Modeling and Pricing Longevity Derivatives with Stochastic Mortality Using the Esscher Transform

Shuo-Li Chuang and Patrick L. Prockett

University of Texas at Austin

September 7, 2012
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Lee and Carter (1992) model the mortality rate as a function of age group $x$ and time $t$ (in years).

$$m_{x,t} = \exp(a_x + b_x k_t + \epsilon_{x,t})$$

$x$: age group  
$t$: time  
$a_x$: the general shape of the mortality curve for age group $x$  
$k_t$: the mortality rate time (year) index  
$b_x$: each age group’s response to the mortality rate index  
$\epsilon_{x,t}$: error term
Example of Lee-Carter Time Trend $k_t$

(a) $k_t$

(b) $\Delta k_t = k_t - k_{t-1}$

Data: Male of England and Wales from 1841 to 2006
Modified Lee-Carter Model


\[ m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t}) \]

\[ \log(m_{x,t}) - \log(m_{x,t-1}) = a_x + b_x k_t + \epsilon_{x,t}. \]

This modified model performs better than

- Lee-Carter model with or without cohort effects
- Logit parametric models
- Several other modification of Lee-Carter models

The paper uses data from 11 countries, including U.S.A, U.K., Canada, and several other countries. This modified model performs better than other models whatever the data are used.
Example of Modified Lee-Carter Time Trend $k_t$

\[(c) \quad \log(m_{65,69}, t) - \log(m_{65,69}, t-1) \quad \text{and} \quad (d) \quad k_t\]

Data: Male of England and Wales from 1841 to 2006
To forecast future mortality rate, Michell et al (2011) suggest fitting $k_t$ in the modified Lee-Carter model with a Normal Inverse Gaussian (NIG) process. The NIG distribution has a density

\[
f_{\text{NIG}}(y; \alpha, \beta, \mu, \delta) = \frac{\alpha}{\pi} \exp(\delta \sqrt{\alpha^2 - \beta^2} + \beta(y - \mu)) \frac{K_1\left(\frac{\alpha \delta \sqrt{1 + \left(\frac{y-\mu}{\delta}\right)^2}}{\sqrt{1 + \left(\frac{y-\mu}{\delta}\right)^2}}\right)}{\sqrt{1 + \left(\frac{y-\mu}{\delta}\right)^2}}\]

- 4 parameters, $(\alpha, \beta, \mu, \delta)$, to control shape and location of the distribution.
- It provides a better fit when the distribution has high kurtosis and fat tails.
Fitted Mortality Rate for Age 65 to 69

Data: Male of England and Wales from 1841 to 2006
Model $k_t$ with a NIG Process

$$m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})$$

The modified Lee-Carter Model
Model \( k_t \) with a NIG Process

\[
m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})
\]

By iteration, we have

\[
= m_{x,0} \exp \left( a_x t + b_x \sum_{i=1}^{t} k_i + \sum_{i=1}^{t} \epsilon_{x,i} \right)
\]
Model \( k_t \) with a NIG Process

\[
m_{x,t} = m_{x,t-1} \exp(a_x + b_x k_t + \epsilon_{x,t})
\]

\[
= m_{x,0} \exp \left( a_x t + b_x \sum_{i=1}^{t} k_i + \sum_{i=1}^{t} \epsilon_{x,i} \right)
\]

Let \( k_i \) and \( \epsilon_{x,i} \) be i.i.d. random variables, then we have

\[
= m_{x,0} \exp(a_x t + b_x \sum_{i=1}^{t} k_1 + \sum_{i=1}^{t} \epsilon_1)
\]

where \( k_1 \) a NIG random variable.
Fit $k_1$ with NIG Distribution

Data: Male of England and Wales from 1841 to 2006
A \((Y_t)_{t \geq 0}\) is a Lévy process if

- Independent increments: For \(0 \leq t_1 < t_2 \leq t_3 < t_4\), \((Y_{t_2} - Y_{t_1})\) and \((Y_{t_4} - Y_{t_3})\) are independent.
- Stationary increments: \((Y_{t+h} - Y_t)\) does not depend on \(t\).
- Right continuous: For all \(\epsilon > 0\), 
  \[\lim_{h \to 0} \mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) = 0.\]

In financial modeling, the stock price process is usually assumed to be

\[S_t = S_0 \exp(r_f t + Y_t)\]

- \(S_t\): stock price at time \(t\)
- \(r_f\): risk free rate.
- \(Y_t\): Lévy process
Mortality Rate with a Stochastic Component

Let the time-varying part be a NIG Lévy process. We have

\[ m_{x,t} = m_{x,0} \exp(a_x t + N_{x,t}), \]

where

\[ N_{x,t} \sim \text{NIG}(\alpha/b_x, \beta/b_x, b_x \mu t, b_x \delta t) \]
\[ \sim \text{NIG}(\alpha', \beta', \mu' t, \delta' t). \]
Esscher Transform

If a stock process follows a exponential Lévy process, the Esscher transform can be used to find a martingale measure for pricing purpose.

The Esscher transform \( \mathbb{P}^{\theta} \) is defined as

\[
\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}} = \frac{e^{\theta Y}}{E(e^{\theta Y})},
\]

provided that \( E(e^{\theta Y}) \), the moment generation function, exists.

- The ratio of two measures.
- With Esscher transform, the martingale measure can be calculated easily through the moment generation function.
In financial modeling, the stock price process is assumed to be an exponential Lévy process

\[ S_t = S_0 \exp(r_f t + Y_t). \]

- Let \( M_Y(\theta) \) be the moment generation function and \( \kappa(\theta) \) is the exponential components of \( M_Y(\theta) \).
- With the Esscher transform, the \( \theta \) value needed to obtain the martingale measure can be found by solving

\[ \kappa(\theta + 1) - \kappa(\theta) = r_f. \]

In our mortality process model, $\kappa(\theta)$ is known and we solve $\kappa(\theta + 1) - \kappa(\theta) = a_x$.

Example: By using mortality rate data of 65 to 69 years old male in England and Wales from 1841 to 2006, the estimation results are

$$(\alpha', \beta', \mu', \delta', a_{[65,69]}, \theta) = (7.2093, -0.5786, -0.0003, 0.0039, -6.3177, -0.0070).$$
Application: Pricing a Q-forward

- A q-forward exchanges fixed mortality for realized mortality at maturity of the contract.
- The LLMA (Life & Longevity Markets Association)’s structure can be used for pricing. The related documents are available at http://www.llma.org/publications.html
LLMA Pricing Structure

Party A is an investment bank. Party B is a pension fund who wants to hedge longevity risk. At maturity, the bank (party A) receives

\[ \text{notional amount} \times (m_{\text{realized}} - m_{\text{fixed}}), \]

and the pension fund (party B) receives

\[ \text{notional amount} \times (m_{\text{fixed}} - m_{\text{realized}}). \]

To price the product at time 0 before the realized rate is known, the bank will construct a modified mortality rate with an adjustment for risk premium and an adjustment in mortality rate for expected mortality rate movement. Therefore, at maturity, the bank receives

\[ \text{notional amount} \times (m_{\text{modified}} - m_{\text{fixed}}). \]
Given discount rate \( r \) and contract duration \( t \) years, the present value of settlement is

\[
\text{notional amount} \times \frac{m_{\text{modified}} - m_{\text{fixed}}}{(1 + r)^t}.
\]

The modified rate is

\[
m_{\text{modified}} = m_0 \times (1 - (m_{\text{predicted}} + \xi))^t,
\]

where \( \xi \) is an adjustment for risk premium. The predicted rate can be obtained by mortality modeling. LLMA suggests an average predicted rate across \( t \) years in an age group or desired ages, denoted as \( \hat{m} \). Therefore, the adjustment value \( \xi \) is the smallest value to solve

\[
m_{\text{fixed}} \leq m_0(1 - (\hat{m} + \xi))^t.
\]
Example: JPMorgan Q-forward

- The notional amount is 50 million GBP.
- 10-year contract (from 12/31/2006 to 12/31/2016).
- The reference group is males in England & Wales who will be 65 to 69 years old in 2016.
- Fixed rate is 1.2%, given by contract.
- Discount rate is 4.37 % (T-bill rate).

Goal: Solve for the adjustment for risk premium $\xi$ in the previous equation.
Data: death rate for the age group 65-69, 1841-2006

The predicted mortality rate is calculated from the modified Lee-Carter model with NIG Lévy process and Esscher transform.

Results:
- The average predicted mortality rate for age group 65 to 69 is \( \hat{m} = 1.70\% \).
- The adjustment for risk premium value is \( \xi = 2.08\% \).
- The present value of settlement is £938,601.38 GBP.
References


