Conditional Monte Carlo Pricing of Path Dependent Options

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Presentation Overview

This presentation extends the previous work of the author, who used conditional integration techniques to price vanilla options under the SABR model (plus extensions), to that of path dependent options.

- Extending Willard [1997] - modelling correlated stochastic volatility as an integral of modified Black-Scholes term structure prices (extension to other asset processes will be discussed briefly);
- Revisiting the notion of mixing models;
- Efficient implementation in a parallel computation environment (multi-core will be presented here - but can extend naturally to GPU also);
- Stochastic rates processes.

From Willard to SABR
Willard [1997]

Given the lognormal asset process

\[ dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \]

under some volatility process \( \sigma(t) \), time dependent drift rate \( \mu(t) \) and Wiener process \( W(t) \), then as usual

\[ d\log S(t) = \mu(t)dt - \frac{1}{2} \sigma(t)^2 dt + \sigma(t)dW(t) \]

Integrating yields

\[ S(T) = S(0) \exp \left( \int_0^T \mu(t)dt - \frac{1}{2} \int_0^T \sigma(t)^2 dt + \int_0^T \sigma(t)dW(t) \right) \]

Assume the volatility process is stochastic, with one Wiener driver \( W_{\sigma}(t) \) correlated to the asset process then

\[ dW(t) = \rho dW_{\sigma}(t) + \bar{\rho}dW_S(t) \]

with correlation \( \rho \) and \( \bar{\rho} = \sqrt{1 - \rho^2} \)
Willard [1997]

The asset can therefore be represented at time $T$ as

$$S(T) = S(0) \exp \left( \int_0^T \mu(t) dt - \frac{1}{2} \int_0^T \sigma(t)^2 dt + \rho \int_0^T \sigma(t) dW_{\sigma}(t) + \rho \int_0^T \sigma(t) dW_S(t) \right)$$

which for a given volatility path $\sigma(0, T)$ can be recast as

$$S(T) \mid \sigma(0, T) = S(0) \exp \left( \int_0^T \overline{\mu}(t) dt - \frac{1}{2} \int_0^T \overline{\sigma}(t)^2 dt + \int_0^T \overline{\sigma}(t) dW_S(t) \right)$$

and so is conditionally lognormal with

$$\overline{\sigma}(t) = \overline{\rho} \sigma(t)$$

$$\overline{\mu}(t) = \mu(t) - \frac{1}{2} \rho^2 \sigma(t)^2 + \rho \sigma(t) \frac{dW_{\sigma}(t)}{dt}$$

where the second expression is shorthand for stating

$$\int_0^T \overline{\mu}(t) dt = \int_0^T \mu(t) dt - \frac{1}{2} \rho^2 \int_0^T \sigma(t)^2 dt + \rho \int_0^T \sigma(t) dW_{\sigma}(t)$$
Willard [1997] - conditional terminal distribution

Rewriting

\[ S(T) | \sigma(0, T) = S(0) \exp \left( \int_0^T \mu(t) - \frac{1}{2} \int_0^T \sigma(t)^2 dt + \int_0^T \sigma(t) dW_S(t) \right) \]

\[ = S(0) \exp \left( \overline{\mu}(0, T) T - \frac{1}{2} \overline{\vartheta}(0, T) T + \overline{\vartheta}(0, T)^{1/2} W'_S(T) \right) \]

where

\[ \overline{\vartheta}(0, T) = \frac{1}{T} \int_0^T \overline{\sigma}(t)^2 dt = \frac{1}{T} (1 - \rho^2) \int_0^T \sigma(t)^2 dt \]

\[ \overline{\mu}(0, T) = \int_0^T \overline{\mu}(t) dt = \int_0^T \mu(t) dt - \frac{1}{2} \rho^2 \int_0^T \sigma(t)^2 dt + \rho \int_0^T \sigma(t) dW_\sigma(t) \]

Note that

- when \( \rho = 0 \) then the terminal asset distribution is determined by \( \overline{\vartheta}(0, T) \) only;
- when \( \rho \neq 0 \) then in general the terminal asset distribution is determined by the entire volatility path
Willard [1997] - conditional terminal distribution

So for the conditional asset, the associated forward rate is

\[
\mathbb{E}[S(T)|\sigma(0, T)] = S(0) \exp(\mu(0, T) T) \\
= F(0, T) \exp \left( -\frac{1}{2} \rho^2 \int_0^T \sigma(t)^2 dt + \rho \int_0^T \sigma(t) dW_{\sigma}(t) \right)
\]

and logvariance

\[
(1 - \rho^2) \int_0^T \sigma(t)^2 dt
\]

Qualitatively,

- as the correlation increases the conditional logvariance decreases;
- when \( \rho = 0 \) the conditional forward is the market forward;
- when \( \rho > 0 \) for an increasing/decreasing volatility path, the conditional forward is below/above the market forward;
- when \( \rho < 0 \) for an increasing/decreasing volatility path, the conditional forward is above/below the market forward.
SABR Example

For the case of SABR ($\beta = 1$) i.e.

$$dS(t) = \mu(t) S(t) dt + \sigma(t) S(t) dW$$

$$d\sigma(t) = \xi \sigma(t) dW_\sigma$$

then

$$\int_0^T \sigma(t) dW_\sigma(t) = \int_0^T \frac{d\sigma(t)}{\xi} = \frac{1}{\xi} (\sigma(T) - \sigma(0))$$

and so the conditional terminal asset distribution is lognormal and dependent only upon $\sigma(T)$ and the mean integrated variance

$$\vartheta = \frac{1}{T} \int_0^T \sigma(t)^2 dt$$

The conditional asset distribution is therefore

$$S(T)|\{\vartheta, \sigma(T)\} = F(0, T) \exp \left( -\frac{1}{2} \rho^2 \vartheta + \frac{\rho}{\xi} (\sigma(T) - \sigma(0)) + \rho \sqrt{\vartheta} W(T) \right)$$

allowing for vanilla option evaluation given the joint density of $\vartheta$ and $\sigma(T)$. This is explored in Islah [2009] and McGhee [2010]
SABR Example

So given a call option $C(K, T)$ evaluated under the SABR model, it can be written

$$C(K, T) = \int_0^\infty d\sigma(T) \psi(\sigma(T)) \int_0^\infty d\vartheta C(K, T, \{\vartheta, \sigma(T)\}) \psi(\vartheta | \sigma(T))$$

where

- $\psi(\sigma(T))$ is the density of $\sigma(T)$ (which in this case is known);
- $\psi(\vartheta | \sigma(T))$ is the density of $\vartheta$ conditional on $\sigma(T)$ (nontrivial); and
- $C(K, T, \{\vartheta, \sigma(T)\})$ is the option value given $\vartheta$ and $\sigma(T)$.

The only unknown is $\psi(\vartheta | \sigma(T))$. Islah [2009] calculates this using Yor [1992] whereas McGhee [2010] approximates this as a lognormal via the calculation of

$$\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma(t)^2 dt \right) \bigg| \sigma(T) \right] \quad \text{and} \quad \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma(t)^2 dt \right)^2 \bigg| \sigma(T) \right]$$

The first is analytic (a Gaussian integration) whereas the second requires at most one numerical integration.
How well is \( \psi(\vartheta) \) approximated?

In order to understand how effective the method of integrating over conditional lognormal distributions is at capturing the distribution of \( \vartheta \), consider

\[
\sigma_k(T) = \left( \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma^2(t) dt \right)^k \right] \right)^{1/2k}
\]

This is know analytically for lognormal volatility with the first few moments being

\[
\mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma^2(t) dt \right] = \frac{\sigma^2(0)}{\xi^2 T} \left( e^{\xi^2 T} - 1 \right)
\]

\[
\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma^2(t) dt \right)^2 \right] = \frac{2\sigma^4(0)}{\xi^4 T^2} \left( \frac{1}{30} e^{6 \xi^2 T} - \frac{1}{5} e^{4 \xi^2 T} + \frac{1}{6} \right)
\]

\[
\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T \sigma^2(t) dt \right)^3 \right] = \frac{4\sigma^6(0)}{\xi^6 T^3} \left( \frac{1}{1890} e^{15 \xi^2 T} - \frac{1}{270} e^{6 \xi^2 T} + \frac{1}{70} e^{3 \xi^2 T} - \frac{1}{90} \right)
\]

and so on .....
How well is $\psi(\vartheta)$ approximated?

Here the analytic calculations are compared with those from the conditional integration for AUDJPY parameters. In addition, a lognormal approximation to $\psi(\vartheta)$ is given for comparative purposes:

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\sigma(0)$</th>
<th>$\xi$</th>
<th>$k$</th>
<th>Analytic $\sigma_k(T)$</th>
<th>Conditional Integration</th>
<th>Lognormal $\psi(\vartheta) \sim \psi_{ln}(\vartheta)$</th>
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<td>24.20%</td>
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<tr>
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<td>34.38%</td>
<td>34.38%</td>
<td>34.38%</td>
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<tr>
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<td>28.23%</td>
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<td>59.14%</td>
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<td>154.92%</td>
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<td>4</td>
<td>1314.30%</td>
<td>1239.95%</td>
<td>675.71%</td>
</tr>
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</table>

- Up to moderate maturities, the conditional integration captures the distribution very well.
- For the longest maturities the conditional integration is beginning to lose accuracy in the higher order moments. This can be corrected for by an intermediate integration step or relaxing the lognormal assumption.
SABR process vs approximation - 2YR AUDJPY

Figure 1: SABR Implied Volatility: T=2Y, σ(0)=20%, ξ=60%, ρ=-70%, β=1.0
SABR process vs approximation - 5YR high vol of vol

Figure 4: SABR Implied Volatility: T=5Y, \( \sigma(0)=10\% \), \( \xi=50\% \), \( \rho=0\% \), \( \beta=1.0 \)

2F PDE Integration SABR Approximation
Long dated integration - 30Y USDJPY ($\xi = 21.87\%; \rho = 46.75\%$)
Even for the longest maturity trades the integration scheme performs very well:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Implied Vol 2F PDE</th>
<th>Implied Vol Integration I</th>
<th>Implied Vol Integration II</th>
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<td>0.3</td>
<td>23.34%</td>
<td>23.41%</td>
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<td>0.5</td>
<td>20.89%</td>
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<td>18.26%</td>
<td>18.14%</td>
<td>18.25%</td>
</tr>
<tr>
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<td>17.53%</td>
<td>17.36%</td>
<td>17.51%</td>
</tr>
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<td>17.04%</td>
<td>16.85%</td>
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</tr>
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<td>16.54%</td>
<td>16.36%</td>
<td>16.43%</td>
</tr>
<tr>
<td>1.9</td>
<td>16.45%</td>
<td>16.29%</td>
<td>16.34%</td>
</tr>
</tbody>
</table>
SABR process ($\beta = 1$) sample calculation times

- Test machine:
  - Intel Xeon CPU 2.33GHz
  - RAM 3.25GB
- 50 integration steps per integral
- Runtime (standard integration scheme)
  - 0.00065s : 1 strike
  - 0.00196s : 10 strike
  - 0.01592s : 100 strike
- From this the volatility and option calculations can be split:
  - Initial volatility calculations: 0.00050s
  - Conditional vanilla calculations: 0.00015s per strike
- If necessary, this can be run many times faster by enhanced integration scheme.
The SABR model for $\beta \neq 1$
Integration under the terminal instantaneous volatility and conditional integrated variance distribution can be applied to other SABR variants

- $\beta = 0$ - replacement of the conditional Black-Scholes lognormal price with normal equivalent;

- $0 < \beta < 1$ and $\rho = 0$ - replacement of the conditional Black-Scholes lognormal price with CEV price (as originally pointed out in Stein and Stein [1991]);

- $0 < \beta < 1$ - replacement of conditional Black-Scholes lognormal price with modified CEV price accounting for non-zero correlation (Islah [2009]).
SABR process ($\beta < 1$) sample calculation times

In this case the conditional vanilla (CEV) valuation is significantly slower requiring better numerical integration techniques:

- **Test machine:**
  - Intel Xeon CPU 2.33GHz
  - RAM 3.25GB
- 50 integration steps per integral
- **Runtime (standard integration scheme)**
  - 0.08445s : 1 strike
  - 0.8593s : 10 strike
  - 8.6891s : 100 strike
- **Runtime (standard integration scheme)**
  - 0.0035s : 1 strike
  - 0.0355s : 10 strike
  - 0.3595s : 100 strike
Path dependency, quadrature, mixing models
Willard [1997] - extended interpretation

Although the focus of Willard [1997] concerns path independent valuation, does it also apply to path dependent options? Explicitly conditioning on the volatility path $\sigma(0, t)$ then the conditional asset process follows:

$$d \ln S(t) = \bar{\mu}(t) dt - \frac{1}{2} \bar{\sigma}(t)^2 dt + \bar{\sigma}(t) dW_S(t)$$

where, as before,

$$\bar{\sigma}(t) = \bar{\rho}\sigma(t)$$

$$\bar{\mu}(t) dt = \mu(t) dt - \frac{1}{2} \rho^2 \sigma(t)^2 dt + \rho \sigma(t) dW_\sigma(t)$$

So the conditional asset process follows that of a term structure lognormal asset. Path dependent contract valuation (excluding contracts with early exercise) can be evaluated under such a one-factor process using analytic, pseudo-analytic or finite difference (potentially state dependent) schemes.
Willard [1997] - extended interpretation - important observations

Most importantly

- *if lognormal term structure pricing is available, it can be reused for correlated stochastic volatility pricing*

Interpretation of stochastic volatility pricing

- stochastic volatility generates a universe of volatility paths;
- these in turn induce an implied volatility and forward curve term structure;
- an exotic price is an expectation over all such volatility and forward curve term structures.

In short, the correlated stochastic volatility price is an integral over Black-Scholes term structure prices.
**Conditional Monte Carlo - Gauss-Herminte Quadrature**

Given a forward curve $F(0, T)$ and implied volatility curve $\sigma(0, T)$ induced by the volatility path $\sigma(0, T)$ with associated contract value $V(F(0, T), \sigma(0, T), \rho)$ then the stochastic volatility price is given by

$$V = \int_{\sigma \in \Omega(0, T)} V(F(0, T), \sigma(0, T), \rho) d\Omega(0, T)$$

If $W_{\sigma}(t)$ is the Wiener process driving the volatility, then it is possible to condition further on the terminal value $W_{\sigma}(T)$ yielding

$$V = \int_{-\infty}^{+\infty} dW_{\sigma}(T) \psi(W_{\sigma}(T)) \int_{\sigma(0, T) \in \Omega(0, T) | W_{\sigma}(T)} V(F(0, T), \sigma(0, T), \rho) d\Omega(0, T)$$

where $\Omega(0, T) | W_{\sigma}(T)$ is the space of volatility paths conditional upon $W_{\sigma}(T)$ and $\psi(W_{\sigma}(T))$ is the density of $W_{\sigma}(T)$. This can then be calculated via a quadrature method of order $k$ say and expressed

$$V = \sum_{p=1}^{k} \omega_{p} \int_{\sigma(0, T) \in \Omega(0, T) | W_{\sigma}^{(p)}(T)} V(F(0, T), \sigma(0, T), \rho) d\Omega(0, T) = \sum_{p=1}^{k} \omega_{p} V_{p}$$
Quadrature - suggests mixing models, implies parallelism

- The expression of the option price as a weighted finite sum of conditional prices

\[
V = \sum_{p=1}^{k} \omega_p \int_{\sigma(0,T) \in \Omega(0,T)|W^{(p)}_\sigma(T)} V(F(0,T), \bar{\sigma}(0,T), \rho) d\Omega(0,T)
\]

is reminiscent of the discussion of mixing models in the papers of Brigo et al [2002, 2003], Johnson and Lee [2002] and Piterbarg [2003]. Can the discussion of exotics pricing in this context be put on a firmer footing?

- Parallelism: Given a conditioning \( W^{(p)}_\sigma(T) \) \((p = 1, ..., k)\) then each associated calculation is independent of the other

\[
V_p = \int_{\sigma(0,T) \in \Omega(0,T)|W^{(p)}_\sigma(T)} V(F(0,T), \bar{\sigma}(0,T), \rho) d\Omega(0,T)
\]

This is especially relevant as it effectively parallelizes the equivalent 2F PDE calculation. Within a multi-core of GPU environment the entire problem can be effectively split across the computational resource.
In Brigo, Mercurio, Rapisarda [2002] and Brigo, Mercurio, Sartorelli [2003] the implied volatility smile is constructed as a weighted sum of lognormals. Two cases are considered

- where the risk neutral drift is common to each lognormal producing a symmetric smile;
- where the drift is unique to each lognormal producing a skew.

No reference is given to Willard [1997] which, as previously, takes the form

\[ F(T,T) = F(0,T) \exp \left( -\frac{1}{2} \sigma^2 T + \rho \int_0^T \sigma(t) dW_\sigma(t) + \rho \int_0^T \sigma(t) dW_F(t) \right) \]

showing condition on the volatility path, the terminal asset distribution is lognormal. The sum of lognormal method is therefore a crude discretization, for both zero and non-zero correlation, of this more rigorous representation (in these papers a combination of only three lognormals is used in the symmetric case and two lognormals in the skewed case).
Mixture models and Piterbarg [2003]

- Following the proposal of generating the smile as a sum of lognormals, the possible extension to exotics pricing was explored in Johnson and Lee [2002].
- Piterbarg [2003] points out that although such an approach is valid for vanilla options, it is not valid (in general) for path dependent options.
- The contracts cited are a compound option (note we have so far explicitly excluded early exercise) and a barrier option.
- For the latter, a two state Black-Scholes model is constructed and the corresponding barrier option prices of the corresponding local volatility model are shown in general to diverge from the two state barrier prices.

The extension of the Brigo et al [2002, 2003] results to exotics is misspecified. Just as the Brigo et al [2002, 2003] result is a crude discretization of Willard [1997] for vanilla options, the correct extension to exotics should be a similar discretization of Willard [1997] for path dependent options i.e. as an integral over Black-Scholes term structure prices.
Mixing Models Reformulated

Correlated SV Process (vanilla pricing):

- Decompose as integral of lognormal densities conditional on volatility path;
- Mean (forward) associated with lognormal is function of correlation;
- Discrete approximation gives rise to smile ($\rho = 0$) and skew ($\rho \neq 0$);
- Brigo et al is one example of this.

Correlated SV Process (exotic pricing):

- Decompose contract value as integral over volatility paths;
- Each volatility path gives rise to a conditional asset process;
- Simple technique (analytic / analytic approx / numerical integration / finite difference) can be used to price contract under conditional asset process;
- Continuous barriers need special treatment;
- Exotic is an integral over conditional contract values;
- Discrete approximation should give rise to exotic price;
- Any combination of models gives rise to a valid exotics price.
Pricing algorithm for barrier exotics
Overview

Here the implementation of the scheme is outlined:

▶ Euler discretization of asset process in crude and conditional Monte Carlo (they are the same but need to be interpreted properly);
▶ Barrier probability;
▶ Conditional PDE and barrier correction;
▶ Numerical results for barrier options and one touch contracts;
▶ Multicore implementation.
Euler discretization and barrier calculation

Given a set of sampling times \( \{ T_1, T_2, \ldots, T_N \} \) then the log-volatility \( (Y) \) and log-asset \( (X) \) increments are given by

\[
Y_{k+1} = Y_k - \frac{1}{2} \xi^2 \Delta T + \xi \Delta W^{(1)}_k; \quad \sigma_k = e^{Y_k}
\]

\[
X_{k+1} = X_k + \mu \Delta T - \frac{1}{2} \sigma_k^2 \Delta T + \sigma_k (\rho \Delta W^{(1)}_k + \bar{\rho} \Delta W^{(2)}_k)
\]

For an upper barrier \( H \) then if \( X_k \geq \ln H \) or \( X_{k+1} > \ln H \) then the probability of survival \( \psi_{\text{survive}}(X_k, X_{k+1}, \ln H) = 0; \) otherwise

\[
\psi_{\text{survive}}(X_k, X_{k+1}, \ln H, \sigma_k) = 1 - \exp \left( -\frac{2}{\sigma_k^2} (\ln H - X_k)(\ln H - X_{k+1}) \right)
\]

For the entire path \( X = \{ X_k : 0 \leq k \leq N \} \)

\[
\psi_{\text{survive}}(X, \ln H) = \prod_{k=0}^{N-1} \psi_{\text{survive}}(X_k, X_{k+1}, \ln H, \sigma_k)
\]
Euler discretization and barrier calculation - conditional process

Given a set of sampling times \( \{ T_1, T_2, ..., T_N \} \) then as before, the conditional asset process is given by

\[
d \ln S = \bar{\mu} dt - \frac{1}{2} \bar{\sigma}^2 dt + \bar{\sigma} dW
\]

with Euler discretization

\[
X_{k+1} = X_k + \mu \Delta T - \frac{1}{2} \sigma_k^2 \Delta T + \sigma_k \Delta W_k
\]

This is the same as

\[
X_{k+1} = X_k + \mu \Delta T - \frac{1}{2} \sigma_k^2 \Delta T + \sigma_k \left( \rho \Delta W_k^{(1)} + \bar{\rho} \Delta W_k^{(2)} \right)
\]

\[
= X_k + \left( \mu \Delta T + \rho \sigma_k \Delta W_k^{(1)} - \frac{1}{2} \rho^2 \sigma_k^2 \Delta T \right) - \frac{1}{2} (\bar{\rho} \sigma_k)^2 \Delta T + (\bar{\rho} \sigma_k) \Delta W_k^{(2)}
\]

from before. However, taken out of context - i.e. applying the same barrier logic to the conditional Euler discretization of the conditional asset process would suggest

\[
\psi_{\text{survive}}^\text{conditional}(X_k, X_{k+1}, \ln H, \bar{\sigma}_k) = 1 - \exp \left( -\frac{2}{\bar{\sigma}_k^2} (\ln H - X_k)(\ln H - X_{k+1}) \right)
\]

This is wrong. The correct barrier volatility is still \( \sigma_k \) not \( \bar{\sigma}_k \).
Barrier handling in conditional PDE scheme

The conditional asset process

\[ dX = \mu dt - \frac{1}{2} \sigma^2 dt + \sigma dW \]

gives rise to the conditional Black-Scholes PDE

\[ \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial X^2} + \left( \mu - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial X} - rV = 0 \]

For upside one touch (pay at hit):

\[ V(-\infty, t) = 0; \quad V(\ln H, t) = 1; \quad V(X, T) = 0 \]

For an up and out call option

\[ V(-\infty, t) = 0; \quad V(\ln H, t) = 0; \quad V(X, T) = (e^X - K)^+ \]

This implicitly handles the barrier using the volatility \( \bar{\sigma} \) rather than \( \sigma \).
Barrier handling in conditional PDE scheme

PDE requires a correction in the vicinity of the barrier. For the barrier option, grid nodes close to the barrier should be replaced with a local integration

\[ V(X_k, T_p) = e^{-r_d \Delta T} \int_{-\infty}^{\ln H} \psi_{\text{survive}}(X_k, X, \ln H, \sigma_p) \psi(X|X_k) V(X, T_{p+1}) dX \]

where \(T_p\) and \(T_{p+1}\) correspond to consecutive timeslices, and

- \(\Delta T = T_{p+1} - T_p\)
- \(\psi(X|X_k)\) is the probability density at \(X\) (at \(T_{k+1}\)) conditional on \(X_k\) (at \(T_k\));
- \(\psi_{\text{survive}}(X_k, X, \ln H, \sigma_p)\) is the probability of survival as before but using \(\sigma\) \textbf{not} \(\bar{\sigma}\).
Full algorithm

The full algorithm combining integration (Gauss-Hermite quadrature); Monte Carlo (volatility path generation) and finite difference (1F conditional PDE with barrier correction) is

- Determine \( N_q \) quadrature points and weights;
- Generate \( M_q \) paths per quadrature point using brownian bridge
- Construct the resulting volatility path;
- For each volatility path, determine conditional asset process parameters;
- Evaluate contract on 1F PDE with barrier correction
- Average over \( M_q \) PDE values per quadrature point to give conditional expected value
- Calculate full expectation via quadrature weights
Up-and-out call option - 2-point quadrature

Spot 1.6500; \( r_d = 0\% \); \( r_f = 0\% \); \( \sigma_0 = 10\% \); \( \xi = 50\% \); \( \rho = -50\% \)

Contract 1 year 1.6500 call / knockout 1.8500

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Up-and-out call option - 4-point quadrature

Spot 1.6500; \( r_d = 0\% ; \) \( r_f = 0\% ; \) \( \sigma_0 = 10\% ; \) \( \xi = 50\% ; \) \( \rho = -50\% \)

Contract 1 year 1.6500 call / knockout 1.8500

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Up-and-out call option - 8-point quadrature

Spot 1.6500; \( r_d = 0\% \); \( r_f = 0\% \); \( \sigma_0 = 10\% \); \( \xi = 50\% \); \( \rho = -50\% \)

Contract 1 year 1.6500 call / knockout 1.8500

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Effectiveness of Conditional Integration \( r_d = 0\% \) and \( r_f = 0\% \)
Effectiveness of Conditional Integration $r_d = 1\%$ and $r_f = 5\%$
One Touch Pricing

Spot 1.6500; \( r_d = 0\% \); \( r_f = 0\% \); \( \sigma_0 = 10\% \); \( \xi = 50\% \); \( \rho = 0\% \)

Contract 1 year; range of upside one touches

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Notes:
- using 128 volatility paths per quadrature point;
- running on 8-core; 3.80Ghz; 8GB RAM; 64-bit OS;
- runtime < 0.5s per OT PV.
1 year upside one touches: $\xi = 50\%, \rho = 0\%$

Accepted tolerance $\pm 0.50\%$. 
1 year upside one touches: $\xi = 50\%, \rho = -50\%$

Accepted tolerance $\pm 0.50\%$. 
Beyond stochastic volatility and conclusions
Stochastic Rates

The same technique can be used to price exotics in a stochastic interest rate model:

\[ dX = (r_d - r_f - \frac{1}{2} \sigma^2)dt + \sigma_X (M_{X,f} dW_f + M_{X,d} dW_d + M_{X,X} dW_X) \]

\[ dr_d = (\theta_d - \chi_d r_d)dt + \sigma_{d} (M_{d,f} dW_d + M_{d,d} dW_f) \]

\[ dr_f = (\theta_f + \rho_{f,s} \sigma_f \sigma_s - \chi_f r_f)dt + \sigma_f dW_f \]

where

\[ M_{d,f} = \rho_{d,f}; \quad M_{d,d} = \sqrt{1 - \rho_{d,f}} \]

\[ M_{X,f} = \rho_{X,f}; \quad M_{X,d} = \frac{\rho_{X,d} - M_{X,f} M_{d,f}}{M_{d,d}} \]

\[ M_{X,X} = \sqrt{1 - M_{X,f}^2 - M_{X,d}^2} \]
Stochastic Rates - conditional asset process

As before, this induces a conditional log-asset process is

\[ d \ln X(t) | r_d(0, T), r_f(0, T) = \bar{\mu} dt - \frac{1}{2} \sigma^2 dt + \sigma dW_X \]

where

\[ \bar{\sigma} = \sigma M_{X,X} \]
\[ \bar{\mu} = (r_d - r_f - \frac{1}{2} \sigma^2) dt + \frac{1}{2} \sigma^2 M_{X,X} + \sigma M_{X,f} \frac{dW_X}{dt} + \sigma M_{X,d} \frac{dW_d}{dt} \]
\[ = (r_d - r_f - \frac{1}{2} M_{X,f}^2 - \frac{1}{2} M_{X,d}^2) dt + \sigma M_{X,f} \frac{dW_X}{dt} + \sigma M_{X,d} \frac{dW_d}{dt} \]

once again leading to a pricing PDE conditional on Monte Carlo generated \( r_d(0, T) \)
and \( r_f(0, T) \) paths as

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \frac{\partial^2 V}{\partial X^2} + \left( \bar{\mu} - \frac{1}{2} \bar{\sigma}^2 \right) \frac{\partial V}{\partial X} - rV = 0 \]

requiring continuous barrier adjustment to reflect \( \sigma(t) \) and not \( \bar{\sigma}(t) \).
Other applications

A number of other applications of the conditional Monte Carlo for path dependent options has been considered

- **Correlation Swaps** Combine Lipton and McGhee [1999] expansion for corr swap under Black-Scholes term structure to investigate correlated stochastic volatility effects. See McGhee [2011].

- **Cross FX Smile** Construction of cross FX smile under correlated stochastic volatility models using conditional Monte Carlo (see McGhee [2012]).

- **Cross Knockout Options** Effect of correlated local volatility and correlated stochastic volatility is explored on the pricing of cross knockout options in McGhee [2012]. Conditional Monte Carlo makes the correlated SV calculations feasible.

- **Stochastic Volatility of Volatility** McGhee and Trabalzini [2013] look at the effect of the correlated stochastic volatility of volatility on the mean integrated variance distribution and in particular the smile wings. Conditional Monte Carlo is used to construct the resulting smiles.

- **Stochastic Volatility with Stochastic Correlation** Models such as Fritz [2006] and Morganti [2013] can be simplified by cond MC simulation of the volatility/correlation paths leaving analytic/FD/etc for conditional asset calculation.
Conclusions

▶ Use of conditional integration techniques very successful for vanilla valuation under SABR model;
▶ Ideas presented here extend such an approach to pricing of exotics under stochastic volatility;
▶ Method leverages numerical integration (quadrature), Monte Carlo and finite difference techniques effective in exotics valuation;
▶ Continous barrier pricing requires special handling;
▶ Early exercise more difficult but can still be handled in this framework;
▶ Principle can be extended to making other parameters - for example interest rates - stochastic;
▶ Although focus here on SABR model, the general principle can be applied to a wide range of processes.

Forthcoming presentations

▶ 13-15th May 2014: ICBI Global Derivatives and Risk Management, Amsterdam;
Acknowledgements

I would like to thank, in particular, Katia Babbar, Han Lee and Romano Trabalzini as well as the many members of the RBS Quantitative Analytics team for their constructive comments during the preparation of this and associated presentations.
References


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Calculation Model (without backtesting)
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