Modelling and Testing for Structural Breaks in Panels with Common and Idiosyncratic Stochastic Trends

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Model

$$
\begin{aligned}
y_{it} &= \alpha_i + \beta' F_t + \gamma' x_{it} + u_{it} \\
F_t &= F_{t-1} + \varepsilon_t \\
x_{it} &= x_{it-1} + \epsilon_{it} \\
z_{it} &= \lambda_i' F_t + e_{it}.
\end{aligned} \tag{1}
$$

where

- y_{it} is a scalar,
  F_t is a $R \times 1$ vector of common factors. Note F_t may not be observable.
  x_{it} is a $p \times 1$ vector of individual specific explanatory variables,
  β and γ are $R \times 1$ and $p \times 1$ vectors of the slope parameters of interest.
  α_i is the individual effect,
  $(u_{it}, \varepsilon_{it}, \epsilon_{it}, e_{it})$ are a iid.

HYPOTHESES OF INTEREST

- The problem of interest is to test the changes in the parameter θ_t,i.e., under the null

$$H_0 : \theta_t = \theta$$

for all t. The alternative hypothesis,

$$H_A : \theta_t = \begin{cases} \theta_1 & \text{for } t = 1, ..., k \\ \theta_2 & \text{for } t = k+1, ..., T \end{cases} \tag{2}$$

where $\theta_t = (\beta_t, \gamma_t)'$.

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where $\theta_t = (\beta_t, \gamma_t)'$.

- Test the constancy of the slope parameter γ_t.
- Test the stability of the factor structure β_t.

Useful Notes:

$w_t = F_t - T^{-1} \sum_{t=1}^{T} F_t$ and $\tilde{x}_{it} = x_{it} - \frac{1}{T} \sum_{t=1}^{T} x_{it}$

- $\frac{1}{nT^2} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t \widetilde{x}_{it}' = o_p(1)$;

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- $\frac{1}{nT^2} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t \widetilde{x}_{it}' = o_p(1)$;
- $\frac{1}{\sqrt{n}T} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t u_{it} \xrightarrow{d} \sigma_u \left(\int \overline{B}_{\varepsilon} \overline{B}_{\varepsilon}' \right)^{1/2} \times Z_1$, where $Z_1 \sim N(0, I_R)$. Recall $\sigma_u^2 = Var(u_{it})$ and

$$\frac{F_t}{\sqrt{T}} \xrightarrow{d} B_{\varepsilon}.$$

$\overline{B}_{\varepsilon}$ is the demeaned B_{ε}.

Block Design Matrix

- $\frac{1}{nT^2}\begin{bmatrix} \sum_{i=1}^{n}\sum_{t=1}^{T} w_t w_t' & \sum_{i=1}^{n}\sum_{t=1}^{T} \tilde{x}_{it} w_t' \\ \sum_{i=1}^{n}\sum_{t=1}^{T} w_t \tilde{x}_{it}' & \sum_{i=1}^{n}\sum_{t=1}^{T} \tilde{x}_{it}\tilde{x}_{it}' \end{bmatrix} = \begin{bmatrix} O_p(1) & o_p(1) \\ o_p(1) & O_p(1) \end{bmatrix}.$

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- $\hat{\beta} - \beta$ and $\hat{\gamma} - \gamma$ are asymptotically independent. $\hat{\beta}$ and $\hat{\gamma}$ are OLS of β and γ.

Mixed Normality and Normality

- $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t u_{it} \xrightarrow{d} \sigma_u \left(\int \overline{B}_\varepsilon \overline{B}_\varepsilon'\right)^{1/2} \times Z_1$. $\overline{B}_\varepsilon$ is a demeaned of B_ε.

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- $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} x_{it} u_{it} \xrightarrow{d} N\left(0, c\Omega_\epsilon \sigma_u^2\right)$.

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- $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} x_{it} u_{it} \xrightarrow{d} N\left(0, c\Omega_\epsilon \sigma_u^2\right)$.
- Mixed normality is due to $\int \overline{B}_\varepsilon \overline{B}_\varepsilon^{'}$ is a random matrix: (1) F_t or w_t is shared by all i and (2) F_t is I(1).

under the null of no structural change

$\hat{\beta}\hat{\gamma}$

- $\sqrt{n}T \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} \begin{pmatrix} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{-1/2} \sigma_u \\ \sqrt{6}\Omega_\epsilon^{-1/2}\sigma_u \end{pmatrix} \times Z$

  where

$$Z \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_R & 0 \\ 0 & I_p \end{pmatrix} \right).$$

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- $\sigma_{\zeta}^2 = \sigma_u^2 + \sigma_{\Pi}^2$ where σ_{Π}^2 is caused by the estimation error, $\hat{F}_t - F_t$.

The Wald Test Statistic

- Let $\hat{w}_t = \hat{F}_t - T^{-1} \sum_{t=1}^{T} \hat{F}_t$ and $\hat{W}_{it} = (\hat{w}_t', \tilde{x}_{it}')'$

$$\hat{\theta}_{1k} = \left(\sum_{i=1}^{n} \sum_{t=1}^{k} \hat{W}_{it} \hat{W}_{it}' \right)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{k} \hat{W}_{it} y_{it},$$

and

$$\hat{\theta}_{2k} = \left(\sum_{i=1}^{n} \sum_{t=k+1}^{T} \hat{W}_{it} \hat{W}_{it}' \right)^{-1} \sum_{i=1}^{n} \sum_{t=k+1}^{T} \hat{W}_{it} y_{it}.$$

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- $\mathrm{W}(k) =$

$$\left(\hat{\theta}_{1k}^{*} - \hat{\theta}_{2k}^{*}\right)'\left[\begin{array}{c} \left(\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right)^{-1} \\ +\left(\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\right)^{-1} \end{array}\right]^{-1}\left(\hat{\theta}_{1k}^{*} - \hat{\theta}_{2k}^{*}\right) \text{ with}$$

$$\hat{\theta}_{jk}^{*} = \left[\begin{array}{cc} \hat{\sigma}_{\zeta}^{2}I_R & 0 \\ 0 & \hat{\sigma}_{u}^{2}I_p \end{array}\right]^{-1}\hat{\theta}_{jk},\ j = 1, 2.$$

Limiting Distribution

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- $$Q_R(r) = \mathbf{s}(r)' \mathbf{V}^{-1}(r)\, \mathbf{s}(r),$$
$$Q_p(r) = \frac{\left[B((1-r)^2) - B(r^2)\right]' \left[B((1-r)^2) - B(r^2)\right]}{r^2 + (1-r)^2}$$

- Let $S_1(r) = \sigma_\zeta^{-1} \int_0^r \bar{B}_\varepsilon dB$ and $S_2(r) = \sigma_\zeta^{-1} \int_r^1 \bar{B}_\varepsilon dB$, where $B(\cdot)$ is a standard Brownian motion. Define

$$\mathbf{s}(r) = \begin{bmatrix} S_1(r) \\ S_2(r) \end{bmatrix},$$

$M_1(r) = \int_0^r \bar{B}_\varepsilon \bar{B}_\varepsilon'$, $M_2(r) = \int_r^1 \bar{B}_\varepsilon \bar{B}_\varepsilon'$, and

$$\mathbf{V}^{-1}(r) = \begin{bmatrix} I_R & 0 \\ 0 & -I_R \end{bmatrix} \begin{bmatrix} M_1^{-1} \\ M_2^{-1} \end{bmatrix} \left[M_1^{-1} + M_2^{-1}\right]^{-1} \begin{bmatrix} M_1^{-1} & M_2^{-1} \end{bmatrix} \begin{bmatrix} I_R \\ 0 \end{bmatrix}$$

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$$\mathbf{V}^{-1}(r) = \left[\begin{array}{cc} I_R & 0 \\ 0 & -I_R \end{array} \right] \left[\begin{array}{c} M_1^{-1} \\ M_2^{-1} \end{array} \right] \left[M_1^{-1} + M_2^{-1} \right]^{-1} \left[\begin{array}{cc} M_1^{-1} & M_2^{-1} \end{array} \right] \left[\begin{array}{c} I_R \\ 0 \end{array} \right.$$

- For a given r, $Q_R(r)$ and $Q_p(r)$ are independent such that

$$Q_R(r) \sim \chi^2(R),$$

$$Q_p(r) \sim \frac{(1-r)^2 - r^2}{(1-r)^2 + r^2} \chi^2(p).$$

Sup, Mean and Exp Wald

- $$SupW(k) = \sup_{[Tr^*] \leq k \leq T - [Tr^*]} W(k),$$

  $$AveW(k) = \frac{1}{T} \sum_{k=[Tr^*]}^{T-[Tr^*]} W(k),$$

  and

  $$ExpW(k) = \log \left\{ \frac{1}{T} \sum_{k=[Tr^*]}^{T-[Tr^*]} \exp \left[\frac{1}{2} W(k) \right] \right\}$$

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  and
  $$ExpW(k) = \log \left\{ \frac{1}{T} \sum_{k=[Tr^*]}^{T-[Tr^*]} \exp\left[\frac{1}{2} W(k) \right] \right\}$$
- Under the null, $SupW([Tr]) \xrightarrow{d} \sup_{r^* \leq r \leq 1-r^*} D(r)$,
  $AveW([Tr]) \xrightarrow{d} \int_{r^*}^{1-r^*} D(r)dr$,
  $ExpW([Tr]) \xrightarrow{d} \log \left\{ \int_{r^*}^{1-r^*} \exp\left[\frac{1}{2} D(r)\right] dr \right\}$.

Monte Carlo Simulations

$$y_{it} = \alpha_i + \beta_t' F_t + \gamma_t' x_{it} \quad + \quad u_{it}$$

$$F_t = F_{t-1} + \varepsilon_t,$$

$$x_{it} = x_{it-1} + \epsilon_{it},$$

and

$$z_{it} = \lambda_i' F_t + e_{it}$$

where

$$\begin{pmatrix} u_{it} \\ \varepsilon_{it} \\ \epsilon_{it} \\ e_{it} \end{pmatrix} \overset{iid}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right). \tag{3}$$

Size SupW

Size at the 5% Level

T/n	20	40	60	120	240	480
20	0.0190	0.0282	0.0294	0.0339	0.0275	0.0252
40	0.1452	0.0563	0.0479	0.0351	0.0331	0.0427
60	0.0640	0.0467	0.0508	0.0492	0.0406	0.0475
120	0.2448	0.0585	0.0614	0.0538	0.0508	0.0410
240	0.1131	0.1207	0.0922	0.0593	0.0562	0.0534
480	0.0742	0.1699	0.0611	0.0658	0.0753	0.0693

Panel A. Size for $SupW$

Size for ExpW

T/n	20	40	60	120	240	480
20	0.0394	0.0538	0.0515	0.0544	0.0550	0.0506
40	0.1824	0.0820	0.0690	0.0526	0.0516	0.0614
60	0.0805	0.0609	0.0645	0.0641	0.0536	0.0615
120	0.2504	0.0680	0.0691	0.0616	0.0597	0.0493
240	0.1135	0.1187	0.0943	0.0624	0.0596	0.0545
480	0.0743	0.1513	0.0621	0.0645	0.0716	0.0657

Panel B Size for *ExpW*

Size for AveW

T/n	20	40	60	120	240	480
20	0.0700	0.0836	0.0797	0.0827	0.0862	0.0734
40	0.2056	0.1109	0.0982	0.0782	0.0817	0.0873
60	0.1114	0.0922	0.0932	0.0911	0.0833	0.0881
120	0.2627	0.0974	0.0967	0.0933	0.0861	0.0810
240	0.1441	0.1438	0.1182	0.0902	0.0899	0.0827
480	0.1010	0.1694	0.0915	0.0913	0.1001	0.0952

Panel C. Size for *AveW*

Conclusion

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- The limiting distributions of the proposed test statistics are derived and they are free of nuisance parameters (depend on the rank of the regressors).
- A set of Monte Carlo experiments are conducted to evaluate finite sample performance of the proposed tests.
- The simulation results that all three test statistics perform well when the cross-sectional dimension is as large the time series dimension. All three test statistics tend to oversize if the cross-sectional dimension is small.