Estimation and Inference in Unstable Nonlinear Least Squares Models

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Economic Question

- How do changes in policy regimes or economic environments affect economic decisions at a macroeconomic level?

Answer in Econometric Literature:

- linear or nonlinear models;
- stationary and nonstationary models;
- known or unknown break;
- single break or multiple breaks;
- singe estimating equation or system of equations;
- estimation of breaks or tests for structural change.

Literature on Stability

Tests in linear models

- stationary: Bai and Perron (1998);
  (sup) F-tests - initially developed by Quandt (1960).
- time series models: Zivot and Andrews (1992), Banerjee, Lumsdaine and Stock (1992), Lumsdaine and Papell (1997), Andreou and Ghysels (2003);
  (inf) t-tests.
- multivariate regressions: Perron and Qu (2006).

Literature for Stability

Tests in nonlinear models

- single known / unknown breakpoint: Andrews (1993), Andrews and Ploberger (1994), Sowell (1996);

  (sup, average, exponential) Wald, LM LR-type tests.

Break-point estimation in linear models

- level shifts in mean: Yao and Au (1988);
- general classes of linear models Bai and Perron (1998), Hall and Han (2005), Perron and Qu (2006).

Our Contribution

- How do changes in policy regimes or economic environments affect economic decisions at a macroeconomic level?

Answer:

- **nonlinear models** that can be estimated via nonlinear least squares (NLS)
- allow for **multiple parameter changes**
- assume changes occur at **unknown dates**

- provide an **estimation method for change points and parameters**, derive their asymptotic distributions
- propose **several stability tests**.

Motivation

- Consider the nonlinear model: $Y = f(X, \theta) + U$;
- **Example 1:** in representative agent models, Y could be the consumption growth, X might include income growth and interest rates, and θ could include a tax parameter that changes over time;
- **Example 2:** in partial adjustment models such as inventory models, Y could be the current change in inventories, X might include the gap between desired and actual past inventories, and θ could include the accelerator parameter that might be unstable over time.

Outline

- 1. Model
- 2. Assumptions
- 3. Asymptotics
- 4. Stability Tests
- 5. Simulations
- 6. Conclusions

- $y_t = f(x_t,\ \theta^0_{i+1}) + u_t$ $\qquad t = [T^0_i$ +1, $T^0_{i+1}]$ $\qquad$ i = 0, 1, . . . , m.

  T^0_i are unknown, m = known
  θ^0_i is a $p \times 1$ vector
  $E\left[u_t|x_t\right] = 0$, u_t i.i.d.

Estimation of break dates and parameters:

- as in Bai and Perron (1998), minimize sum of squared residuals
- over all possible partitions of the [1, T] interval
- and over the parameters defined for each partition.

Estimators:

- $$(\hat{T}_1, \ldots, \hat{T}_m) = \underset{(T_1,\ldots,T_m)}{\operatorname{argmin}} \ \underset{(\theta_1,\ldots,\theta_{m+1})}{\operatorname{argmin}} T^{-1} \sum_{i=0}^{m} \sum_{t=T_i+1}^{T_{i+1}} [y_t - f(x_t, \theta_{i+1})]^2$$

  where $(T_1, \ldots, T_m)$ are possible m-partitions of the [1,T] interval.

- $\hat{\theta}_i = \hat{\theta}_i(\hat{T}_1, \ldots, \hat{T}_m)$

  where $\hat{\theta}_i(T_1, \ldots, T_m)$ are NLS estimators for a given partition.

- **A1: Break Fractions**
  $T_i^0 = [T\lambda_i^0]$, *where* $0 < \lambda_1^0 < \cdots < \lambda_m^0 < 1$.

- **A2: Parameter Space**
  Θ *is a compact and convex subset of* $\mathbb{R}^p$.

- **A3: Underlying Memory of Processes**
  Let $f_t(\theta) = f(x_t, \theta)$ *and* $\psi_t(\theta) = u_t f_t(\theta)$.
- *(i) Assume* $f_t(\theta)$ and $\psi_t(\theta)$ *are strictly stationary processes,* β*-mixing, where the* β*-mixing coefficients,* β_1 for $f_t(\theta)$ and β_2 for $\psi_t(\theta)$, *satisfy* $\beta_i(s) \leq D_i s_i^{-A}$, *with* $D_i > 0$ *and also* $A_i > 2 + 4\xi_i$, *for some* $\xi_i > 0$, $i = 1, 2$;
- *(ii)* $\sup_\theta E|f_t(\theta)|^{2+\delta_1} < \infty$, $\sup_\theta E|\psi_t(\theta)|^{2+\delta_2} < \infty$ *for some* $\delta_i > 0$, $i = 1, 2$.

- **A4: Smoothness**
  Let $F_t(\theta) = \partial f(x_t, \theta) \,/\, \partial\theta$.
- *(i)* $f(x_t, \theta)$ *is twice continuously differentiable in* Θ, *for each* x_t, *where* $E[sup_\theta\, f_t(\theta)]^2$, $E[sup_\theta\, F_t(\theta)]$ *and* $E[sup_\theta\, \partial F_t(\theta)/\partial\theta']$ *exist and are bounded;*
- *(ii)* $T^{-1}\sum_{t=1}^{[Tr]} F_t(\theta)F_t(\theta)' \xrightarrow{p} rW(\theta)$, *a positive definite matrix of constants, uniformly in* $\theta \times r$.
- **A5: Error Process**
  Let $u_t(\theta) = y_t - f_t(\theta)$.
  (i) $E[\sup_\theta|u_t(\theta)|$ *given* $x_t] < \infty$;
  (ii) u_t *i.i.d.*;
  (iii) $E[\, u_t \mid x_t \,] = 0$ *and* $Var[\, u_t \mid x_t \,] = \sigma^2 < \infty$;
  (iv) $E[\sup_t|u_t|] < \infty$.

- **A6: Break Identification**
  $E[f_t(\theta_j^0)] \neq E[f_t(\theta_{j+1}^0)]$ *for each* $j = 1, 2, \ldots, m.$

- **A7: Parameter Identification**
  $\bar{S}(\theta_1, \ldots, \theta_{m+1}) = (m+1)\sigma^2 + \sum_{i=1}^{m+1} [\lambda_i^0 - \lambda_{i-1}^0] E[f_t(\theta_i) - f_t(\theta_i^0)]^2$ *has a unique minimizer at* $(\theta_1^0, \ldots, \theta_{m+1}^0)$.

Linear vs. nonlinear models:

- similarity:

  $$OLS = (X'X)^{-1}X'y; \qquad NLS = (F'F)^{-1}F'y + o_p(T^{-1/2}).$$

  where $F = \partial f(X, \theta^0)/\partial\theta$.
- difference in our setting:

  the above approximation cannot be legitimately performed prior to obtaining T-rate consistent estimators of the change points.

MAIN RESULT 1: Consistency of Break Fractions

Let the estimated break-fractions $\hat{\lambda}_i$ be such that $\hat{T}_i = [T\hat{\lambda}_i]$.

- *Under A1-A5:* $\hat{\lambda}_i \xrightarrow{p} \lambda_i^0$, *for* $i = 1, ..., m$.

Consistency of Break Fraction Estimates:

- By means of two lemmas:
- Let $d_t = \hat{u}_t - u_t$, and use inequality:

$$T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 = T^{-1}\sum_{t=1}^{T}u_t^2 + T^{-1}\sum_{t=1}^{T}d_t^2 + 2T^{-1}\sum_{t=1}^{T}d_t u_t \leq T^{-1}\sum_{t=1}^{T}u_t^2$$

- **Lemma 1.** $T^{-1}\sum_{t=1}^{T} d_t u_t \xrightarrow{p} 0.$
- **Lemma 2.** If estimated break fraction $\hat{\lambda}_j \overset{p}{\nrightarrow} \lambda_j^0$ for some j
$\Rightarrow \limsup P\left[\, T^{-1}\sum_{t=1}^{T} d_t^2 > C \,\right] > \epsilon$, for some $C, \epsilon > 0.$

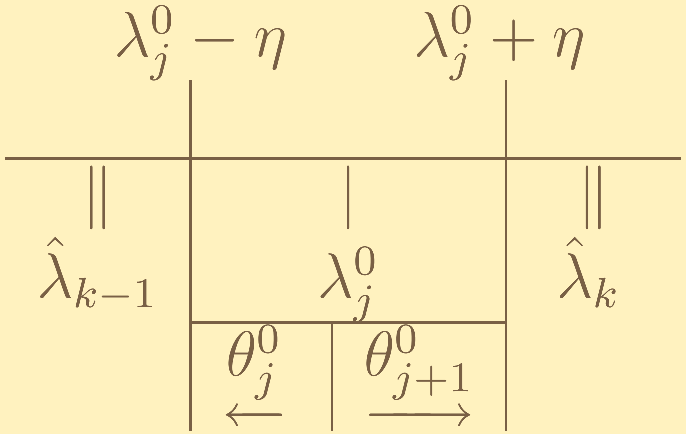
Lemma 1. $T^{-1}\sum_{t=1}^{T} d_t u_t \xrightarrow{p} 0.$

- Note that:

$$T^{-1}\sum_{t=1}^{T} u_t d_t = \underbrace{T^{-1}\sum_{i=0}^{m}\sum_{T_i^0+1}^{T_{i+1}^0} u_t f(x_t,\theta_i^0)}_{A} - \underbrace{T^{-1}\sum_{i=0}^{m}\sum_{\hat{T}_i+1}^{\hat{T}_{i+1}} u_t f(x_t,\hat{\theta}_i)}_{B}$$

- $A = o_p(1)$ by pointwise laws of large numbers.
- to show $B = o_p(1)$, we borrow a proof by Caner (2005), that uses **empirical process theory**.

Lemma 2. If estimated break fraction $\hat{\lambda}_j \overset{p}{\nrightarrow} \lambda_j^0$ for some j
$\Rightarrow \limsup P\left[\, T^{-1}\sum_{t=1}^{T} d_t^2 > C \,\right] > \epsilon$, for some $C, \epsilon > 0$.



From inequality below, Lemma 2 follows:

$$T^{-1}\sum_{t=1}^{T} d_t^2 \geq T^{-1}\sum_{1}[f_t(\hat{\theta}_k) - f_t(\theta_j^0)]^2 + T^{-1}\sum_{2}[f_t(\hat{\theta}_k) - f_t(\theta_{j+1}^0)]^2.$$

By contradiction, we get consistency of break-point estimates.

MAIN RESULT 2: T-Rate Convergence of Break Fractions

Under A1-A6, for every $\eta > 0$, *there exists a finite* $C > 0$, *such that for all large T,* $P(|\, T(\hat{\lambda}_k - \lambda_k^0)\,| > C) < \eta \; (k = 1, \ldots, m)$.

- crucial result because we will encounter in all estimation / inference procedures sums of the form:

$$T^{-1} \sum_{[T\hat{\lambda}_i]+1}^{[T\hat{\lambda}_{i+1}]} O_p(1), \qquad \text{but we need} \qquad T^{-1} \sum_{[T\lambda_i^0]+1}^{[T\lambda_{i+1}^0]} O_p(1).$$

- the result above allows us to approximate the first sum with the second
- if we couldn't do so, then the difference between those two sums would not disappear in the limit as T grows large.

MAIN RESULT 3: Asymptotic Normality

Under A1-A7, $T^{1/2}(\hat{\theta} - \theta^0) \xrightarrow{d} \mathcal{N}\left(0, [W(\theta^0)]^{-1}\right)$, *where* $[W(\theta^0)]$ *is a block diagonal matrix whose i-i-th block is* $\sigma^2[\lambda_i^0 - \lambda_{i-1}^0]\, E\left[F_t(\theta_i^0)' F_t(\theta_i^0)\right]$.

- we obtain normality of parameters if we consistently estimate the break fractions, at a T-rate;
- to show this result, we use mean value expansions of partial sums of squares, where the end points of these sums are the estimated change points;
- the exciting part is that given the T-rate convergence of break fractions, we can replace the estimated change points with the true ones;

- **even in unstable nonlinear models of type NLS, we can find the breaks and estimate the parameters we need.**

Stability Tests

Similar to Bai and Perron (1998), we propose 3 classes of test that detect instability.

These are hypotheses of interest:

- **Test 1:** no breaks vs. a known number of breaks;
- **Test 2:** no breaks vs. an unknown number of breaks;
- **Test 3:** l vs. l+1 breaks.
- **tests have non-standard distributions**, but they carry over from the linear setting of Bai and Perron (1998), where critical values can be found.

Stability Tests

Computation:

- as Bai and Perron (2003) show, independent of the number of breaks, we only need to search over $T(T+1)/2$ partitions of the sample;
- furthermore, we need to bound the candidate change points away from the end-points of the sample (cut-offs usually 5%-15%);
- by doing so, we further reduce the number of partitions we need to search over.

1. A Test of No Break vs. A Known Number of Breaks

- Hypothesis:

$$H_0 : m = 0 \qquad vs. \qquad H_A : m = k$$

- **Sup F-type Test:**

$$\sup_{(\lambda_1,\ldots,\lambda_k)\in\Lambda_\epsilon} F_T(k;p) = \sup_{(\lambda_1,\ldots,\lambda_k)\in\Lambda_\epsilon} \frac{(SSR_0 - SSR_k)/kp}{SSR_k/[T-(k+1)p]}$$

where $\Lambda_\epsilon = \{(\lambda_1, \ldots, \lambda_k) : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon\}$

- Distribution under the Null: :

$$\frac{1}{kp} \sup_{(\lambda_1,\ldots,\lambda_k)\in\Lambda_\epsilon} \sum_{i=1}^{k} \frac{\|\lambda_i W_p(\lambda_{i+1}) - \lambda_{i+1} W_p(\lambda_i)\|^2}{\lambda_i \lambda_{i+1}(\lambda_{i+1} - \lambda_i)}$$

- test is consistent for its alternative;
- **it does not depend on nuisance parameters**.

2. A Test of No Break vs. A Unknown Number of Breaks

- Hypothesis:

$$H_0 : m = 0 \qquad vs. \qquad H_A : \text{ m unknown}, m < M, M \text{ fixed}$$

- **Double Maximum Test:**

$$D\max F_T(M, a_1, \ldots, a_M) = \max_{1 \le m \le M} a_m \sup_{\Lambda_\epsilon} F_T(m; p)$$

- Distribution under the Null:

$$\max_{1 \le m \le M} \frac{a_m}{kp} \sup_{(\lambda_1, \ldots, \lambda_k) \in \Lambda_\epsilon} \sum_{i=1}^{k} \frac{\| \lambda_i W_p(\lambda_{i+1}) - \lambda_{i+1} W_p(\lambda_i) \|^2}{\lambda_i \lambda_{i+1} (\lambda_{i+1} - \lambda_i)}$$

- test is consistent for its alternative;
- **depends on choice of weights.**

2. A Test of No Break vs. A Unknown Number of Breaks

Choice of weights:

- equal weights over the possible number of breaks;
- give more weight to some number of breaks according to some prior;
- since for any fixed number of parameters p, the critical values of $\sup_{(\lambda_1,\ldots,\lambda_k)\in\Lambda_\epsilon} F_T(m;p)$ decreases as m increases, this implies that if we have a large number of breaks, we may get a test with low power, because the marginal p-values decrease with m;
- one way to keep marginal p-values of the tests equal across number of breaks is to use weights that depend on p and the significance level of the test
- for example, let $c(p,\alpha,m)$ be the asymptotic critical value of the test $\sup_{(\lambda_1,\ldots,\lambda_k)\in\Lambda_\epsilon} F_T(m;p)$ and assign:
  $a_1 = 1$ and $a_m = c(p,\alpha,1)/\,c(p,\alpha,m)$ for $1 < m \leq M$.

3. Test for an Additional Break

- Hypothesis:

$$H_0 : m = l \qquad vs. \qquad H_A : m = l + 1.$$

- **Test each (l+1)-segment for an additional break by means of:**

$$\{S_T(\hat{T}_1, \ldots, \hat{T}_l) - \min_{1 \le i \le l+1} \inf_\tau S_T(\hat{T}_1, \ldots, \hat{T}_{i-1}, \tau, \hat{T}_i, \ldots, \hat{T}_l)\}/\hat{\sigma}^2$$

- Distribution under the Null:

$$\lim P(F_T(l+1|l) \le x) = G_{p,\eta}^{l+1}$$

where $G_{p,\eta}$ *is the cdf of* $\sup_{\eta \le \mu \le 1-\eta} \frac{\|W_q(\mu) - \mu W_q(1)\|^2}{\mu(1-\mu)}$.

- test is consistent and provides insight for constructing sequential rather than global methods for estimation.

3. Sequential Estimation of Break-Points

- if there is evidence for one break, then estimate it and split sample into 2 parts;
- if there is evidence of an additional break, then search each sub-sample to find the estimated second break;
- iterate procedure until there is no evidence of an additional break.

Model:

- $f(x_t, \theta) = \theta_i^1 + \theta_i^2 \, e^{\,x_t \, \theta_i^3} \qquad t \in [T_i^0, T_{i+1}^0].$

Table 1 : m = 1, 100 simulations, break fraction : 0.4

True Parameter Values			θ^1		θ^2		θ^3	
			before	after	before	after	before	after
			+1	-1	-10	+10	+1	-1
T	True breaks	MC Est Breaks	$\hat{\theta}^1$		$\hat{\theta}^2$		$\hat{\theta}^3$	
30	12	12.06	1.21 (1.12)	-1.05 (.74)	-10.14 (1.58)	10.08 (.81)	1.12 (1.33)	-.99 (.05)
50	20	20.09	1.05 (.91)	-1.12 (.76)	-9.93 (.92)	10.06 (.61)	1.00 (.06)	-.99 (.02)
100	40	40.01	1.00 (.42)	-.99 (.35)	-9.98 (.44)	10.02 (.37)	1.00 (.02)	-.99 (.02)

Model:

- $f(x_t, \theta) = \theta_i^1 + \theta_i^2 \, e^{\, x_t \, \theta_i^3} \qquad t \in [T_i^0, T_{i+1}^0].$

Table 2 : m = 2, 100 simulations, break fractions : [0.4, 0.7]

T	True breaks	MC Estimated breaks	
30	[0.40, 0.70]	0.399	0.70
50		0.40	0.70
50		0.40	0.70

Conclusions

- **we provided a comprehensive treatment of estimation and inference in NLS models** similar to Bai and Perron (1998) results for linear models;
- **key difference** comes from using a mean value expansion rather than an exact formula for parameter estimates;
- as a consequence of nonlinearity, we use **nonlinear asymptotics** and **empirical process theory**;

- **the method we develop is useful for detecting breaks in nonlinear economic models, by means of proposed tests;**
- **it also offers a solution on estimating the model when breaks are present.**

In progress

- derive asymptotic distributions of change point estimates and Wald-like tests;
- study finite sample behavior of estimates and tests;
- extend to more general nonlinear models;
- use it for estimating consumption models in the presence of tax changes
- and inventory models in the presence of flexible accelerator instability.

Thank You!