Yaari’s LifeCycle Model in the 21st Century: Consumption Under a Stochastic Force of Mortality
(Joint work with H. Huang and T.S. Salisbury)

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Main Research Question

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- His model is at the foundation of much of modern micro-economics.
- The Yaari (1965) model was based on a deterministic force of mortality in which the entire survival curve is known at time zero.
- In this paper we extend the Yaari (1965) model – with no annuities – to a world with stochastic mortality rates.
"...As far as I am aware, no one has challenged the view that if people were capable of it, they ought to plan their consumption, saving and retirement according to the principles enunciated by Modigliani and Brumberg in 1950s...”
Professor Angus S. Deaton, Princeton University, 2005
Let $\lambda(t)$ denote the mortality rate of a cohort of a population. Let $\mathcal{F}_t = \sigma\{\lambda(q) \mid q \leq t\}$ be the filtration determined by $\lambda$. Then individuals in the population have lifetimes of length $\zeta$ satisfying

$$P(\zeta > s \mid \zeta > t, \mathcal{F}_\infty) = e^{-\int_t^s \lambda(q) \, dq}. \quad (1)$$
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Assume further that $\lambda(t)$ is a Markov process, and define the survival function $p(t, s, \lambda)$ by

$$p(t, s, \lambda) = E \left[ e^{-\int_t^s \lambda(q) \, dq} \mid \lambda(t) = \lambda \right]. \quad (2)$$
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This gives the conditional probability of surviving from time $t$ to time $s$, given knowledge of the mortality rate at time $t$. Therefore

$$P(\zeta > s \mid \zeta > t, \mathcal{F}_t) = E \left[ e^{-\int_t^s \lambda(q) \, dq} \mid \mathcal{F}_t \right] = p(t, s, \lambda(t)).$$ \hspace{1cm} (3)

If $t = 0$ then we write $p(s, \lambda)$ for $p(0, s, \lambda)$. 
A very popular law of mortality is the Gompertz law of mortality.

\[
\lambda(t) = \frac{1}{b} \exp \left( \frac{x + t - m}{b} \right),
\]

Notes: \( x = 65, m = 89.3, b = 9.5 \) and \( p(0, 35, 0.0081) = 5\% \)
Objective Function:

\[ J = \max_c E \left[ \int_0^D e^{-\rho t} u(c(t)) 1_{\{t \leq \zeta\}} dt \right], \]

where \( \zeta \leq D \) is the remaining lifetime satisfying \( \Pr[\zeta > t] = p(t, \lambda_0) \).
The Yaari (1965) LifeCycle Model

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The budget constraint is:

\[ F_t(t) = v(t, F(t)) F(t) + \pi_0 - c(t), \]

with \( F(0) = W > 0 \) and \( F(D) = 0 \).
The Yaari (1965) LifeCycle Model

- **Objective Function:**
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  \[ F_t(t) = v(t, F(t)) F(t) + \pi_0 - c(t), \]

  with \( F(0) = W > 0 \) and \( F(D) = 0 \).

- The investment return \( v = v(t, F) \) is defined by:
  \[ v(t, F) = \begin{cases} 
  r + \zeta \lambda(t), & F \geq 0, \\
  R + \lambda(t), & F < 0,
  \end{cases} \]
Graphical View of the Solution

Four Different Wealth Trajectories
When $v(t) = r$, and $u(c) = c^{(1-\gamma)/(1 - \gamma)}$ then by the Euler-Lagrange Theorem, the optimal $F(t)$ must satisfy a second-order non-homogenous differential equation in regions where $F(t) \neq 0$. 

The PDE to solve is:

$$F_{tt}(t) = r(\rho \lambda(t)^{\gamma} + rF_t(t) + r\rho \lambda(t)^{\gamma}F(t)) = r(\rho \lambda(t)^{\gamma} + \pi_0).$$

In general the PDE can't be solved explicitly (unless $\lambda$ is constant).
Solution to LCM with DfM

- When \( v(t) = r \), and \( u(c) = c^{(1-\gamma)}/(1-\gamma) \) then by the Euler-Lagrange Theorem, the optimal \( F(t) \) must satisfy a second-order non-homogenous differential equation in regions where \( F(t) \neq 0 \).

- The PDE to solve is:

\[
F_{tt}(t) - \left( \frac{r - \rho - \lambda(t)}{\gamma} + r \right) F_t(t) + r \left( \frac{r - \rho - \lambda(t)}{\gamma} \right) F(t) = - \left( \frac{r - \rho - \lambda(t)}{\gamma} \right) \pi_0.
\]
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We were able to solve for Gompertz mortality.
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\]

In general the PDE can’t be solved explicitly (unless \( \lambda \) is constant). We were able to solve for Gompertz mortality.

Analytic Solution to Yaari (1965)

- When \( \lambda(t) \) is Gompertz we obtain an explicit expression for \( c^*(t) \) and \( F(t) \).
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Optimal consumption rate (when $\pi_0 = 0$) is:

$$c^*(t) = c^*(0)e^{kt}(p(t, \lambda_0))^{1/\gamma},$$
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Optimal trajectory of wealth is:

$$F(t) = \left( F(0) - c^*(0) \int_0^t e^{ks}(p(s, \lambda_0))^{1/\gamma}e^{-rs} \, ds \right) e^{rt}$$

$$= \left( F(0) - c^*(0) a_x^t(r - k, m^*, b) \right) e^{rt}$$
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  = \left( F(0) - c^*(0) a_x^t (r - k, m^*, b) \right) e^{rt}
  
- Initial consumption rate is...
  \[
  c^*(0) = \frac{F(0)}{a_x^D (r - k, m^*, b)},
  
  where $k = (r - \rho)/\gamma$ and $m^* = m + b \ln[\gamma]$. 
Daily Yield on U.S. Inflation-adjusted (TIPS) Bonds:
January 2003 to March 2010

- 10 yr
- 5 yr
Optimal Consumption Rate

<table>
<thead>
<tr>
<th>Coefficient of Relative Risk Aversion (CRRA) $\gamma = 4$</th>
</tr>
</thead>
</table>

Nest Egg of $100$ Invested at Following REAL Rates...

<table>
<thead>
<tr>
<th>r = 0.5%</th>
<th>r = 1.5%</th>
<th>r = 2.5%</th>
<th>r = 3.5%</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Age 65</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5 Years Later</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 Years Later</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 Years Later</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 Years Later</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Numerical Results (DfM) #1

### Coefficient of Relative Risk Aversion (CRRA) \( \gamma = 4 \)

<table>
<thead>
<tr>
<th>Nest Egg of $100 Invested at Following REAL Rates...</th>
<th>( r = 0.5% )</th>
<th>( r = 1.5% )</th>
<th>( r = 2.5% )</th>
<th>( r = 3.5% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age 65</td>
<td>$3.330</td>
<td>$3.941</td>
<td>$4.605</td>
<td>$5.318</td>
</tr>
<tr>
<td>5 Years Later</td>
<td>$3.286</td>
<td>$3.888</td>
<td>$4.544</td>
<td>$5.247</td>
</tr>
<tr>
<td>10 Years Later</td>
<td>$3.212</td>
<td>$3.801</td>
<td>$4.442</td>
<td>$5.130</td>
</tr>
<tr>
<td>20 Years Later</td>
<td>$2.898</td>
<td>$3.429</td>
<td>$4.007</td>
<td>$4.627</td>
</tr>
<tr>
<td>30 Years Later</td>
<td>$2.156</td>
<td>$2.552</td>
<td>$2.982</td>
<td>$3.444</td>
</tr>
</tbody>
</table>
Optimal Initial Withdrawal Rate (IWR) from $100

As a Function of Pension Income $\pi_0$

Depending on the Coefficient of Relative Risk Aversion

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\gamma = 1$</th>
<th>$\gamma = 2$</th>
<th>$\gamma = 4$</th>
<th>$\gamma = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Pension</td>
<td>6.330%</td>
<td>5.301%</td>
<td>4.605%</td>
<td>4.121%</td>
</tr>
<tr>
<td>$\pi_0 = $1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_0 = $2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_0 = $5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Gompertz Mortality ($m = 89.3$, $b = 9.5$) and $r = 2.5\%$
Numerical Results (DfM) #2

<table>
<thead>
<tr>
<th>Optimal Initial Withdrawal Rate (IWR) from $100</th>
<th>As a Function of Pension Income $\pi_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depending on the Coefficient of Relative Risk Aversion</td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>$\gamma = 2$</td>
</tr>
<tr>
<td>No Pension</td>
<td>6.330%</td>
</tr>
<tr>
<td>$\pi_0 = 1$</td>
<td>6.798%</td>
</tr>
<tr>
<td>$\pi_0 = 2$</td>
<td>7.162%</td>
</tr>
<tr>
<td>$\pi_0 = 5$</td>
<td>8.015%</td>
</tr>
</tbody>
</table>

Note: Gompertz Mortality ($m = 89.3, b = 9.5$) and $r = 2.5\%$
Retire with a $100 Nest Egg and a $5 per year pension...

Figure #1: Optimal Consumption: $5 Pension Income with Investment Rate = 2.5%

- CRRA = 1, Spend 8.01% @65
- CRRA = 2, Spend 6.55% @65
- CRRA = 4, Spend 5.55% @65
- CRRA = 8, Spend 4.84% @65
Good advice, bad reason?

The New York Times

Sunday Business

New Advice to Retirees: Spend More at First, Cut Back Later

By ILANA POLYAK

H

OW much can you take out of your retirement nest egg each year without running out of money?

Not much, according to the standard, conservative advice of many financial planners. They often say that people who retire at age 65 can safely remove only about 4 percent of their portfolio each year, along with adjustments for inflation. On that basis, the initial withdrawal from a portfolio worth $1 million would be just $40,000.

But some experts have been making waves by suggesting that it may make more sense to withdraw bigger amounts in the early years of retirement.

Ty Bernick, a financial planner in East Claire, Wis., for example, says retirees generally spend less as they age, so that it is reasonable for them to spend more when they are in retirement’s early stages. Mr. Bernick’s conclusions, which relied on data from the Bureau of Labor Statistics’ Consumer Expenditure Survey for 2002, were published in June in The Journal of Financial Planning. (www.journaloffinancialplanning.org/article.aspx?article=2008–06–06–012508.cfm).

Spendings in practically every category, from housing to clothing to entertainment, declined with age, the data showed. The only category in which spending rose with age was health care, he said.

It’s almost a tug of war between income, pushing costs up, and human nature, pulling them down,” Bernick said.

George and Kathy Magaw, both 56 and clients of Mr. Bernick in East Claire, expect to spend less as they age, and so it is reasonable for them to spend more when they are in retirement’s early stages. Mr. Magaw said. He has decided to take early retirement from his job as a training manager for a manufacturing company in June 2008, when he will be 61. Mrs. Magaw is not employed. His plans to spend time on his fishing and to go on waterfowl hunting trips in remote parts of Wisconsin. The couple also expect to visit grandchildren in Wisconsin and Connecticut.

“Aren’t going to end up spending a little bit more money up front?” Yes,” Mr. Magaw said, but the period of higher expenditures should be relatively brief. “It won’t be more than the first two or three years,”

Then he expects to reduce spending gradually on things like travel and entertainment — making up for increases in health care, Mr. Magaw said.

The traditional advice that calls for an initial withdrawal of 4 percent is based on several assumptions. To compensate for inflation, the withdrawal rate would increase 1 percent every year. Moreover, a $1-million nest egg could last 40 years, if they live 80 years.

And the rate would generally be invested in stocks — 5 percent in stocks — in a further hedge against inflation — and the remainder in fixed-income investments and cash.

The approach is based on risk-adjusted studies using all kinds of hypothetical examples of market returns. The withdrawal rates are intended to leave very little chance of running out of money.

“Yet the whole premise is that if you spend $40,000 in the first year of retirement, we have to lower it,” said Christine Fidley, senior financial planner with K. H. Beach Associates in Minneapolis.

The couple plan to retire in 2008 from their jobs as a training manager and a sales manager, respectively, and to live off their nest egg.

Mr. Magaw said: “It’s usually the last $40,000 that puts the quality in quality of life.”

In order to take out more than 4 percent that first year, Mr. Magaw said, investors need to follow a few rules. To generate income, they must always sell winning stocks before losing stocks. They cannot add more than 4 percent a year to their withdrawal even if inflations is higher than that. And no successions are permitted immediately after a year of investment losses.

Mr. Magtan’s research can be found at www Изисгй.г.ор/journal/article.aspx?article=2008–06–06–012508.cfm

Two planners challenge the traditional approach for managing that nest egg.

To Mr. Bernick, a couple who spend $40,000 in their first year of retirement may need to spend as much when they are in their 70s. People who are 55 and older spend an average of $1,200 a year on apparel and services, for example, while those who are 60 to 74 spend twice as much, based on the consumer survey she used. Those 75 and up spend an average of $1,600 a year for entertainment, compared with $1,275 for ages 65 to 74.

According to the calculations, a couple in the first year of retirement at age 55, with expenditures of $60,000, might be able to safely withdraw that much from a nest egg worth $1 million — at 4 percent initial withdrawal rate. They would not run out of money so long as they reduced their spending later on according to the pattern shown in the survey.

“Of course it depends on the mix of stocks and bonds in someone’s portfolio,” Mr. Bernick said. “But a 6 percent withdrawal rate becomes more realistic.”

He said that the rate could vary because of many factors, including a retiree’s health and the rate of inflation.

Others advocate lowering the purse strings in retirement, but for other reasons. In the October 2004 issue of The Journal of Financial Planning, Jonathan Guyton, a planner at Cornerstone Wealth Advisors in Minneapolis, advocated an initial withdrawal rate as high as 6 percent, drawing his conclusion from a study of market returns from 1973 to 2003. Mr. Guyton found that a person who retired in 1973, in the middle of a depression, had a 68 percent chance of running out of money, but had a 4.1 percent initial withdrawal rate of 8 percent with a portfolio that was 80 percent stocks.

A portfolio with 60 percent stocks could have another advantage.

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Wealth Trajectory

**Figure #2**

Financial Capital: $5 Pension Income and Investment Rate = 2.5%

- CRRA = 1, WDT = 24.6 years
- CRRA = 2, WDT = 29.6 years
- CRRA = 4, WDT = 34.9 years
- CRRA = 8, WDT = 40.5 years
<table>
<thead>
<tr>
<th>Percent</th>
<th>Risk Aversion $\gamma = 4$</th>
<th>Risk Aversion $\gamma = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pensionized</td>
<td>Age 65</td>
<td>Age 80</td>
</tr>
<tr>
<td>0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td></td>
<td></td>
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<tr>
<td>40%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100%</td>
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</tbody>
</table>
### How Does Pensionization Impact Consumption?

<table>
<thead>
<tr>
<th>Percent</th>
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<tbody>
<tr>
<td>Pensionized</td>
<td>Age 65</td>
<td>Age 80</td>
</tr>
<tr>
<td>0%</td>
<td>$4.605$</td>
<td>$4.007$</td>
</tr>
<tr>
<td>20%</td>
<td>$5.263$</td>
<td>$4.580$</td>
</tr>
<tr>
<td>40%</td>
<td>$5.795$</td>
<td>$5.042$</td>
</tr>
<tr>
<td>60%</td>
<td>$6.227$</td>
<td>$5.419$</td>
</tr>
</tbody>
</table>
Up until now we have assumed that $\lambda(t)$ is deterministic.

What happens when we allow for a stochastic mortality rate $\lambda(t)$? In particular, what if

$$d\lambda(t) = \mu(t)\lambda(t)dt + \sigma\lambda(t)dB(t)$$

How does optimal consumption behavior change and what is the impact of "longevity risk aversion" on the optimal plan?
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\[
10p_{90} = \frac{35p_{65}}{25p_{65}}
\]

<table>
<thead>
<tr>
<th>Conditional Survival Probability:</th>
<th>(x = 65)</th>
<th>(x = 75)</th>
<th>(x = 85)</th>
<th>(x = 90)</th>
<th>(x = 95)</th>
<th>(x = 100)</th>
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<tbody>
<tr>
<td>To 65</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>To 75</td>
<td>0.8659</td>
<td>1.000</td>
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<tr>
<td>To 85</td>
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<td>0.6620</td>
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<tr>
<td>To 90</td>
<td><strong>0.3696</strong></td>
<td>0.4268</td>
<td>0.6447</td>
<td>1.000</td>
<td></td>
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<tr>
<td>To 95</td>
<td>0.1758</td>
<td>0.2031</td>
<td>0.3067</td>
<td>0.4757</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>To 100</td>
<td><strong>0.0500</strong></td>
<td>0.0577</td>
<td>0.0872</td>
<td><strong>0.1353</strong></td>
<td>0.2844</td>
<td>1.000</td>
</tr>
<tr>
<td>(\lambda_x)</td>
<td>0.0081</td>
<td>0.0232</td>
<td>0.0667</td>
<td>0.1129</td>
<td>0.1911</td>
<td>0.3234</td>
</tr>
</tbody>
</table>
**Conditional Survival Probability:**

<table>
<thead>
<tr>
<th>Survival Probability</th>
<th>$x = 65$</th>
<th>$x = 75$</th>
<th>$x = 85$</th>
<th>$x = 90$</th>
<th>$x = 95$</th>
<th>$x = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>To 65</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To 75</td>
<td>0.8659</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To 85</td>
<td>0.5733</td>
<td>?</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To 90</td>
<td>0.3696</td>
<td>?</td>
<td>?</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>To 95</td>
<td>0.1758</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>To 100</td>
<td>0.0500</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>1.000</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>0.0081</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
The objective function is now written as:

$$J(t, \lambda, F) = \max_c(s) \left( \int Z \left. T \left. t \left. e \left. R \left. s \left. t \left( r + \lambda(q) \right) \right) dq \right) \left( c(s) \right) ds \right)$$

We condition on wealth $F(t)$ and the mortality rate $\lambda(t)$.
The objective function is now written as:

$$J(t, \lambda, F) = \max_{c(s) \text{ adapted}} \mathbb{E} \left[ \int_t^T e^{-\int_t^s (r + \lambda(q)) dq} u(c(s)) ds \middle| \lambda(t) = \lambda, F(t) = F \right]$$
The objective function is now written as:

\[ J(t, \lambda, F) = \max_{c(s) \text{ adapted}} \mathbb{E} \left[ \int_t^T e^{-\int_t^s (r + \lambda(q)) \, dq} u(c(s)) \, ds \, \middle| \, \lambda(t) = \lambda, F(t) = F \right] \]

- We condition on wealth \( F(t) \) and the mortality rate \( \lambda(t) \).
The objective function is now written as:

\[ J(t, \lambda, F) = \max_{c(s) \text{ adapted}} E \left[ \int_t^T e^{-\int_t^s (r+\lambda(q)) dq} u(c(s)) ds \right| \lambda(t) = \lambda, F(t) = F \]

We condition on wealth \( F(t) \) and the mortality rate \( \lambda(t) \).

The wealth process (budget constraint) still satisfies
\[ dF(t) = (rF(t) - c(t)) dt. \]
Solution: Optimal Consumption under SfM

- We can’t use Calculus of Variations and must resort to dynamic programming.
Solution: Optimal Consumption under SfM

- We can’t use Calculus of Variations and must resort to dynamic programing.
- Under the optimal control $c(t)$, the expectation

$$E \left[ \int_0^T e^{-\int_t^s (r+\lambda(q)) \, dq} u(c(s)) \, ds \right| \mathcal{F}_t],$$

is a martingale and...
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  \[
  E \left[ \int_0^T e^{-\int_t^s (r+\lambda(q)) \, dq} u(c(s)) \, ds \mid \mathcal{F}_t \right],
  \]
  is a martingale and...
- It can be written as:
  \[
  e^{-\int_0^t (r+\lambda(q)) \, dq} J(t, \lambda(t), F(t)) + \int_0^t e^{-\int_t^s (r+\lambda(q)) \, dq} u(c(s)) \, ds.
  \]
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By Ito’s lemma, we have the following HJB equation for the value function:

$$\sup_c \{u(c) - cJ_F\} + J_t - (r + \lambda) J + rFJ_F + \mu(t)\lambda J_\lambda + \frac{\sigma^2 \lambda^2}{2} J_{\lambda\lambda} = 0$$
Solve the HJB equation under CRRA utility as follows:
Solve the HJB equation under CRRA utility as follows:

Let

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad J = \frac{F^{1-\gamma}}{1-\gamma} a(t, \lambda) \]
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Let

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad J = \frac{F^{1-\gamma}}{1-\gamma} a(t, \lambda) \]

Apply the 1st order condition \( c^* = J^{\frac{1}{1-\gamma}}_F \). We obtain

\[ c^* = Fa^{\frac{1}{\gamma}} \]

and get the following equation for \( a(t, \lambda) \):

\[ a_t - (r\gamma + \lambda)a + \gamma a^{1-\frac{1}{\gamma}} + \mu(t)\lambda a_{\lambda} + \frac{\sigma^2 \lambda^2}{2} a_{\lambda\lambda} = 0 \]

with boundary condition \( a(T, \lambda) = 0 \).
Main Question: How does the volatility of mortality ($\sigma$), impact the optimal initial withdrawal rate? The drift $\mu(t)$ of the mortality rate process is calibrated to fit a Gompertz survival curve ($m = 89.3, b = 9.5$), such that $p(35, 0.0081) = 5\%$

<table>
<thead>
<tr>
<th>Volatility</th>
<th>$\gamma = 0.5$</th>
<th>$\gamma = 1.0$</th>
<th>$\gamma = 1.5$</th>
<th>$\gamma = 3$</th>
<th>$\gamma = 5$</th>
<th>$\gamma = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma = 15%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma = 25%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: Retirement age 65, interest rate $r = 2\%$, mortality $\lambda_0 = 0.0081$
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<th>(\gamma = 0.5)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(\sigma = 0)</td>
<td>7.59%</td>
<td>6.12%</td>
<td>5.58%</td>
<td>5.02%</td>
<td>4.78%</td>
<td>4.61%</td>
</tr>
<tr>
<td>(\sigma = 15%)</td>
<td>7.52%</td>
<td>6.12%</td>
<td>5.60%</td>
<td>5.04%</td>
<td>4.80%</td>
<td>4.62%</td>
</tr>
<tr>
<td>(\sigma = 25%)</td>
<td>7.44%</td>
<td>6.12%</td>
<td>5.62%</td>
<td>5.06%</td>
<td>4.82%</td>
<td>4.63%</td>
</tr>
</tbody>
</table>

Notes: Retirement age 65, interest rate \(r = 2\\%\), mortality \(\lambda_0 = 0.0081\)
Main Theorem:

You might have noticed that the optimal IWR was invariant to mortality volatility $\sigma$ when $\gamma = 1$ (which is logarithmic utility). This is not a coincidence. It is a theorem.
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- Denote by $c^{SfM}(t, \lambda, F)$ the optimal consumption at time $t$, given $\lambda(t) = \lambda$ and $F(t) = F$, under a stochastic force of mortality (SfM) model. Denote by $c^{DfM}(t, F)$ the optimal consumption at time $t$, when $F(t) = F$, under a deterministic force of mortality (DfM) model.

**THEOREM**: Assume that the survival functions for the two models agree: $p^{SfM}(t, \lambda_0) = p^{DfM}(t, \lambda_0)$ for every $t > 0$, and that utility is CRRA$(\gamma)$. There are three regimes: (a) $\gamma > 1$ ($c^{SfM}(0, \lambda_0, F) > c^{DfM}(0, F)$). (b) $\gamma = 1$ ($c^{SfM}(0, \lambda_0, F) = c^{DfM}(0, F)$). (c) $0 < \gamma < 1$ ($c^{SfM}(0, \lambda_0, F) < c^{DfM}(0, F)$).

Proof in the paper...
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- Proof in the paper...
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Conclusion

- Stochastic mortality will change consumer behavior, but not as much as one might expect, when properly calibrated.
- Any horse race (i.e. comparison) between deterministic and stochastic mortality models, should ensure *rational mortality expectations*. 
Stochastic mortality will change consumer behavior, but not as much as one might expect, when properly calibrated. Any horse race (i.e. comparison) between deterministic and stochastic mortality models, should ensure rational mortality expectations. Future research will examine the impact of annuities in such a model.