PIVOTAL STRUCTURAL CHANGE TESTS IN LINEAR SIMULTANEOUS EQUATIONS WITH WEAK IDENTIFICATION

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STRUCTURAL AND REDUCED FORM EQUATIONS:

$$y_{i1} = \beta_1' Y_{i2} + u_i, \quad i \leq [nr],$$

$$y_{i2} = \beta_2' Y_{i2} + u_i, \quad i > [nr],$$

$i = 1, 2 \cdots, n$, n represents sample size.

$r \in \Xi$, Ξ is a set whose closure lies in (0,1).

$$Y_{i2} = x_i' \Pi + V_{i2},$$

Y_{i2}: $l \times 1$, x_i: $k \times 1$. Π: $k \times l$.

LITERATURE:

Bai (1997, 1998,1999), Bai and Perron (1998), Perron and Zhu (2005), Andrews (1993).

Caner (2007): Boundedly Pivotal Structural Change Tests in Weakly Identified GMM.

Moreira (2003), Kleibergen (2002): LM, LR type of tests in weak IV.

Stock and Wright (2000): Anderson-Rubin type of test in weak IV.

STRUCTURAL CHANGE TESTS:

$H_0 : \beta_1 = \beta_2.$

THE PROBLEMS IN WEAK IV CASE:

1. Tests in weak IV involve $H_0 : \beta_1 = \beta_{10}$,

2. The estimates of parameters are inconsistent, standard test statistics are not asymptotically pivotal.

3. There can be asymptotic bounds in certain tests in GMM, likelihood ratio test, Anderson-Rubin and Kleibergen like tests.

4. There may be waste of power.

Reparametrized Structural Equation:

$$y_{i1} = \delta' Y_{i2r} + \beta_2' Y_{i2} + u_i,$$

$\delta = \beta_1 - \beta_2$

$Y_{i2r} = Y_{i2} 1_{\{i \leq [nr]\}}$. $(l \times 1)$

REDUCED FORM SYSTEM:

$$y_1 = X_r \Pi \delta + X \Pi \beta_2 + v_{1r},$$

$$Y_{2r} = X_r \Pi + V_{2r},$$

X_r is formed by stacking vectors $x_i' 1_{\{i \leq [nr]\}}$
$v_{1r} = V_{2r}\delta + V_2 \beta_2 + u$.

NEW TEST:

$H_0 : \delta = 0,$

COMPACT REDUCED FORM SYSTEM:

$$Y_r = X_r \Pi_* + X[\Pi\beta_2, 0] + V_r,$$

$V_r = [v_{1r}, V_{2r}].$

$Y_r = [y_1, Y_{2r}]$

$\Pi_* = \Pi A'$, $A = [\delta, I_l]'.$

INVARIANT SUFFICIENT STATISTICS:

$$S_r = X_r^{*'} Y_r d_0 = X_r^{*'} y_1,$$

$$T_r = X_r^{*'} Y_r \Omega_r^{-1} A_0.$$

$X_r^* = M_X X_r$

$Y_r = [y_1, Y_{2r}]$

$A_0 = [0_l, I_l]'.$

$d_0 = [1, 0_l']'$

Every column of A_0 is orthogonal to d_0.

Assume at this point V_r is normal with mean zero with nonsingular variance covariance matrix:

$\Omega_r = [\omega_{ij,r}].$

Distributions:

$$S_r \sim N(X_r^{*'} X_r^* \Pi\delta, X_r^{*'} X_r^* \omega_{11r}),$$

$$vec(T_r) \sim N(vec(X_r^{*'} X_r^* \Pi A' \Omega_r^{-1} A_0, (A_0' \Omega_r^{-1} A_0) \otimes X_r^{*'} X_r^*)).$$

S_r and T_r are independent.

S_r has a null distribution not dependent on δ, Π.

T_r, under the null of $\delta = 0$, is sufficient statistic for Π.

TEST STATISTICS:

ANDERSON-RUBIN TYPE OF TEST: $H_0 : \delta = 0$

$$AR = \frac{S_r'(X_r^{*'} X_r^*)^{-1} S_r}{d_0' \hat{\Omega}_r d_0},$$

where

$$\hat{\Omega}_r = Y_r' M_{Z_r} Y_r / (n - k),$$

$$Z_r = [X, X_r],$$

$$M_{Z_r} = I - Z_r (Z_r' Z_r)^{-1} Z_r'.$$

Asymptotics: IID CASE

Theorem 1. Under the null hypothesis of no structural change and standard regularity assumptions

$$\sup_{r \in \Xi} AR \xrightarrow{d} \sup_{r \in \Xi} \frac{[W_k(r) - rW_k(1)]'[W_k(r) - rW_k(1)]}{r(1-r)},$$

where $W_k(r)$ is the k dimensional standard Brownian Motion. k represents number of instruments

Kleibergen type of test:

$$LM = \frac{S_r'\hat{\Pi}[\hat{\Pi}'X_r^{*'}X_r^*\hat{\Pi}]^{-1}\hat{\Pi}'S_r}{d_0'\hat{\Omega}_r d_0},$$

where

$$\hat{\Pi} = (X_r^{*'}X_r^*)^{-1}T_r(A_0'\hat{\Omega}_r^{-1}A_0)^{-1}.$$

Theorem 2. Under the null hypothesis and regularity assumptions, with weak instrument asymptotics $\Pi = C/\sqrt{n}$

$$\sup_{r\in\Xi} LM \xrightarrow{d} \sup_{r\in\Xi} \frac{[W_l(r) - rW_l(1)]'[W_l(r) - rW_l(1)]}{r(1-r)}.$$

l represents number of parameters.

Smoothing Idea, Time Series Case:

Let θ denote the parameters, z_i represents the data, $g(z_i, \theta)$ represents moments. Denote $g(z_i, \theta)$ as $g_i(\theta)$.

Guggenberger and Smith (2006), Otsu (2006), Smith (2004) propose the following:

$$g_{in}(\theta) = S_n^{-1} \sum_{j=i-n}^{i-1} k(j/S_n) g_{i-j}(\theta),$$

where S_n is a bandwidth parameter, $S_n \to \infty$. $k(.)$ is kernel, GS use truncated kernel.

We allow for triangular array of random variables that are near epoch dependent.

ANDERSON-RUBIN TEST:

$$AR = \frac{(y_1' X_r^*)^\omega \hat{\Omega}_{11,r}^{-1} (X_r^{*'} y_1)^\omega}{2n},$$

where

$$(X_r^{*'} y_1)^\omega = (X_r' y_1)^\omega - (X_r' X)^\omega [(X'X)^\omega]^{-1} (X' y_1)^\omega.$$

$$(X_r' y_1)^\omega = \sum_{i=1}^{n} S_n^{-1} \sum_{j=i-n}^{i-1} k(j/S_n) x_{i-j,r} y_{i-j,1}.$$

$$\hat{\Omega}_{11,r} = \frac{S_n}{n} \sum_{i=1}^{n} (x_{ir} y_{i1})^\omega (y_{i1} x_{ir}^{*'})^\omega.$$

Asymptotics:

Lemma 3. Under regularity assumptions and under the null, uniformly over r

$$\hat{\Omega}_{11,r} \xrightarrow{p} 2r(1-r)\Sigma_{xv1},$$

where $\Sigma_{xv1} = \lim_{n\to\infty} var[n^{-1/2}\sum_{i=1}^{n} x_i v_{i1}]$.

Extend GS (2006) from full sample case to partial sample. The result changes now we have extra multipliers $2r(1-r)$ in the limit compared to GS (2006).

Corollary 1. Under the null of no structural change and regularity assumptions

$$\sup_{r\in\Xi} AR \xrightarrow{d} \sup_{r\in\Xi} \frac{[W_k(r) - rW_k(1)]'[W_k(r) - rW_k(1)]}{r(1-r)}.$$

VARIANT OF ANDERSON-RUBIN TEST IN CANER (2007):

STEP 1: OBTAIN THE RESTRICTED PARTIAL SAMPLE LIML ESTIMATOR

$$\tilde{\beta}(r) = \arg\min_{r \in \Xi} [nr] A_n(\beta, r) + [n(1-r)] A_n(\beta, 1-r),$$

where

$$A_n(\beta, r) = \frac{[\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i]' [\sum_{i=1}^{[nr]} x_i x_i']^{-1} [\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i]}{[\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i]' [\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i] - [\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i]' [\sum_{i=1}^{[nr]} x_i x_i']^{-1} [\sum_{i=1}^{[nr]} (y_{i1} - \beta' Y_{i2}) X_i}$$

where $A_n(\beta, 1-r)$ is the other partial sample in the sum from $[nr]+1$ to n.

STEP 2: USE THIS IN TEST STATISTIC FOR $H_0 : \beta_1 = \beta_2$ (BOUNDEDLY PIVOTAL)

$$\sup_{r \in \Xi} S_n = \sup_{r} \{ [nr][1 + A_n(\tilde{\beta}(r), r)^{-1}]^{-1} + [n(1-r)][1 + A_n(\tilde{\beta}(r), 1-r)^{-1}]^{-1} \}.$$

ASYMPTOTICS:

$\sup S_n$ is asymptotically bounded by the following distribution

$$\sup_{r \in \Xi} \frac{[W_k(r) - rW_k(1)]'[W_k(r) - rW_k(1)]}{r(1-r)} + \chi_k^2.$$

Compared with Corollary 1, this may be conservative with number of instruments (k) getting large.

Kleibergen Type of Test:

$$LM = \bar{S}'_{hac,r} P_{\bar{T}_{hac,r}} \bar{S}_{hac,r},$$

where

$$\bar{S}_{hac,r} = 2^{-1/2} n^{-1/2} \hat{\Omega}_{11,r}^{-1/2} (X_r^{*\prime} y_1)^{\omega},$$

$$\bar{T}_{hac,r} = \hat{\Omega}_{11,r}^{-1/2} [n^{-1/2} \sum_{i=1}^{n} (x_{ir}^{*} Y'_{i2r})^{\omega} - \hat{\Omega}_{21,r} \hat{\Omega}_{11,r}^{-1} n^{-1/2} \sum_{i=1}^{n} (x_{ir}^{*} y_{i1})^{\omega}].$$

Corollary 2. Under the null and regular assumptions with weak instrument asymptotics ($\Pi = C/\sqrt{n}$)

$$\sup_r LM \xrightarrow{d} \sup_r \frac{[W_l(r) - rW_l(1)]'[W_l(r) - rW_l(1)]}{r(1-r)}.$$

COMPARISON:

1. The limit depends on number of parameters to be tested "l".

2. It can have better power than the other AR based test where the limit in Corollary 1 depends on number of instruments.

3. This is the same limit as in Andrews (1993) LM test. But that test only works when there is standard identification.

Boundedly Pivotal Kleibergen Type Test:

Step 1. Estimate

$$\tilde{\beta}_K(r) = \arg\min K_n(\beta, r),$$

Step 2. Use that in test statistic for $H_0 : \beta_1 = \beta_2$

$$\sup_r K_n(\tilde{\beta}_K(r), r).$$

$$\begin{aligned} K_n(\beta, r) &= [[nr]^{-1/2} \sum_{i=1}^{[nr]} \psi_i(\beta)]'(\hat{\Omega}_1)^{-1/2} P_{(\hat{\Omega}_1)^{-1/2} \tilde{D}_n^1(\beta, r)} (\hat{\Omega}_1)^{-1/2} [[nr]^{-1/2} \sum_{i=1}^{[nr]} \psi_i(\beta)] \\ &+ [[n(1-r)]^{-1/2} \sum_{i=[nr]+1}^{n} \psi_i(\beta)]'(\hat{\Omega}_2)^{-1/2} P_{(\hat{\Omega}_2)^{-1/2} \tilde{D}_n^2(\beta, r)} (\hat{\Omega}_2)^{-1/2} [[n(1-r)]^{-1/2} \sum_{i=[nr]+1}^{n} \psi_i(\beta)]. \end{aligned}$$

Terms:

$$\tilde{D}_n^1(\beta, r) = (n^{-1/2}\sum_{i=1}^{[nr]} x_i Y_{i2}') - \hat{\Omega}_{q1}(\hat{\Omega}_1)^{-1}(n^{-1/2}\sum_{i=1}^{[nr]} \psi_i(\beta)).$$

$$\psi_i(\beta) = (y_{i1} - \beta' Y_{i2})x_i,$$

$$\hat{\Omega}_{q1} = [nr]^{-1}\sum_{i=1}^{[nr]}(x_i Y_{i2}' - \bar{X}\bar{Y}_2)(\psi_i(\beta) - \bar{\psi})'.$$

$$\hat{\Omega}_1 = [nr]^{-1}\sum_{i=1}^{[nr]}(\psi_i - \bar{\psi})(\psi_i - \bar{\psi})'.$$

ASYMPTOTICS :

$\sup K_n(\beta, r)$ is asymptotically bounded by the following limit

$$\sup_r \frac{[W_l(r) - rW_l(1)]'[W_l(r) - rW_l(1)]}{r(1-r)} + \chi_2^l.$$

This clearly shows that the limit in Corollary 2 is more useful in small samples than the bound limit in Caner (2007).

Table 1: Size at 5% level

Tests	$\Pi = .1, k = 2$	$\Pi = 1, k = 2$	$\Pi = .1, k = 5$	$\Pi = 1, k = 5$
$\sup AR$	**4.0**	**3.0**	**7.3**	**7.1**
$\sup LM$	**6.0**	**3.4**	**18.9**	**5.9**
$\sup S_n$	**0.0**	**0.8**	**0.1**	**0.9**
$\sup K_n(\beta, r)$	**0.1**	**0.1**	**0.0**	**0.0**

Note: $n = 100$, *1000 iterations are used to generate the table.*

Table 2: Power, $\Pi = .1, k = 2$

Tests	$\delta = 5$	$\delta = 1$	$\delta = -1$	$\delta = -5$
$\sup AR$	**14.5**	**7.9**	**7.8**	**13.6**
$\sup LM$	**20.6**	**10.1**	**8.5**	**16.3**
$\sup S_n$	**0.7**	**0.0**	**0.0**	**0.5**
$\sup K_n(\beta, r)$	**0.7**	**0.3**	**0.1**	**0.5**

Note: $n = 100$, *1000 iterations are used to generate the table.*

$\delta = \beta_1 - \beta_2$.

Table 2: Power, $\Pi = .1, k = 5$

Tests	$\delta = 5$	$\delta = 1$	$\delta = -1$	$\delta = -5$
$\sup AR$	**24.7**	**10.2**	**12.5**	**22.8**
$\sup LM$	**49.8**	**19.3**	**13.2**	**37.3**
$\sup S_n$	**0.7**	**0.0**	**0.0**	**0.4**
$\sup K_n(\beta, r)$	**2.5**	**0.0**	**0.3**	**2.6**

Note: $n = 100$, *1000 iterations are used to generate the table.*

$\delta = \beta_1 - \beta_2$.