} \tilde{C}'^2}{\tilde{C}} \]

where \(\mathcal{N}_2(a, b, \rho)\) is the standardized cumulative bivariate normal distribution function with correlation \(\rho (\neq \pm 1)\).
Heuristics of the approximation and the proxy model

\[ X_t = \ln(S_t e^{-\int_0^t (r_s - q_s) ds}) \]

\[ \Rightarrow dX_t = \sigma_X(t, X_t) dW_t^S - \frac{1}{2} \sigma_X^2(t, X_t) dt \]

\[ \sigma_X(t, x) = \sigma_S(t, e^{x + \int_0^t (r_s - q_s) ds}). \]

Our main objective is to provide an accurate analytical approximation of the expected payoff

\[ \mathbb{E}[h(X_T, I_T)] \]

for the best-of payoff \( h \) and for a fixed maturity \( T \).

Proxy process:

\[ dX_t^P = \sigma_{X,t} dW_t^S - \frac{1}{2} \sigma_{X,t}^2 dt, \quad X_0^P = X_0 = \log(S_0), \]

\[ \sigma_{X,t} = \sigma_X(t, X_0) = \sigma_S(t, S_0 e^{\int_0^t (r_s - q_s) ds}). \]
Pricing and the approximation formula

Justifications:
- Small variations for $\sigma_X(.)$ i.e $\sigma_X(t, X_t) \approx \sigma_X(t, X_0)$.
- $|\sigma_X|_{\infty} \ll 1$
- $T \ll 1 \Rightarrow X_T \approx X_0$

Expansion around the proxy model:

$$\mathbb{E}[h(X_T, I_T)] = \mathbb{E}[h(X_T^P, I_T)] + \text{Correction terms} + \text{error}.$$

Definitions, assumptions and notations

**Definition (Integral Operator)**

For any integrable function $l$, we set

$$\omega(l)_t^T = \int_t^T l_u du$$

for $t \in [0, T]$. Similarly, for integrable functions $(l_1, l_2)$, we put for $t \in [0, T]$

$$\omega(l_1, l_2)_t^T = \omega(l_1 \omega(l_2)_t^T) = \int_t^T l_1_r \left( \int_r^T l_2_s ds \right) dr.$$
**Pricing and the Approximation Formula**

**Definition (Greeks)**

We set

\[ \text{Greek}_{j,k} = \partial_{\varepsilon_x}^j \partial_{\varepsilon_i}^k \mathbb{E}(\varphi(b_S, T + X_T^P + \varepsilon_x, \ln(I_T) + \varepsilon_i)) \bigg|_{\varepsilon_x = \varepsilon_i = 0} \]

\[ = \partial_{\varepsilon_x}^j \partial_{\varepsilon_i}^k \mathbb{E}(\bar{\varphi}(\int_0^T \sigma_{X,t} dW_t^S + \varepsilon_x, \int_0^T \sigma_{I,t} dW_t^I + \varepsilon_i)) \bigg|_{\varepsilon_x = \varepsilon_i = 0}. \]

with the shifted function defined by

\[ \bar{\varphi}(x, i) = \varphi \left( x + \ln(S_0) + b_{S,T} - \frac{1}{2} \int_0^T \sigma_{X,t}^2 dt, i + \ln(I_0) + b_{I,T} - \frac{1}{2} \int_0^T \sigma_{I,t}^2 dt \right) \]

and

\[ \text{Greek}_{j,k} = \partial_{S}^j \partial_{I}^k \mathbb{E}(\phi(S e^{b_{S,T} + \int_0^T \sigma_{S,t} dW_t^S} - \frac{1}{2} \int_0^T \sigma_{S,t}^2 dt, I e^{b_{I,T} + \int_0^T \sigma_{I,t} dW_t^I} - \frac{1}{2} \int_0^T \sigma_{I,t}^2 dt)) \bigg|_{S=S_0, I=I_0} \]
**Hypothesis**

\[ R_n : \sigma_X(t, x) \text{ is of class } C^n \text{ w.r.t } x. \] In addition, this function and their derivatives are uniformly bounded.

\[ H : \Phi \text{ belongs to the space of real infinitely differentiable functions with compact support.} \]

And let's note by

\[
M_1 = \max_{1 \leq i \leq 4} \sup_{(t, x) \in [0, T] \times \mathbb{R}} \left| \frac{\partial^i}{\partial x^i} \sigma_X(t, x) \right| < \infty,
\]

\[
M_0 = \max \left( M_1, \sup_{(t, x) \in [0, T] \times \mathbb{R}} \sigma_X(t, x) \right) < \infty.
\]
Pricing and the approximation formula

**Theorem**

*(Second order approximation price formula).*

Under \((R_4)\) and \((H)\), we have

\[
\mathbb{E}(\Phi(S_T, I_T)) = \mathbb{E}(\Phi(S^P_T, I_T)) + \alpha_{2,0} S_0^2 \text{Greek}^{\phi}_{2,0} + \alpha_{1,1} S_0 I_0 \text{Greek}^{\phi}_{1,1} \\
+ \alpha_{3,0} S_0^3 \text{Greek}^{\phi}_{3,0} + \alpha_{2,1} S_0^2 I_0 \text{Greek}^{\phi}_{2,1} + \alpha_{1,2} S_0 I_0^2 \text{Greek}^{\phi}_{1,2} + \text{Resid}_2.
\]

with

\[
\alpha_{1,1} = \frac{1}{2} \omega(\sigma_X^2, \sigma_X^2 \rho_{S,I})_0^T + \omega(\sigma_X \sigma_X \rho_{S,I}, \sigma_X^2 \rho_{S,I})_0^T,
\]

\[
\alpha_{1,2} = \omega(\sigma_X \sigma_X \rho_{S,I}, \sigma_X^2 \rho_{S,I})_0^T, \quad \alpha_{2,0} = \frac{3}{2} \omega(\sigma_X^2, \sigma_X \sigma_X)_0^T,
\]

\[
\alpha_{2,1} = \omega(\sigma_X \sigma_X \rho_{S,I}, \sigma_X^2 \rho_{S,I})_0^T + \omega(\sigma_X^2, \sigma_X \sigma_X \rho_{S,I})_0^T, \quad \alpha_{3,0} = \omega(\sigma_X^2, \sigma_X \sigma_X)_0^T.
\]

and

\[
|\text{Resid}_2| \leq c \sup(\|\varphi_x^{(1)}\|_\infty, \|\varphi_{xx}^{(2)}\|_\infty) M_1 M_0^2 (\sqrt{T})^3
\]
Pricing and the approximation formula

**Theorem**

*(Third order approximation price formula).* Under \((R_6)\) and \((H)\), we have

\[
\mathbb{E}(\Phi(S_T, I_T)) = \mathbb{E}(\Phi(S_T^P, I_T)) + \sum_{n=2}^{6} \sum_{i=1}^{n} \eta_{i,n-i} S_0^i I_0^{n-i} \text{Greek}_{i,n-i} + \text{Resid}_3.
\]

\[
| \text{Resid}_3 | \leq c \sup(| \varphi_x^{(1)} | \infty, | \varphi_{xx}^{(2)} | \infty, | \varphi_{xxx}^{(3)} | \infty) M_1 M_0^3 (\sqrt{T})^4
\]

with \(\varphi(x, i) = \Phi(e^x, e^i)\).

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Expansion Formulas for bivariate payoffs with application to best-
Pricing and the approximation formula

remarks:

- In the multiplicative case: \( \sigma_X(t, x) = \Delta s(t, x) \to M_0, M_1 = O(\Delta) \).

\[
|\text{Resid}_2| = O(\Delta^3(\sqrt{T})^3), \quad |\text{Resid}_3| = O(\Delta^4(\sqrt{T})^4)
\]

\( \Rightarrow \) approximation of order 2 (resp 3) w.r.t the amplitudes of the data.

- These estimates can provide information regarding the impact of local volatility level, slope. Example: CEV model, \( \sigma(t, x) = \nu x^{\beta - 1} \). Locally at the money \( S_0 = 1, M_0 \approx c\nu \) and \( M_1 \approx c\nu |\beta - 1| \). Approximations errors have magnitudes \( \nu^3 T^{3/2} |\beta - 1| \) and \( \nu^4 T^2 |\beta - 1| \).

- Less smooth payoff like call, put or best-of, we get the same approximation formulas with the same order for the residual terms. For the error analysis, we need a complementary ellipticity hypothesis on the local volatility function \( \sigma_X \) and it involves Malliavin calculus artillery. The proof is similar to (Gobet and Miri 09).
Proofs

Parameterized process:

\[ dX_t^n = \eta(t, X_t^n) dW_t^S - \frac{1}{2} \sigma_X^2(t, X_t^n) dt, \quad X_0^n = X_0. \]

Let’s define \( X_{j,t} \equiv \frac{\partial^j X_t^n}{\partial \eta^j} \big|_{\eta=0} \), expansion for the perturbed process \( X_t^n \):

\[
X_T = X^1_T = X_{0,T} + X_{1,T} + \frac{1}{2} X_{2,T} + \ldots
\]

with

\[
\begin{align*}
    dX_{1,t} &= \sigma_X t dW_t^S - \frac{1}{2} \sigma_X^2 t dt, \quad X_{1,0} = 0, \\
    dX_{2,t} &= 2X_{1,t} (\sigma'_X t dW_t^S - \sigma_X t \sigma'_X t dt), \quad X_{2,0} = 0, \\
    dX_{3,t} &= 3[X_{2,t} (\sigma'_X t dW_t^S - [\sigma_X t \sigma'_X t] dt) + (X_{1,t})^2(\sigma''_X t dW_t^S - [\sigma'_X t]^2 + \sigma_X t \sigma''_X t] dt)], \quad X_{3,0} = 0.
\end{align*}
\]

where

\[
\begin{align*}
    \sigma'_X t &= \partial_x \sigma_X(t, X_0) \\
    \sigma''_X t &= \partial_{xx}^2 \sigma_X(t, X_0)
\end{align*}
\]
Proofs

\[ X_T^P = x_0 + X_1, T, \]

and

\[ X_T = X_T^P + \frac{1}{2} X_2, T + \frac{1}{6} X_3, T \ldots \]

Price expansion

\[ \text{Price} = B(0, T) \mathbb{E}(\tilde{S}_T, I_T) \]

Let's denote the payoff in the logarithmic variables by

\[ \varphi(s, i) = \Phi(e^s, e^i) \]

and do an expansion:

\[
\frac{P}{B(0, T)} = \mathbb{E}[\varphi(b_S, T + X_T, \ln(I_T))] \\
= \mathbb{E}[\varphi(b_S, T + X_T^P, \ln(I_T))] + E[\varphi_1^{(1)}(b_S, T + X_T^P, \ln(I_T)) \frac{X_2}{2}, T] + \text{Resid}_2
\]
Proofs

Computation for the Greeks coefficients

**Lemma**

For any continuous (or piecewise continuous) function $f$, any continuous semimartingale $Z$ vanishing at $t=0$, one has:

$$
\int_0^T f_t Z_t \, dt = \int_0^T \left( \int_t^T f_s \, ds \right) \, dZ_t = \int_0^T \omega(f)_t \, dZ_t.
$$
Let $W$ be a standard linear Brownian motion and let $u$ be a square integrable progressively measurable process. For any $j, k \geq 0$, one has:

\[
\mathbb{E}\left[\left(\int_0^T u_t dW_t\right) \partial_x^j \partial_i^k \bar{\varphi}\left(\int_0^T \sigma_{X,t} dW_t^S, \int_0^T \sigma_{I,t} dW_t^I\right)\right]
\]

\[
= \mathbb{E}\left[\left(\int_0^T \sigma_{X,t} u_t \langle W, W^S_t \rangle \partial_x^j \partial_i^k \bar{\varphi}\left(\int_0^T \sigma_{X,t} dW_t^S, \int_0^T \sigma_{I,t} dW_t^I\right)\right]\right]
\]

\[
+ \mathbb{E}\left[\left(\int_0^T \sigma_{I,t} u_t \langle W, W^I_t \rangle \partial_x^j \partial_i^{k+1} \bar{\varphi}\left(\int_0^T \sigma_{X,t} dW_t^S, \int_0^T \sigma_{I,t} dW_t^I\right)\right]\right].
\]

If $u, \langle W, W^S \rangle$ and $\langle W, W^I \rangle$ are deterministic, then

\[
\mathbb{E}\left[\left(\int_0^T u_t dW_t\right) \partial_x^j \partial_i^k \bar{\varphi}\left(\int_0^T \sigma_{X,t} dW_t^S, \int_0^T \sigma_{I,t} dW_t^I\right)\right]
\]

\[
= \left(\int_0^T \sigma_{X,t} u_t \langle W, W^S_t \rangle \right) \text{Greek}^{\varphi}_{j+1, k} + \left(\int_0^T \sigma_{I,t} u_t \langle W, W^I_t \rangle \right) \text{Greek}^{\varphi}_{j, k+1}.
\]
Proofs

\begin{align*}
\mathbb{E}[\varphi'_x(b_S, T + X^P_T, \ln(I_T)) \frac{X_{2,T}}{2}] \\
= \mathbb{E}[\varphi'_x(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \frac{X_{2,T}}{2}] \\
= \mathbb{E}[\varphi'_x(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T X_{1,t}(\sigma'_{X,t} dW^S_t - \sigma_{X,t} \sigma'_{X,t} dt)] \\
= \mathbb{E}[\varphi''_{x,x}(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \sigma_{X,t} \sigma'_{X,t} X_{1,t} dt] \\
\quad + \mathbb{E}[\varphi''_{x,i}(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \sigma_{I,t} \sigma'_{X,t} \rho_{S,I} X_{1,t} dt] \\
\quad - \mathbb{E}[\varphi'_x(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \sigma_{X,t} \sigma'_{X,t} X_{1,t} dt] \\
= \mathbb{E}[\varphi''_{x,x}(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \omega(\sigma_X \sigma'_X)^T dX_{1,t}] \\
\quad + \mathbb{E}[\varphi''_{x,i}(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \omega(\sigma_I \sigma'_X \rho_{S,I})^T dX_{1,t}] \\
\quad - \mathbb{E}[\varphi'_x(\int_0^T \sigma_{X,t} dW^S_t, \int_0^T \sigma_{I,t} dW^I_t) \int_0^T \omega(\sigma_X \sigma'_X)^T dX_{1,t}]
\end{align*}
Proofs

\begin{align*}
\mathbb{E}[\bar{\varphi}''_{x,x} & (\int_0^T \sigma_X, t dW^S_t, \int_0^T \sigma_I, t dW^I_t) \int_0^T \omega(\sigma_X \sigma'_X)^T dX_1, t] \\
= \mathbb{E}[\bar{\varphi}''_{x,x} (\int_0^T \sigma_X, t dW^S_t, \int_0^T \sigma_I, t dW^I_t) \int_0^T \omega(\sigma_X \sigma'_X)^T (\sigma_X, t dW^S_t - \frac{1}{2} \sigma^2_{X}, t dt)] \\
= \omega(\sigma^2_X, \sigma_X \sigma'_X)^T \mathbb{E}[\bar{\varphi}'''_{x,x,x} (\int_0^T \sigma_X, t dW^S_t, \int_0^T \sigma_I, t dW^I_t)] \\
&+ \omega(\sigma_1 \sigma_X \rho_{S,I}, \sigma_X \sigma'_X)^T \mathbb{E}[\bar{\varphi}'''_{x,x,i} (\int_0^T \sigma_X, t dW^S_t, \int_0^T \sigma_I, t dW^I_t)] \\
&- \frac{1}{2} \omega(\sigma^2_X, \sigma_X \sigma'_X)^T \mathbb{E}[\bar{\varphi}''_{x,x} (\int_0^T \sigma_X, t dW^S_t, \int_0^T \sigma_I, t dW^I_t)]
\end{align*}
Numerical experiments

**Constant Elasticity of variance model :**

\[
\frac{dS_t}{S_t} = (r - q) dt + \nu S_t^{\beta - 1} dW_t, \quad S_0 > 0.
\]

with \( \nu > 0 \) and \( 0 < \beta < 1 \).

- \( \nu \) : the diffusion level or the stock price volatility.
- \( \beta \) : the diffusion skew.

The volatility for the log discounted asset price :

\[
\sigma_X(t, x) = \nu e^{(\beta - 1)x + (\beta - 1)(r - q)t}
\]

\[
\sigma_X(t, X_0) = \gamma e^{\alpha t}
\]

\[
\sigma_X'(t, X_0) = (\beta - 1)\gamma e^{\alpha t}
\]

\[
\sigma_X''(t, X_0) = (\beta - 1)\sigma_X'(t, X_0)
\]

with \( \gamma = \nu e^{(\beta - 1)X_0} \), \( \alpha = (\beta - 1)(r - q) \) and \( X_0 = \log(S_0) \).
**Numerical experiments**

\( T = 1, 3, 6 \) with \( K \in [S_0 e^{-2.\nu.T}, S_0 e^{2.\nu.T}] \). Number of Monte Carlo paths: 2.5 millions, 600 time steps for the Euler discretisation.

Set of parameters with \( \beta = 0.3, \nu = 0.2 \) and \( \rho_{si} = 0.3 \)

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>( I_0 )</th>
<th>( r )</th>
<th>( q )</th>
<th>( \nu )</th>
<th>( \beta )</th>
<th>( i_0 )</th>
<th>( a )</th>
<th>( \theta )</th>
<th>( \sigma_i )</th>
<th>( \rho_{si} )</th>
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<td>0.88</td>
<td>0.98</td>
<td>0.03</td>
<td>0.02</td>
<td>0.2</td>
<td>0.3</td>
<td>0.02</td>
<td>0.04</td>
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<td>0.05</td>
<td>0.3</td>
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Range for the strike and maturity \( T \setminus K \):

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<th>0.7</th>
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<td>1</td>
<td>1.8</td>
<td>2.34</td>
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</table>

MC prices (confidence intervals at 95\%), second and third order approximation prices:
Numerical experiments

1 year Maturity

-0.0001 0 0.0001 0.0002 0.0003 0.0004 0.0005 0.0006 0.0007

0.5 0.7 0.9 1.1 1.3

K

Absolute errors

2nd order approximation

3rd order approximation

J. Hok joint work with E. Gobet

Expansion Formulas for bivariate payoffs with application to best-Of Option on Equity and Inflation 28/37
Numerical experiments

3 years Maturity

Absolute errors
2nd order approximation
3rd order approximation

J. Hok joint work with E. Gobet

Expansion Formulas for bivariate payoffs with application to best-
Numerical experiments

-0.003
-0.001
0.001
0.003
0.005
0.007
0.009
0.3 0.8 1.3 1.8 2.3

6 years Maturity

Absolute errors

2nd order approximation

3rd order approximation

J. Hok joint work with E. Gobet
Numerical experiments

Very good accuracy for the approximation formulas:

<table>
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<th>T=3</th>
<th>T=6</th>
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<tbody>
<tr>
<td><strong>Average errors (in bp)</strong></td>
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<tr>
<td>Second order approximation</td>
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<td>19</td>
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<td>Third order approximation</td>
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<table>
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<tr>
<td><strong>Maximum errors (in bp)</strong></td>
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<tr>
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<tr>
<td>Third order approximation</td>
<td>1</td>
<td>12</td>
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Errors increase with the maturity.

Errors for far out of the money options are more important than deep in the money option. With the third order approximation, average errors for deep in the money option are less than 1 bp, compared to 17 bps for far out of the money option.
Extension to stochastic interest rate

\[
\frac{dS^d_t}{S^d_t} = \sigma_S(t, S^d_t) dW^S_t
\]

\[
\frac{dI_t}{I_t} = i_t dt
\]

\[
di_t = a(\theta_t - i_t) dt + \sigma_I dW^I_t
\]

\[
r_t = f(0, t) - \int_0^t \gamma^t_s \cdot \Gamma^t_s ds + \int_0^t \gamma^t_s dW^r_s
\]

\[
d\langle W^S, W^I \rangle_t = \rho_{S,I} dt, \quad d\langle W^I, W^r \rangle_t = \rho_{I,r} dt, \quad d\langle W^S, W^r \rangle_t = \rho_{S,r} dt
\]

with \( S^d_t = S_t e^{-\int_0^t (r_s - q_s) ds} \), \((\Gamma^T_t)_{0 \leq t \leq T}\) deterministic volatility of the ZC \((B(t, T))_{0 \leq t \leq T}\) and \(\gamma^T_t = -\partial_T \Gamma^T_t\) the volatility process of the forward rates.
Extension to stochastic interest rate

\[ E_Q[e^{-\int_0^T r_s ds} \Phi(S_T, I_T)] = B(0, T) E_T[\varphi(\int_0^T (r_s - q_s) ds + X_T, \log(I_T))], \quad (5) \]

and under \( Q_T \), the SDE becomes

\[ dX_t = \sigma_X(t, X_t) dW_t^{T,S} + \left[ -\frac{1}{2} \sigma_X^2(t, X_t) + \Gamma_t T \rho_{S,r} \sigma(t, X_t) \right] dt, \quad (6) \]

\[ di_t = [a(\theta_t - i_t) + \sigma_I \Gamma_t T \rho_{I,r}] dt + \sigma_I dW_t^{T,I}, \quad (7) \]

\[ r_t = f(0, t) - \int_0^t \gamma_s \cdot (\Gamma_s - \Gamma_s T) ds + \int_0^t \gamma_s dW_s^{T,r}, \quad (8) \]

\((W_t^{T,S}, W_t^{T,I}, W_t^{T,r})\) is a \( Q_T \)-Brownian motion with the same correlations as \((W_t^S, W_t^I, W_t^r)\).
Let’s introduce the proxy process for the log-discounted price

$$dX^p_t = \sigma_{X,t} dW_t^{T,S} + \left[-\frac{1}{2}\sigma_{X,t}^2 + \Gamma_T \rho_{S,r}\sigma_{X,t}\right]dt, \quad X^p_0 = X_0 \quad (9)$$

the parametrised process for $\eta \in [0, 1]$

$$dX^\eta_t = \eta[\sigma_{X}(t, X^\eta_t) dW_t^{T,S} + \left(-\frac{1}{2}\sigma_{X}^2(t, X^\eta_t) + \Gamma_T \rho_{S,r}\sigma_{X}(t, X^\eta_t)\right)]dt, \quad X^\eta_0 = x_0. \quad (10)$$

and perform $X_{1,t}, X_{2,t}$ with $X^p_T = x_0 + X_{1,T}$,

$$dX_{2,t} = 2X_{1,t} \left(\sigma'_{X,t} dW_t^{T,S} - \sigma'_{X,t}(\sigma_{X,t} - \Gamma_T \rho_{S,r})dt\right), \quad X_{2,0} = 0. \quad (11)$$
Extension to stochastic interest rate

By Taylor expansion:

\[ E_T[\varphi(\int_0^T (r_s - q_s) ds + X_T, \log(I_T))] = E_T[\varphi(\int_0^T (r_s - q_s) ds + X_T^P, \log(I_T))] \]

\[ + E_T[\varphi_x^{(1)}(\int_0^T (r_s - q_s) ds + X_T^P, \log(I_T)) \frac{X_{2,T}}{2}] \]

\[ + \text{error}_2. \]

By projection

\[ \int_0^T (r_s - q_s) ds + X_T^P = \int_0^T (\sigma_{X,t} - \Gamma_{t}^{T} \rho_{S,r} ) dW_{t}^{T,S} + D_T, \]

with \( D_T \) independent of \( W_{T,S}. \)
**Theorem (Second order approximation price formula)**

Assume (R4) and (H). Then

$$
E_Q e^{-\int_0^T r_s ds} \Phi(S_T, I_T) = B(0, T) \left( E_T \Phi(S_T^P, I_T) + \alpha_{2,0} S_0^2 \text{Greek}^{\phi}_{2,0} + \alpha_{1,1} S_0 I_0 \text{Greek}^{\phi}_{1,1} \\
+ \alpha_{3,0} S_0^3 \text{Greek}^{\phi}_{3,0} + \alpha_{2,1} S_0^2 I_0 \text{Greek}^{\phi}_{2,1} + \alpha_{1,2} S_0 I_0^2 \text{Greek}^{\phi}_{1,2} \right) \\
+ \text{error}_2, \tag{13}
$$

with

$$
\begin{align*}
\alpha_{1,1} &:= \frac{1}{2} \omega(\sigma_X^2, \sigma_X^2 \sigma_I \rho_S, I)^T_0 + \omega(\sigma_X \sigma_I \rho_S, I, \sigma_X^2 \sigma_I \rho_S, I)^T_0, \\
\alpha_{1,2} &:= \omega(\sigma_X \sigma_I \rho_S, I, \sigma_X^2 \sigma_I \rho_S, I)^T_0, \\
\alpha_{2,0} &:= \omega(\sigma_X (\frac{3}{2} \sigma_X - \Gamma^T \rho_S, r), \sigma_X^0 (\sigma_X - \Gamma^T \rho_S, r))^T_0, \\
\alpha_{2,1} &:= \omega(\sigma_X \sigma_I \rho_S, I, \sigma_X^0 (\sigma_X - \Gamma^T \rho_S, r))^T_0 + \omega((\sigma_X - \Gamma^T \rho_S, r) \sigma_X, \sigma_X^0 \sigma_I \rho_S, I)^T_0, \\
\alpha_{3,0} &:= \omega((\sigma_X - \Gamma^T \rho_S, r) \sigma_X, \sigma_X^0 (\sigma_X - \Gamma^T \rho_S, r))^T_0, \\
| \text{error}_2 | &\leq c \sup(\| \varphi^{(1)}_x \|_{\infty}, | \varphi^{(2)}_{xx} \|_{\infty}) M_1 M_0^2 T^{3/2}.
\end{align*}
$$
Conclusions

- Intuitive and practical framework for the valuation of hybrid option where one of the underlying follows the local volatility model and the other asset value a log-normal process.

- Because of the generality of the local volatility function, the pricing is computed numerically, which is time consuming. We provided an approximation formula of Black-Scholes type for the price and its implementation is simple. Experiments show a very good accuracy.

- Natural subsequent work: derivation of approximation formulas to account for inflation smile/skew.