Estimation and inference in an integrated model with a trend

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1 Problem

Consider a simple linear regression model

$$X_t = \alpha + \beta t + u_t, \quad t = 1, 2, ..., n.$$

where errors $\{u_t\}$ may be
a) a stationary (short or long-memory process) process
b) non-stationary $I(d)$ process $d > 1/2$.

Problem: how to estimate β? Known results:

a) if $\{u_t\}$ is a long memory process, then LS estimator of β is consistent and asymptotically normal , Yajima (1988, 1991)
b) Weighted least squares estimator (polynomial regression) also can be used, Dahlhaus (1995)
c) general estimation method discussed by Robinson (2004)

The main objectives:

1. Feasible confidence intervals for α, β
2. Estimation of d, when $-1/2 < d < 3/2$
3. Application to GDP data

Related problem: estimation the location μ in the model

$$X_t = \mu + u_t$$

Properties of sample mean and confidence intervals for μ we discussed by Beran (1989), Hall, Jing and Lahiri (1998)

2 The class $I(d)$ process

Definition involves memory parameter d and generating process $\{\xi_t\}$, which is

a) a stationary sequence,
b) has the spectral density

$$f_\xi(\lambda) = c_\xi \, |\lambda|^{-2d_\xi} + o(|\lambda|^{-2d_\xi}), \text{ as } \lambda \to 0$$

with parameter $|d_\xi| < 1/2$.

We denote by $d_\xi \in (-0.5, 0.5)$ the memory parameter of a generating process.

Definition of I(d) process $\{u_t\}$ (with memory parameter d):

a) If $d \in (-0.5, 0.5)$, then

$$u_t = \xi_t$$

with

$$d_\xi = d.$$

b) $d = k + d_\xi > 0.5$, ($|d_\xi| < 0.5$) where $k$is integer, then

$$(1 - \mathrm{L})^k u_t = \xi_t.$$

c) If $d = k + d_\xi < -0.5$, ($|d_\xi| < 0.5$) where k is integer, then

$$u_t = (1 - \mathrm{L})^k \xi_t.$$

Assumption A (On generating process):

1) $\{\xi_t\}$ is a linear sequence

$$\xi_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j},$$

$$\sum_{j=0}^{\infty} a_j^2 < \infty, \qquad \{\varepsilon_j\} \text{ i.i.d.} \qquad E\varepsilon_j = 0, \qquad E\varepsilon_j^4 < \infty.$$

b) $\{\xi_t\}$ has spectral density

$$f_\xi(\lambda) = |\lambda|^{-2d_\xi}(c_\xi + b_\xi \lambda^2 + o(\lambda^2)), \text{ as } \lambda \to 0,$$

for some $d_\xi \in (-1/2, 1/2)$, $c_\xi > 0$

3 Estimation of d

How to estimate the memory parameter d_0 of a sequence

$$X_t = \alpha + u_t$$

where (u_t) is an $I(d_0)$ process, and α unknown mean?

Assumption B. $d_0 \in (-1.5, 2.5)$ and $d_0 \neq -0.5, 0.5, 1.5$.

Extended Local Whitle estimator

$$\hat{d}_X := \operatorname{argmin}_{d \in [-1.5, 2.5]} U_n(d),$$

minimizes the modified local Whittle objective function

$$U_n(d) := \log\left(\frac{1}{m}\sum_{j=1}^{m} j^{2d}\ I_n(\lambda_j; d)\right) - \frac{2d}{m}\sum_{j=1}^{m} \log j,$$

$$\text{bandwidth } m: \qquad m \to \infty,\ m = o(n).$$

It involves

<u>Modified periodogram</u>

$$I_n(\lambda_j; d) := |w_X(\lambda_j) + k_s(\lambda_j)|^2, \qquad d \in [s-0.5, s+0.5),\ s = -1, 0, 1, 2,$$

$$\lambda_j = 2\pi j/n,\ j = 1, \ldots, n \text{ Fourier frequencies},$$

$$w_X(\lambda_j) := (2\pi n)^{-1/2} \sum_{t=1}^{n} \mathrm{e}^{\mathrm{i}t\lambda_j} X_t, \qquad \text{discrete Fourier transform}$$

<u>correction terms</u>

$$k_s(\lambda_j) := \begin{cases} 0, & \text{if } s = 0, \\ (2\pi n)^{-1/2} \mathrm{e}^{\mathrm{i}\lambda_j} (1 - \mathrm{e}^{\mathrm{i}\lambda_j})^{-1} (X_n - X_0), & \text{if } s = 1, \\ (2\pi n)^{-1/2} \mathrm{e}^{\mathrm{i}\lambda_j} \left(\frac{X_n - X_0}{1 - \mathrm{e}^{\mathrm{i}\lambda_j}} + \frac{\nabla X_n - \nabla X_0}{\left(1 - \mathrm{e}^{\mathrm{i}\lambda_j}\right)^2} \right), & \text{if } s = 2, \end{cases}$$

take constant values on the intervals $[s - 0.5, s + 0.5)$

<u>Note</u>: Estimations uses observations $X_{-1}, X_0, ..., X_n$.

Properties: as $n \to \infty$,

$$2\sqrt{m}(\hat{d}_X - d_0) \xrightarrow{d} \mathrm{N}(0,1), \quad \text{if } m = o(n^{4/5}),$$

and

$$(m/n)^2(\hat{d}_X - d_0) \xrightarrow{p} 2b_\xi/c_\xi \neq 0, \quad \text{if } n^{4/5}/m = o(1).$$

4 Estimation of parameters d, α and β

Consider a model

$$X_t = \alpha + \beta t + u_t$$

where (u_t) is $I(d_0)$ process

Standard LS estimators can be used:

$$\hat{\beta} = \frac{\sum_{t=1}^n (X_t - \bar{X})(t - \bar{t})}{\sum_{t=1}^n (t - \bar{t})^2}$$

and

$$\hat{\alpha} = \bar{X} - \hat{\beta}\bar{t},$$

where

$$\bar{X} = n^{-1}\sum_{t=1}^n X_t, \qquad \bar{t} = n^{-1}\sum_{t=1}^n t = (n+1)/2$$

Assumption A.3. $\{u_t\} \sim \mathrm{I}(d_0)$, with $d_0 \in (-0.5, 1.5)$, $d_0 \neq 0.5$.

Assumption A.3 covers most empirical applications.

Estimation of d_0 in a model with a linear trend

Apply the modified local Whittle estimator

$$\hat{d}_{\hat{u}} = \text{argmin}_{[-0.5,1.5]}\, U_n(d),$$

to residuals $\hat{u}_t = X_t - \hat{\beta}t$.

Then,

$$2\sqrt{m}(\hat{d} - d_0) \xrightarrow{d} \mathrm{N}(0,1), \quad \text{if } m = o(n^{4/5}),$$

and

$$(m/n)^2(\hat{d} - d_0) \xrightarrow{p} const, \quad \text{if } n^{4/5}/m = o(1).$$

4.1 Asymptotic distribution of LS estimator

Asymptotic distribution and rate of LS estimators $\hat{\alpha}$ and $\hat{\beta}$ depend on

a) memory parameter d_0
b) the long-run variance

$$s_\xi^2 = \lim_n \mathrm{E}\left(n^{-1/2-d_\xi}\sum_{t=1}^n \xi_t\right)^2$$

Estimation of long-run variance. s_ξ^2 can be estimated by consistently by the estimator

$$\hat{s}_q^2(d)$$

where

$$\hat{s}_q^2(d) := \begin{cases} p(d)\hat{c}_q(d), & \text{if } d \in [-0.5, 0.5), \\ p(d-1)\hat{c}_q(d), & \text{if } d \in [0.5, 1.5], \end{cases}$$

where $p(d)$ known function of d,

$$p(d) := \begin{cases} 2\frac{\Gamma(1-2d)\sin(\pi d)}{d(1+2d)}, & \text{if } d \neq 0, \\ 2\pi, & \text{if } d = 0. \end{cases}$$

$$\hat{c}_q(d) := \begin{cases} q^{-1}\sum_{j=1}^q \lambda_j^{2d}\, I_{n,\hat{u}}(\lambda_j), & \text{if } d \in [-1/2, 1/2), \\ q^{-1}\sum_{j=1}^q \lambda_j^{2(d-1)}\, I_{n,\nabla X}(\lambda_j), & \text{if } d \in [1/2, 3/2], \end{cases}$$

$$I_{n,\xi}(\lambda_j) := (2\pi)^{-1}\left|\sum_{t=1}^n \mathrm{e}^{\mathrm{i}t\lambda_j}\xi_t\right|^2, \quad \lambda_j = 2\pi j/n$$

Consistency: $\hat{s}_q^2(d)$ performs well with the bandwidth $n^{0.5} \leq q \leq n^{0.7}$:

$$\hat{s}_q^2(d) \to s_\xi^2.$$

Theorem 4.1. *Suppose that $d_0 \in (-0.5, 1.5)$, and Assumptions A and A.3 are satisfied. Then, as $n \to \infty$,*

$$\frac{n^{3/2-\hat{d}}}{\hat{s}_q(\hat{d})\sigma_\beta(\hat{d})}(\hat{\beta}-\beta) \xrightarrow{d} \mathrm{N}(0,1),$$

where

$$\sigma_\beta(d) := \begin{cases} 12\left(\frac{1}{2d+3}-\frac{1}{4}\right)^{1/2}, & \text{if } d \in [-0.5, 0.5), \\ 12\left(\frac{1}{4}\frac{2d-1}{2d(2d+1)(2d+3)}\right)^{1/2}, & \text{if } d \in [0.5, 1.5], \end{cases}$$

If $d_0 \in (-0.5, 0.5)$, then

$$\frac{n^{1/2-\hat{d}}}{\hat{s}_q(\hat{d})\sigma_\alpha(\hat{d})}(\hat{\alpha}-\alpha) \xrightarrow{d} \mathrm{N}(0,1),$$

where

$$\sigma_\alpha(d) := \left(1 + 36\left(\frac{1}{2d+3}-\frac{1}{4}\right)\right)^{1/2}.$$

<u>Asymptotic confidence interval</u> of size $1-\gamma$ for β:

$$\left[\hat{\beta} - \frac{\sigma_\beta(\hat{d})\hat{s}_q(\hat{d})}{n^{3/2-\hat{d}}}c_\gamma,\ \hat{\beta} + \frac{\sigma_\beta(\hat{d})\hat{s}_q(\hat{d})}{n^{3/2-\hat{d}}}c_\gamma\right],$$

c_γ is the upper quantile: $\Pr(|\mathrm{N}(0,1)| > c_\gamma) = \gamma$.

<u>**Remark**</u>. Results on estimation of β hold also for non-linear processes under very weak assumptions.

5 Simulation results

5.1 Linear error processes

I. Estimation of d_0

Generating process $\{\xi_t\}$: fractional ARIMA$(0, d_\xi, 0)$ noise

Estimation of $d \in (-0.5, 0.5)$ of model $I(d)$.

Sample size $n = 500$, $m = \lfloor n^{.65} \rfloor = 56$ and $q = \lfloor n^{.7} \rfloor = 77$.

10,000 replications

Table 1: Bias and mean squared error of the modified Whittle estimators $\hat{d}_X$ and $\hat{d}_{\hat{u}}$.

	Bias	MSE	Bias	MSE
	$\hat{d}_X$		$\hat{d}_{\hat{u}}$	
$d = -1.3$	0.00709	0.00610		
$d = -1.0$	-0.00116	0.00621		
$d = -0.7$	-0.00099	0.00601		
$d = -0.3$	-0.00516	0.00650	-0.00842	0.00670
$d = 0$	-0.00590	0.00597	-0.01019	0.00702
$d = 0.3$	-0.00517	0.00613	-0.01488	0.00703
$d = 0.7$	-0.00464	0.00639	-0.01350	0.00703
$d = 1.0$	-0.00590	0.00616	-0.01274	0.00614
$d = 1.3$	-0.00513	0.00625	-0.00865	0.00609

Findings: Model

$$X_t = \alpha + \beta t + u_t, \qquad u_t \sim I(d)$$

d_ξ is estimated very accurately over the range considered.

MSE is fairly constant across different values of d_X,

Estimation procedure does uniformly well over different d

II. Estimation of β:

Findings: $\hat{\beta}$ when $d_u \in (-0.5, 1.5)$, convergence rates is functions of d_u

Precision of is very high for low values of d_u, decreases as $d_u \to 1.5$.

Coverage probabilities indicate very little size distortion

Table 2: Bias, 95% confidence intervals and coverage probabilities of the LS estimator $\hat{\beta}$.

	Bias[a]	95% C.I.	C.P	Bias	95% C.I.	C.P.
		$d_u = -.4$			$d_u = .6$	
$\beta = -0.3$	0	[-0.30014, -0.29986]	0.97	-0.00010	[-0.31012, -0.29008]	0.93
$\beta = 0$	0	[-0.00014, 0.00014]	0.97	-0.00002	[-0.01004, 0.01000]	0.92
$\beta = 0.3$	0	[0.29986, 0.30014]	0.97	0.00005	[0.29003, 0.31006]	0.93
		$d_u = -.2$			$d_u = .8$	
$\beta = -0.3$	0	[-0.30026, -0.29974]	0.95	-0.00026	[-0.33183, -0.26869]	0.95
$\beta = 0$	0	[-0.00026, 0.00026]	0.95	0.00043	[-0.03114, 0.03199]	0.95
$\beta = 0.3$	0	[0.29973, 0.30026]	0.95	-0.00009	[0.26834, 0.33147]	0.95
		$d_u = 0$			$d_u = 1$	
$\beta = -0.3$	0	[-0.30061, -0.29939]	0.95	-0.00043	[-0.39645, -0.20441]	0.95
$\beta = 0$	0	[-0.00061, 0.00061]	0.95	-0.00022	[-0.09624, 0.09580]	0.95
$\beta = 0.3$	0	[0.29940, 0.30061]	0.95	0.00031	[0.20429, 0.39633]	0.95
		$d_u = .2$			$d_u = 1.2$	
$\beta = -0.3$	-0.00002	[-0.30155, -0.29848]	0.95	-0.00312	[-0.62176, 0.02802]	0.95
$\beta = 0$	0.00001	[-0.00152, 0.00154]	0.95	0.00219	[-0.32270, 0.32708]	0.96
$\beta = 0.3$	0.00001	[0.29848, 0.30154]	0.95	0.00072	[-0.02417, 0.62561]	0.95
		$d_u = .4$			$d_u = 1.4$	
$\beta = -0.3$	0	[-0.30402, -0.29598]	0.95	-0.00741	[-1.73378, 1.11896]	0.93
$\beta = 0$	0	[-0.00402, -0.00402]	0.95	-0.01169	[-1.43806, 1.41468]	0.94
$\beta = 0.3$	-0.00001	[0.29596, 0.30400]	0.95	0.00871	[-1.11765, 1.73509]	0.94

[a] Whenever the obtained bias is smaller than 10^{-5} the value 0 is reported.

III. Estimation of α:

Table 4. Bias, 95% CIs and CPs of $\hat{\alpha}$ when $\alpha = 0$.

	Bias	95% C.I.	C.P.
$d_u = -.4$	-0.00019	[-0.03784, 0.03746]	0.97
$d_u = -.2$	0.00042	[-0.07142, 0.07226]	0.95
$d_u = 0$	0.00030	[-0.17501, 0.17561]	0.95
$d_u = .2$	0.00239	[-0.48634, 0.49111]	0.95
$d_u = .4$	0.00521	[-1.70901, 1.71944]	0.94

Findings:

1. CIs are wider than for the sample mean, because of the extra scaling term $\sigma_\alpha(d_u)$.

2. CPs are very close to the nominal ones, with some small differences close to the boundaries of d_u.

6 Nonlinear error processes

Data generated by the model

$$y_t = \alpha + \beta t + u_t, \quad t = 1, \ldots, n \qquad (EXP)$$

and

$$y_t = \alpha + \beta t + v_t, \quad t = 0, \ldots, n \qquad (UR)$$

where

$$a) \qquad u_t = \exp(\xi_t) - \mathrm{E}\left(\exp(\xi_t)\right) \qquad\qquad b) \quad v_t = \sum_{j=0}^{t} u_t.$$

The partial sums of the error term converge to a Gaussian process, so we can use normal limiting quantiles for $\hat{\beta}$.

Table 3: Bias, 95% confidence intervals and coverage probabilities of the LS estimator $\hat{\beta}$ for models (EXP) and (UR).

	Bias	95% C.I.	C.P	Bias	95% C.I.	C.P.
		$d_\xi = -.4, \quad d_u = 0$			$d_\xi = -.4, \quad d_v = 1$	
$\beta = -0.3$	-0.00004	[-0.30113, -0.29894]	0.93	-0.00204	[-0.45858, -0.14550]	0.93
$\beta = 0$	-0.00002	[-0.00111, 0.00109]	0.92	-0.00248	[-0.15902, 0.15406]	0.92
$\beta = 0.3$	0.00001	[0.29891, 0.30110]	0.92	0.00102	[0.14448, 0.45755]	0.92
		$d_\xi = -.2, \quad d_u = 0$			$d_\xi = -.2, \quad d_v = 1$	
$\beta = -0.3$	-0.00002	[-0.30098, -0.29905]	0.93	-0.00168	[-0.43915, -0.16421]	0.93
$\beta = 0$	0.00001	[-0.00096, 0.00097]	0.92	0.00073	[-0.13673, 0.13820]	0.92
$\beta = 0.3$	-0.00001	[0.29902, 0.30095]	0.93	-0.00195	[0.16059, 0.43551]	0.93
		$d_\xi = 0, \quad d_u = 0$			$d_\xi = 0, \quad d_v = 1$	
$\beta = -0.3$	-0.00002	[-0.30133, -0.29871]	0.95	-0.00296	[-0.51001, -0.09591]	0.95
$\beta = 0$	0.00001	[-0.00131, 0.00131]	0.95	-0.00023	[-0.20729, 0.20683]	0.95
$\beta = 0.3$	-0.00002	[0.29866, 0.30128]	0.95	-0.00098	[0.09196, 0.50607]	0.95
		$d_\xi = .2, \quad d_u = .2$			$d_\xi = .2, \quad d_v = 1.2$	
$\beta = -0.3$	0.00001	[-0.30274, -0.29725]	0.94	0.01004	[-0.81763, 0.23771]	0.93
$\beta = 0$	0.00006	[-0.00281, 0.00268]	0.94	0.00910	[-0.51856, 0.53677]	0.93
$\beta = 0.3$	0.00006	[0.29726, 0.30275]	0.94	0.01375	[-0.21391, 0.84142]	0.93
		$d_\xi = .4, \quad d_u = .4$			$d_\xi = .4, \quad d_v = 1.4$	
$\beta = -0.3$	0.00017	[-0.31610, -0.28357]	0.95	-0.02983	[-4.59255, 3.93289]	0.94
$\beta = 0$	0.00021	[-0.01606, 0.01647]	0.95	-0.03443	[-4.29715, 4.22829]	0.95
$\beta = 0.3$	0.00004	[0.28377, 0.31630]	0.95	0.02940	[-3.93332, 4.59212]	0.94

Findings:

Bias of $\hat{\beta}$ and the confidence intervals for β are larger than in linear case.

Empirical coverage probabilities close to the nominal ones

7 Empirical application: the case of US quarterly GDP

Quarterly series of US GDP values in billions of dollars.

from 1947 to 2003, $n = 228$ observations.

Series transformed taking logarithms

The following model fitted:

$$y_t = \alpha + \beta t + u_t, \quad t = 1, \ldots, 228,$$

u_t is $I(d)$ process with $d \in (-0.5, 1.5)$.

Bandwidth parameters:

Bandwidth for d: $m = \lfloor n^{.65} \rfloor = 34$,
For q, we used $q = \lfloor n^{.7} \rfloor = 77$

The resulting estimator is

$$\underline{\hat{d} = 0.775}.$$

The corresponding 95% confidence interval is

$$d \in [0.607, 0.943].$$

The fitted model is

$$\hat{y}_t = 7.4046 + 0.0084t + \ u_t,$$

95% confidence intervals for β is

$$[0.0061, 0.0106]$$

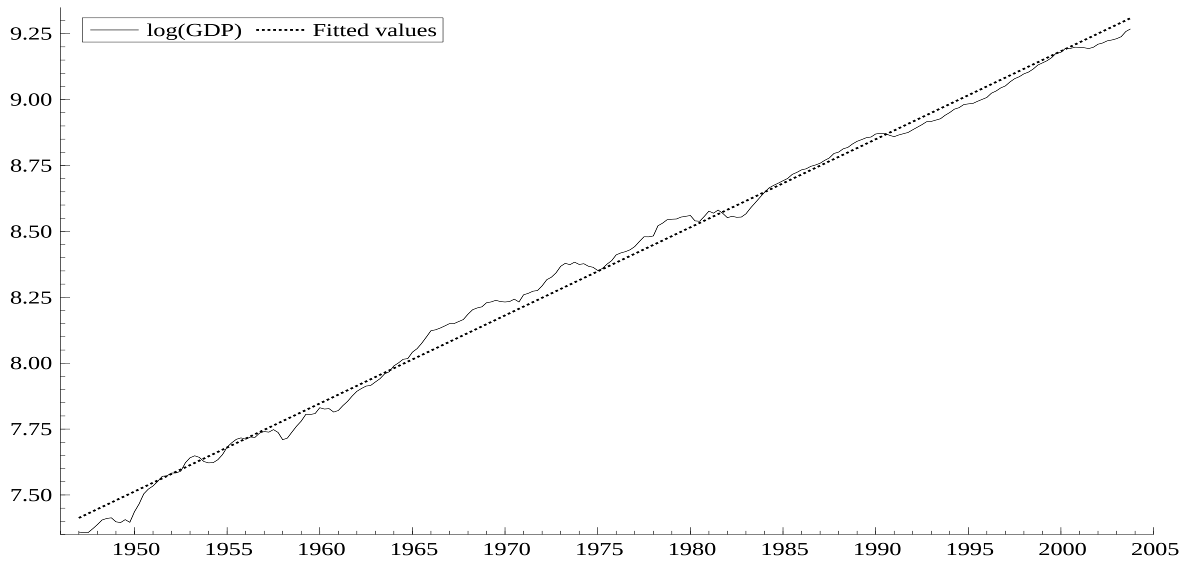

Figure 1: Fitted linear trend and actual values of log(GDP).

Conclusion:

- The series is nonstationary around a linear trend,

  Shocks u_t have an effect that dies out

- Results reject hypothesis of driftless unit-root nonstationarity.

- Graphical analysis confirms statistical analysis, indicating a pattern of business cycles around a linear trend.

8 LS estimation for general processes

Assume that

$$X_t = \alpha + \beta t + u_t$$

where (u_t) is $I(d_0)$ process, $d_0 \in (-0.5, 1.5)$.

Question? Can we relax assumption of linearity on generating process ξ_t.

Answer: It can be replaced by much weaker assumption on partial sums:

Assumption B. Finite-dimensional distributions of the process

$$Y_n(s) = n^{-1/2-d_\xi} \sum_{t=1}^{\lfloor ns \rfloor + 1} \xi_t, \quad 0 \le s \le 1$$

converge to those of some process $(Y_\infty(r))$:

$$Y_n(r) \xrightarrow{d} Y_\infty(r), \ \ as\ n \to \infty$$

Note:

$$\mathrm{E}\left(Y_\infty^2(1)\right) = s_\xi^2.$$

Write

$$Y_\infty(r) = s_\xi J_{1/2+d_\xi}(r).$$

Then

$$\mathrm{E}\left(J_{1/2+d_\xi}(r) J_{1/2+d_\xi}(s)\right) = \frac{1}{2}(r^{1+2d_\xi} + s^{1+2d_\xi} - |r-s|^{1+2d_\xi}).$$

and

$$\mathrm{E}\left(Y_n(r) Y_n(s)\right) \to \mathrm{E}\left(Y_\infty(r) Y_\infty(s)\right) = s_\xi^2 \mathrm{E}\left(J_{1/2+d_\xi}(r) J_{1/2+d_\xi}(s)\right),$$

Note: The limit $Y_\infty(r)$ can be Gaussian and non-Gaussian.

If $Y_\infty(r)$ is Gaussian, then $J_{1/2+d_\xi}(r)$ is a fractional Brownian motion.

(Non-Gaussian) limit distributions. Set

$$Z_\beta(d_0) := \begin{cases} \int_0^1 (J_{1/2+d_\xi}(r) - rJ_{1/2+d_\xi}(1))\mathrm{d}r, & \text{if } d_0 = d_\xi \in (-0.5, .0, 5), \\ \int_0^1 \int_0^r \left(J_{1/2+d_\xi}(s) - \int_0^1 J_{1/2+d_\xi}(u)\mathrm{d}u\right) \mathrm{d}s\mathrm{d}r, & \text{if } d_0 = 1 + d_\xi \in (0.5, 1.5), \end{cases}$$

$$Z_\alpha(d_\xi) := 6\int_0^1 (J_{1/2+d_\xi}(r) - rJ_{1/2+d_\xi}(1))\mathrm{d}r + J_{1/2+d_\xi}(1),$$

Define

$$Z_\beta(0,1) := \frac{Z_\beta(d)}{(Var(Z_\beta(d))^{1/2}}, \quad Z_\alpha(0,1) := \frac{Z_\alpha(d)}{(Var(Z_\alpha(d))^{1/2}}.$$

Remark 1. Note that

$$EZ_\beta(0,1) = EZ_\alpha(0,1) = 0$$
$$EZ_\beta^2(0,1) = EZ_\alpha^2(0,1) = 1$$

2. When the limit $Y_\infty(u)$ is Gaussian, then

$$Z_\beta(0,1) = N(0,1), \quad Z_\alpha(0,1) = N(0,1)$$

are standard normal.

THEOREM 8.1. *Assume that*

$$X_t = \alpha + \beta t + u_t, \quad t = 1, ..., n$$

$\{u_t\}$ *is an* $\mathrm{I}(d_0)$ *process with* $d_0 \in (-0.5, 0.5)$.

Then, uniformly in $n \geq 1$,

$$\mathrm{E}\left[(\hat\beta - \beta)^2\right] \leq Cn^{-3+2d_0}, \quad \mathrm{E}\left[(\hat\alpha - \alpha)^2\right] \leq Cn^{-1+2d_0}.$$

In addition, if $\{\xi_t\}$ *satisfies B, then*

$$\frac{n^{3/2-d_0}}{s_\xi \sigma_\beta(d_0)}(\hat{\beta} - \beta) \xrightarrow{d} -Z_\beta(0,1),\ \textit{if}\ d_0 \in (-0.5, 1.5)$$

and

$$\frac{n^{1/2-d_0}}{s_\xi \sigma_\alpha(d_0)}(\hat{\alpha} - \alpha) \xrightarrow{d} Z_\alpha(0,1),\ \textit{if}\ d_0 \in (-0.5, 0.5).$$

Application. To apply theorem, we need consistent estimates for parameters

$$d_0, \quad s_\xi.$$

1. Fully extended local whittle estimator has property

$$\hat{d} - d_0 = o(1 \log n)$$

under very general assumptions, Abadir, Distaso and Giraitis (2005).

2. Newey-West type estimator and P Robinson (2004) estimator for long-run variance are consistent under very weak assumptions, Abadir, Distaso and Giraitis (2005).

PROPOSITION 8.1. *Assume that*

$$\hat{d} - d_0 = o_p(1/\log n)$$

and

$$\hat{s}_q^2(\hat{d}) \xrightarrow{p} s_\xi^2.$$

and assumptions A, B hold. Then

$$\frac{n^{3/2-\hat{d}}}{\hat{s}_q(\hat{d})\sigma_\beta(\hat{d})}(\hat{\beta} - \beta) \xrightarrow{d} -Z_\beta(0,1), \ \textit{if } d_0 \in (-1/2, 3/2)$$

and

$$\frac{n^{1/2-\hat{d}}}{\hat{s}_q(\hat{d})\sigma_\alpha(\hat{d})}(\hat{\alpha} - \alpha) \xrightarrow{d} Z_\alpha(0,1), \ \textit{if } d_0 \in (-1/2, 1/2).$$

Remark:

1. Results apply for a wide class of generating processes $\{\xi_t\}$. It includes:

(a) linear sequences $\{\xi_t\} \sim \mathrm{I}(d_\xi)$,

(b) EGARCH processes $\{\xi_t\}$,

(c) nonlinear sequences $\xi_t := h(\zeta_t)$, where $\zeta_t \sim \mathrm{I}(d_\zeta)$ is Gaussian, with $d_\zeta \in [0, 1/2)$.