Mortality Surface by Means of Continuous Time Cohort Models

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Motivation

- **Insurance companies** and **pension funds** are exposed to **mortality risks**.

- The development of a **liquid** and **transparent** mortality-linked capital market is desired.

- Mortality-risk appraisal consisting in an **accurate**, yet **easy-to-handle** description of **human survivorship** is fundamental in this respect.

- A number of proposals have been put forward, some of which have great potential, but still without any consensus being reached on the best approach to mortality risk modeling.
Goals

We have developed our model with the following goals in mind:

1. Analytical tractability
2. Parsimoniousness
3. Fit to historical data
4. Null or low probability of negative intensities (specific to our model)
5. Possibility and ability of deterministic forecasting
6. Possibility and ability of stochastic forecasting
7. Possibility and ability of measuring correlation among different generations
A single generation

The standard uni-dimensional framework

Stochastic mortality of a given generation is described by means of a Cox process

- \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), where \(\mathbb{P}\) is the real-world probability measure
- \(\{\mathcal{F}_t : 0 \leq t \leq T\}\) satisfies the usual technical conditions
- \(\mu(t, x)\) - the mortality intensity of an individual belonging to a given generation, initial age \(x\), at calendar time \(t\)
- \(n\) - number of state processes
- \(X(t) = [X_1(t), ..X_n(t)]^T\) - the vector of state processes
- \(R : D \rightarrow \mathbb{R}, \ D \subset \mathbb{R}^n\)

We define:

\[\mu(t) \overset{\text{def}}{=} R(X(t))\]

Consequently, the survival probability from \(t\) to \(T\), conditional on being alive at \(t\) is:

\[S(t, T) = \mathbb{E}_t \left[ e^{\int_t^T \mu(s)ds} \right] = \mathbb{E}_t \left[ e^{\int_t^T R(X(s))ds} \right]\]
A single-generation DPS framework

The Duffie, Pan and Singleton (2000) framework

If \( dX(t) = \lambda(X(t))dt + \sigma(X(t))dZ(t) \), having

- \( Z \) a \((\mathcal{F}_t)\)-standard Brownian motion in \( \mathbb{R}^n \),
- \( \lambda : D \to \mathbb{R}^n \), \( \sigma : D \to \mathbb{R}^{n \times n} \), \( \lambda \), \( \sigma \), and \( R : D \to \mathbb{R} \) are affine,
- \( \lambda(x) = K_0 + K_1 x \), for \( K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \),
- \( (\sigma(x)\sigma(x)^T)_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x \), for \( H = (H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n} \),
- \( R(x) = r_0 + r_1 x \), where \( (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n \).

we have that

\[
\mathbb{E}[e^{-\int_t^T R(X(s))ds} | \mathcal{F}_t] = e^{\alpha(t; T) + \beta(t; T) \cdot X(t)},
\]

where \( \alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n \) and \( \beta(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n \) satisfy the complex-valued ODEs

\[
\beta'(t; T) = r_1 - K_1^T \beta(t; T) - \beta(t; T)^T H_1 \beta(t; T)/2,
\]
\[
\alpha'(t; T) = r_0 - K_0 \beta(t; T) - \beta(t; T)^T H_0 \beta(t; T)/2,
\]

with boundary conditions \( \alpha(T, T) = \beta(T, T) = 0 \).
Transition to the entire mortality surface

Transiting from single-generation to mortality surface

- We label each generation with a proper index \( i \in \mathbb{I} \subset \mathbb{N} \),
- Each generation has its own mortality intensity,
- We assume that the state processes of each of the generations are driven by Brownian motions that have a correlation \textbf{unique} for that generation.
- In effect, each generation is assigned its own correlation matrix.

Given the \( n \) state processes driving the mortality intensity of generation \( i \), we have:

\[
dX^i(t) = \lambda(X^i(t))dt + \sigma(X^i(t))dW^i(t)
\]

where

- \( W(t) = [W^i_1(t), W^i_2(t), ..., W^i_n(t)] \)
- \( \rho^i_{n \times n} = \{\rho^i_{lm}\}_{1 \leq l, m \leq n} \) - instantaneous correlation matrix proper of generation \( i \)
- \( \rho^i_{lm}dt = \langle dW^i_l(t), dW^i_m(t) \rangle \)
Using the DPS framework for a mortality surface

In order to use the DPS framework in case of a mortality surface, we need to:

- Use the Cholesky decomposition of the correlation matrix $\rho^i$
  \[
  \rho^i = H^i(H^i)^T, \text{ and}
  \]

- Transform $W(t)$ to
  \[
  dW^i(t) = H^i dZ(t),
  \]
  where $Z(t)$ is a vector of uncorrelated Brownian motions.

Finally, we obtain:
\[
  dX^i(t) = \lambda(X^i(t))dt + \sigma(X^i(t))H^i dZ^i(t)
\]
allowing us to make use of the DPS framework.
Model construction

Additional assumptions

We make the following additional assumptions:

- For any generation \( i \), each of its state processes follows an Ornstein-Uhlenbeck dynamic
  \[
dX_k^i(t) = \psi_k X_k^i dt + \sigma_k dW_k^i(t), \quad k = 1, n,
\]
  where \( \psi_k \in \mathbb{R}, \sigma_k > 0 \) and \( W_1^i(t), W_2^i(t), \ldots, W_n^i(t) \) are correlated.

- Using a more compact form, we have:
  \[
dX^i(t) = \Psi X^i(t) dt + \Sigma dW^i(t),
\]
or
  \[
dX^i(t) = \Psi X^i(t) dt + \Sigma H^i dZ^i(t),
\]
  where \( \Psi = \text{diag}[\psi_1, \psi_2, \ldots, \psi_n] \), and \( \Sigma = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_n] \).

- Finally, we set
  \[
R(X^i(t)) = 1 \cdot X^i(t).
\]
The general solution

- In the survivorship context, it is convenient to set the valuation time \( t = 0 \), and reason in terms of remaining life \( \tau = T - t \). With this transformation, using the DPS framework, we need to solve \( n \) systems of the form:

\[
\hat{\beta}'(\tau) = -1 + \Psi \hat{\beta}(\tau)
\]

\[
(\hat{\alpha}^i)'(\tau) = \frac{1}{2} \hat{\beta}(\tau)^T \Sigma \rho^i \Sigma^T \hat{\beta}(\tau)
\]

\[
(\hat{\alpha}^i)'(0) = \hat{\beta}'(0) = 0.
\]

- Finally, we have

\[
S^i(0, \tau) = \mathbb{E} \left[ \exp \left( - \int_0^\tau 1 \cdot X^i(s) ds \right) \right] = e^{\hat{\alpha}^i(\tau) + \hat{\beta}(\tau) \cdot X^i(0)}.
\]

- The \( n \) solutions of the \( n \) systems are given by:

\[
\hat{\beta}(\tau) = -\int_0^\tau e^{\Psi (\tau - s)} \cdot 1 ds
\]

\[
\hat{\alpha}^i(\tau) = \int_0^\tau \frac{1}{2} \hat{\beta}(s)^T \Sigma \rho^i \Sigma^T \hat{\beta}(s) ds
\]
Two models

Reminder: Previously, we have omitted the argument \( x \) for notational convenience.

Two models of different complexities

The General OU APC Model

- \( \psi_1, \psi_2, \ldots, \psi_n, \sigma_1, \sigma_2, \ldots, \sigma_n \) depend on \( x \)
- \( \hat{\alpha}_x(\tau), \hat{\beta}_x(\tau) \) depend on \( x \)
- \( S_i^x(0, \tau) = e^{\hat{\alpha}_x(\tau)+\hat{\beta}_x(\tau)\cdot X_i^x(0)} \)
- \( S_{x+t}^i(t, T) = e^{\hat{\alpha}_x(\tau)+\hat{\beta}_x(\tau)\cdot X_i^x(t)} \)

The Simple OU APC Model

- All of the coefficients are not age-dependent.
- \( S^i(0, \tau) = e^{\hat{\alpha}^i(\tau)+\hat{\beta}(\tau)\cdot X^i(0)} \)

Onward, we will be examining the Simple OU APC Model.
An important question:

How many factors do we actually need for calibration?

Our calibration data set:

- The male population of the United Kingdom,
- Cohort death rates for life age $x = 40$,
- We examine them until they have reached the age of 59 having $\tau = 1, \ldots, 19$
- The generations $i$ span from 1900-1950, with a 5-year increment,
- 11 cohorts in total.

$$
\tilde{S}_x^i(0, \tau) = \prod_{s=1}^{\tau} (1 - q_i(x + s - 1, x + s))
$$
Choosing the appropriate number of factors

**The survival probability surface**

*Figure 1:* The survival probability surface representing the data set
Choosing the appropriate number of factors

The survival probability surface - the relevant segment

Figure 2: The relevant segment of the survival probability surface representing the data set
Choosing the appropriate number of factors

Principal Component Analysis

Application of the PCA

We use the observed survival probabilities to compute the corresponding average mortality intensity for generation $i$.

\[
\bar{\mu}_x(0, \tau_j) = -\frac{1}{\tau_j} \log \bar{S}_x(0, \tau_j) = -\frac{1}{\tau_j} \sum_{s=1}^{\tau_j} \log (1 - q_i(x + s - 1, x + s))
\]

Results from the PCA:

- The **mean** and the **first** principal component account for **95.72%** of the data,
- The **mean**, the **first** and the **second** principal component account for **99.83%** of the data.

Conclusion

The obtained results indicate that having **two factors** is a rational initial choice.
The two-factor model

- Within the simple APC model, we have, for a given generation $i$
  
  \[
  dX_1(t) = \psi_1 X_1 dt + \sigma_1 dZ_1(t) \\
  dX_2(t) = \psi_2 X_2 dt + \sigma_2 \rho^i dZ_1(t) + \sigma_2 \sqrt{1 - (\rho^i)^2} dZ_2(t)
  \]

- The mortality intensity of generation $i$ is given by
  
  \[
  \mu^i(t) = X_1(t) + X_2^i(t).
  \]

- By solving the ODEs, we obtain that:
  
  \[
  \hat{\beta}_j(\tau) = -\int_0^\tau e^{\psi_j(\tau - s)} ds = \frac{1}{\psi_j} \left( 1 - e^\psi_j \tau \right), \quad j \in \{1, 2\}
  \]

  \[
  \hat{\alpha}^i(\tau) = \sum_{j=1}^2 \frac{\sigma_j^2}{2 \psi_j^3} \left( \psi_j \tau - 2 e^{\psi_j \tau} + \frac{1}{2} e^{2 \psi_j \tau} + \frac{3}{2} \right) + \frac{\rho^i \sigma_1 \sigma_2}{\psi_1 \psi_2} \left( \tau - \frac{e^{\psi_1 \tau}}{\psi_1} - \frac{e^{\psi_2 \tau}}{\psi_2} \right. \\
  \left. + \frac{e^{(\psi_1 + \psi_2) \tau}}{\psi_1 + \psi_2} + \frac{\psi_2^2 + \psi_1 \psi_2 + \psi_2^2}{\psi_1 \psi_2 (\psi_1 + \psi_2)} \right).
  \]
The simple APC model

Visual representation of the model - An example with three generations

Main characteristics:

- every generation characterized by its own $\rho$

$$dX_1(t) = \psi_1 X_1 dt + \sigma_1 dZ_1(t)$$
$$dX_i^j(t) = \psi_2 X_2 dt + \sigma_2 \rho^i dZ_1(t) + \sigma_2 \sqrt{1 - (\rho^i)^2} dZ_2(t)$$
The simple APC model

**Visual representation of the model - An example with three generations**

Main characteristics:

- every generation characterized by its own $\rho$
- $\psi_1, \psi_2, \sigma_1, \sigma_2$ same for all generations

\[ dX_1(t) = \psi_1 X_1 dt + \sigma_1 dZ_1(t) \]
\[ dX_i^j(t) = \psi_2 X_2 dt + \sigma_2 \rho^j dZ_1(t) + \sigma_2 \sqrt{1 - (\rho^j)^2} dZ_2(t) \]
The Simple APC Model

Calibration

Achievements and Conclusions

The simple APC model

Visual representation of the model - An example with three generations

Main characteristics:

- every generation characterized by its own $\rho$
- $\psi_1, \psi_2, \sigma_1, \sigma_2$ same for all generations
- $Z_1(t), Z_2(t)$ are orthogonal BMs

$$dX_1(t) = \psi_1 X_1 dt + \sigma_1 dZ_1(t)$$
$$dX_2^i(t) = \psi_2 X_2 dt + \sigma_2 \rho^i dZ_1(t) + \sigma_2 \sqrt{1 - (\rho^i)^2} dZ_2(t)$$
The specified model for the mortality intensity of one generation is known in the interest-rate domain as the **two-factor Gaussian model**, or G2++ (Brigo and Mercurio, 2006). Each intensity is Gaussian, and can become negative with a positive probability:

$$Pr(\mu^i(\tau) < 0) = \Phi \left( - \frac{E(\mu^i(\tau))}{\sqrt{Var(\mu^i(\tau))}} \right)$$

where $\Phi(\cdot)$ is the CDF of the **standard normal distribution**,

$$E(\mu^i(\tau)) = f^i(0, \tau) + \frac{\sigma_1^2}{2\psi_1^2} \left( 1 - e^{\psi_1 \tau} \right)^2 + \frac{\sigma_2^2}{2\psi_2^2} \left( 1 - e^{\psi_2 \tau} \right)^2 + \rho \frac{\sigma_1 \sigma_2}{\psi_1 \psi_2} \left( 1 - e^{\psi_1 \tau} \right) \left( 1 - e^{\psi_2 \tau} \right),$$

$$Var(\mu^i(\tau)) = - \frac{\sigma_1^2}{2\psi_1} \left( 1 - e^{2\psi_1 \tau} \right) - \frac{\sigma_2^2}{2\psi_2} \left( 1 - e^{2\psi_2 \tau} \right) - 2\rho \frac{\sigma_1 \sigma_2}{\psi_1 + \psi_2} \left( 1 - e^{(\psi_1 + \psi_2) \tau} \right),$$

and $f^i(0, \tau)$ is the **forward mortality intensity** for the instant $\tau$:

$$f^i(0, \tau) = - \frac{\partial \log S^i(0, \tau)}{\partial \tau} = e^{\psi_1 \tau} X_1(0) + e^{\psi_2 \tau} X_2^i(0)$$

$$- \sum_{j=1}^{2} \frac{\sigma_j^2}{2\psi_j^3} \left( \psi_j - 2\psi_j e^{\psi_j \tau} + \psi_j e^{2\psi_j \tau} \right) - \frac{\rho^i \sigma_1 \sigma_2}{\psi_1 \psi_2} \left( 1 - e^{\psi_1 \tau} - e^{\psi_2 \tau} + e^{(\psi_1 + \psi_2) \tau} \right)$$
Correlations

Benefits of the model

The model enables the derivation of formulas for instantaneous \textbf{correlations} among intensities of different generations.

- For the generation $i$ the instantaneous mortality intensities follows the SDE:

$$d\mu^i(t) = [\psi_1 X_1(t) + \psi_2 X^i_2(t)]dt + (\sigma_1 + \rho^i \sigma_2) dZ_1(t) + \sigma_2 \sqrt{1 - (\rho^i)^2} dZ_2(t)$$

- **Instantaneous correlation** between $\mu^i(\cdot)$ and $\mu^j(\cdot)$ is equal to

$$\text{Corr}[d\mu(t_i), d\mu(t_j)] = \frac{(\sigma_1 + \rho^i \sigma_2)(\sigma_1 + \rho^j \sigma_2) + \sigma^2_2 \sqrt{(1 - (\rho^i)^2)(1 - (\rho^j)^2)}}{\sqrt{(\sigma_1 + \sigma_2 \rho^i)^2 + \sigma^2_2(1 - (\rho^i)^2)} \sqrt{(\sigma_1 + \sigma_2 \rho^j)^2 + \sigma^2_2(1 - (\rho^j)^2)}}$$
The simple APC model

Simulation

Benefits of the model

The model enables us to make **stochastic forecasting** of the survival probability curve in an **arbitrary future time** $p > 0$.

- Simple two-factor model – no dependence on age.
- When viewed from time 0, survival curve at time $p > 0$, for a head in generation $i$, is the random object:

$$S^i(p, \tau) = e^{\alpha^i(\tau-p) + \beta(\tau-p) \cdot X^i(p)}.$$

- Given the calibrated parameters, we simulate $X_1(p)$ and $X_2(p)$:

$$X_1(t) = \exp(\psi_1 t)X_1(0) + \sigma_1 \exp(\psi_1 t) \sqrt{\frac{1}{2\psi_1}(1 - \exp(-2\psi_1 t))} Z_1$$

$$X_2(t) = \exp(\psi_2 t)X_2(0) + \sigma_2 \rho_i \exp(\psi_2 t) \sqrt{\frac{1}{2\psi_2}(1 - \exp(-2\psi_2 t))} Z_1$$

$$+ \sigma_2 \sqrt{1 - (\rho_i)^2} \exp(\psi_2 t) \sqrt{\frac{1}{2\psi_2}(1 - \exp(-2\psi_2 t))} Z_2$$
Parameters

Initial values of factors are calibrated, because they cannot be observed!

Fixing the number of relevant factors to two results in:

- four parameters common to all generations: \([\psi_1, \psi_2, \sigma_1, \sigma_2]\)
- three parameters specific to each generation: \([\rho^i, X^i_1(0), X^i_2(0)]\)
- given \(k = 11\) generations, we have in total: \(4 + 3k = 37\) parameters

We collect them in:

\[
\theta = [\psi_1, \psi_2, \sigma_1, \sigma_2, \rho^1, \rho^2, \ldots, \rho^k, X^1_1(0), X^2_1(0), \ldots, X^k_1(0), X^1_2(0), X^2_2(0), \ldots, X^k_2(0)].
\]

Given the meaning, we restrict to:

\[
\Theta = \left\{ \psi_1, \psi_2 \in [-1, 1], \sigma_1, \sigma_2 \in [0, 1], \rho \in [-1, 1]^k, X_1(0) \in [-1, 1]^k \right\}.
\]
The simple APC model

Parameters

Controlling for low probability of negative mortality intensity

Given an a priori decision of $\Pr(\mu^i < 0) \leq 1\%$, for each generation $i$ we choose $X_2^i(0), i \in \{1, \ldots, k\}$ during calibration.

- If we treat $\Pr(\mu^i < 0)$ as one more parameter, then given the equations describing the theoretical constraints, and all of the other parameters, we can find $X_2^i(0), i \in 1, 2, \ldots, k$ for any given $\tau$.
- We need it for $\tau$ between 1 and the extreme age, which is 69.

For each generation $i$ we choose $X_2^i(0)$ such that the constraint is satisfied for all $\tau$. 
Error (or cost) definition

We minimize mean square error between the actual and estimated parameters with the mean computed across $k = 11$ generations and durations $\tau = 19$.

\[ \theta^* = \arg \min_{\theta \in \Theta} \sqrt{\frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{\tau} (\tilde{S}^i(j) - S^i(j; \theta))^2} \]
Differential evolution - Our use of the algorithm

Figure 3: The Differential Evolution algorithm
Methodology

Differential evolution - What have we done?

- 30 experiments, with $n_I = 100000$ iterations per experiment.
- Grid Computing Platform of The Wharton School, University of Pennsylvania
- Each experiment conducted on a single 2.5 GHz core with 4GB RAM
- Average duration of an experiment: 24h
## Cost and Parameters

### Table 1: Calibration results

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<th>$\psi_1$</th>
<th>$\psi_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho$</th>
<th>$X_1(0)$</th>
<th>$X_2(0)$</th>
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Probability of negative intensities

Figure 4: Probability of negative mortality intensity for the generation 1950
Residuals

Figure 5: Calibration residuals plot
### Correlation

**Table 2: Correlation table (%)**

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<td>95.84</td>
<td>96.93</td>
<td>99.61</td>
<td>100.00</td>
</tr>
</tbody>
</table>
Percentage Absolute Relative Error

Figure 6: Percentage Absolute Relative Error
Figure 7: Survival probability curve at $p=1$, $S(1, \tau)$, given the mean and the 90% confidence interval.
Conclusion

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- Both in-sample and out-of-sample deterministic forecasts have been examined.
  - In-sample errors up to age 59 are very small.
  - Out-of-sample errors remain small at least until age 65.
The ex-post in-sample performance of stochastic forecasts is very satisfactory. We anticipate that the increase in the error at later ages for out-of-sample forecast would probably be rectified with the introduction of a third factor, and will pursue this extension after further empirical investigation.
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- The possibility of **capturing this correlation**, thanks to a generation-based model, together with the use of the **Differential Evolution** algorithm, which permits an efficient calibration, are the major contributions of this work.
Thank you for your attention!