PcLiRE: An Ox program for the analysis of linear Rational Expectations Models

Sophocles Mavroeidis*

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Abstract

PcLiRE is an Ox class that implements a method for the solution of multivariate dynamic Linear Rational Expectations models based on the generalized Schur Decomposition, proposed by Klein (2000). The program can be used to analyze the stability of small-scale RE models, to estimate them with Full Information Maximum Likelihood, to perform dynamic simulations and to solve forward-looking optimal control problems. These features are illustrated using two examples: a New Keynesian Phillips Curve and an optimal monetary policy rule.

Keywords: Schur Decomposition, Stability, (FIML, Optimal Control - *in progress*).

1 Introduction

Solution methods for rational expectations models have been the subject of considerable research since the seminal work of Blanchard and Kahn (1980), see Binder and Pesaran (1995) for a review. One of the most commonly used solution algorithms in monetary economics is that of Anderson and Moore (1985). Here, we implement a recent approach proposed by Klein (2000), that is based on

*Department of Quantitative Economics, and UvA-Econometrics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, the Netherlands. Please send any comments to s.mavroeidis@uva.nl.

the generalized Schur decomposition. This approach is simpler and relatively more intuitive than the others.

In this note, we provide an exposition of the issues involved in solving multivariate linear Rational Expectations models, such as conditions for existence and uniqueness of solutions, and explain the computational procedures in `PcLiRE`.

2 Setup

2.1 Definitions and notation

2.1.1 Generic LiRE model

A multivariate (infinite-horizon) Linear Rational Expectations model (LiRE) has the generic form (see Binder and Pesaran 1995 and Broze, Gouriéroux, and Szafarz 1995):

$$\sum_{i=0}^{k}\sum_{j=0}^{l} M_{ij}\,\mathrm{E}(y_{t+j-i}|\mathcal{F}_{t-i}) = Q\,z_t, \qquad t = 0, 1, \ldots \tag{1}$$

where y_t is a n-dimensional vector of decision or 'endogenous' variables, z_t is a m-dimensional vector of 'forcing variables' or 'driving processes', M_{ij} and Q are fixed coefficient matrices of dimension $n \times n$ and $n \times m$ respectively, and $\mathcal{F}_t$ denotes the (non-decreasing) information set at time t, such that y_t, z_t are adapted to it. This implies that $\mathcal{F}_t$ contains *at least* current and lagged values of (y_t, z_t).[1] Using the notation $Y_T^1 = \{y_t; t = 1, 2, \ldots, T\}$ and $Z_T^1 = \{z_t; t = 1, 2, \ldots, T\}$ for the history of (y_t, z_t) with Y_0 and Z_0 denoting initial conditions, $\mathcal{F}_t = \{Y_t^1, Y_0; Z_t^1, Z_0\}$. Also, $\mathrm{E}(y_t|\mathcal{F}_s)$ denotes the conditional expectation of y_t given $\mathcal{F}_s$, which is a (possibly non-linear) function of the variables in the information set. A stylized example could be:

$$M_{00}\,y_t + M_{10}\,y_{t-1} + M_{20}\,y_{t-2} + M_{01}\,\mathrm{E}(y_{t+1}|\mathcal{F}_t) + M_{11}\,\mathrm{E}(y_t|\mathcal{F}_{t-1}) + M_{21}\,\mathrm{E}(y_{t+1}|\mathcal{F}_{t-1}) = Q\,z_t$$

[1] Additional information such as sunspot shocks or bootstrap effects may be relevant when the model has a non-unique solution. Sunspots are independent variables that influence the decisions variables because agents expect them to do so.

2.1.2 Solution

Next, we explain what is meant by a solution to a Rational Expectations model. Since the terms $\mathrm{E}(y_{t+j-i}|\mathcal{F}_{t-i})$ in (1) are functions of $(Z_{t-i}^1, Y_{t-i}^1, Z_0, Y_0)$, it follows that y_t is an implicit function of z_t and the rest of the variables in the information set $\mathcal{F}_{t-1}$, parameterized by the coefficients $\{M_{ij}\}_{i,j}$ and Q, say

$$y_t = f(z_t, Z_{t-1}^1, Y_{t-1}^1, Z_0, Y_0; \theta),$$

where θ is a vector containing all the elements of $\{M_{ij}\}_{i,j}$ and Q. Thus, conditional on the history $(Y_{t-1}^1, Z_{t-1}^1, Y_0, Z_0)$, the distribution of y_t can be determined by a change of variables given the conditional distribution of z_t.

However, the definition of f appears to be circular: on the one hand, we need to know $\mathrm{E}(y_{t+j-i}|\mathcal{F}_{t-i})$ before we can get f, but on the other hand, we need to know f in order to derive the conditional expectations. A solution to the LiRE model (1) involves finding a function f such that the resulting stochastic process y_t satisfies equation (1).

Except in ill-conditioned cases, such a stochastic process will always exist, and will typically be non-unique.[2] It is therefore common to impose the additional requirement that the process y_t should be non-explosive, since explosive processes are rarely encountered in economics (speculative bubbles are an important exception). This implies that $\mathrm{E}(y_{t+\tau}|\mathcal{F}_t)$ is bounded for all $\tau > 0$ and hence rules out speculative bubbles ('no-bubbles condition').

The above discussion is summarized in the following definition:

Definition 2.1 (Solution to LiRE model) *Given a law for the stochastic process z_t in (1), a solution to the multivariate linear Rational Expectations model (1) is a stochastic process y_t that: (i) satisfies the difference equation (1); and (ii) is non-explosive.*

2.1.3 Predetermined variables

For the solution, it is useful to define the concept of a predetermined variable.

[2]See for instance Binder and Pesaran (1995), who characterize the entire set of solutions by means of martingales. Note that unique solutions arise in models without future expectations of the endogenous variables y_t.

Definition 2.2 (Predetermined variable) *A variable* x_t *is* predetermined *if* $\mathrm{E}(x_t|\mathcal{F}_s) = x_t$ *for some* $s < t$, *where* $\mathcal{F}_s$ *is an information set containing at least current and past values of* x_s. *Setting* $\tau = \min\{s \,:\, \mathrm{E}(x_t|\mathcal{F}_s) = x_t\}$, *we say that* x_t *is predetermined for* $t - \tau$ *periods.*

An example of a predetermined variable is one that lies in the $(t-1)$-dated information set or earlier, e.g., $\mathrm{E}(y_{t+j}|\mathcal{F}_{t-i})$ for $i > 0$. In contrast, all variables dated t, such as y_t, $\mathrm{E}(y_{t+j}|\mathcal{F}_t)$, $j > 0$, are non-predetermined.

2.2 The forcing variables

For most of the models in the monetary economics literature (and elsewhere), the forcing variables z_t follow a stable linear process.[3] Thus, we restrict attention to such processes in our implementation. Without loss of generality, we assume that z_t can be represented by a first-order Vector Autoregression:

$$z_t = c_z + \Phi\, z_{t-1} + u_t \tag{2}$$

where $u_t \sim (0, \Sigma_u)$ is a vector mean innovation process w.r.t. $\mathcal{F}_{t-1}$. This can be seen as a companion form of a more general VARMA(p,q) specification. For instance, let e_t be a vector white noise process, $e_t \sim iid(0, I)$, and suppose that the driving process is x_t and follows a VARMA(2,1):

$$C_0\, x_t - C_1\, x_{t-1} - C_2\, x_{t-2} = D_0\, e_t + D_1\, e_{t-1}$$

One way of writing this in companion form is:

$$z_t = \Phi_0^{-1}\Phi_1\, z_{t-1} + \Phi_0^{-1}\, \mathbf{e}_t,$$

$$z_t = \begin{pmatrix} e_t \\ x_t \\ x_{t-1} \end{pmatrix} \quad \mathbf{e}_t = \begin{pmatrix} e_t \\ 0 \\ 0 \end{pmatrix} \quad \Phi_0 = \begin{pmatrix} I & 0 & 0 \\ -D_0 & C_0 & 0 \\ 0 & 0 & I \end{pmatrix} \quad \Phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ D_1 & C_1 & C_2 \\ 0 & I & 0 \end{pmatrix}$$

such that $\Phi = \Phi_0^{-1}\Phi_1$ and $u_t = \Phi_0^{-1}\mathbf{e}_t$.

Notice that we have also assumed that z_t is *not Granger-caused* by y_t. There is no loss of generality in that either, since if there is an element of z_t that does not fulfil this requirement, it can be absorbed

[3]There is no loss of generality in ruling out unit roots in z_t. If there was an integrated variable in z_t, it can be categorized as an endogenous variable by including it in y_t, treating only its stationary part as a forcing variable. This will also enable a more thorough cointegration analysis of the system.

into the endogenous variables y_t. The Granger-non-causality of y_t for z_t is used to simplify the solution, as we see below, because it implies that $\mathrm{E}(z_{t+j}|\mathcal{F}_t)$ is only function of the history of z_t.

We summarize the above discussion in the following statement.

Assumption 1 *The forcing variable z_t in the LiRE model (1) follows a stationary VAR(1) process, and is not Granger-caused by y_t.*

3 The PcLiRE model

Even though the above setup allows the treatment of models with back-dated expectational terms such as $\mathrm{E}(y_{t+j-i}|\mathcal{F}_{t-i})$ for $i > 0$ and $j > 0$, as shown by Binder and Pesaran (1995), our implementation focuses only on contemporaneous expectations, i.e., we consider only models with $M_{ij} = 0$ for $i, j > 0$ in (1):

$$\sum_{i=0}^{k} A_{-i}\, y_{t-i} + \sum_{j=1}^{l} A_j\, \mathrm{E}(y_{t+j}|\mathcal{F}_t) = Q\, z_t, \qquad t = 0, 1, \ldots \tag{3}$$

As we discuss in section 3.6.2 below, there is no loss of generality in this restriction, since any model with $M_{ij} \neq 0$ for some $i, j > 0$ can be cast in the form (3). However, the presence of back-dated expectations necessitates some additional calculations not yet implemented by the program, see below.

Example: Consider the New Keynesian Phillips curve model of Galí and Gertler (1999):

$$\begin{aligned} \pi_t &= \lambda s_t + \gamma_f \mathrm{E}(\pi_{t+1}|\mathcal{F}_t) + \gamma_b \pi_{t-1} + \epsilon_t \\ s_t &= \rho s_{t-1} + \varphi \pi_{t-1} + \epsilon_{2t}, \end{aligned} \tag{4}$$

with $(\epsilon_t, \epsilon_{2t})' \sim NID(0, I)$. This can be cast in the form (3) as follows:

$$\underbrace{\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}}_{A_0} \underbrace{\begin{pmatrix} \pi_t \\ s_t \end{pmatrix}}_{y_t} + \underbrace{\begin{pmatrix} -\gamma_f & 0 \\ 0 & 0 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} \pi_{t+1|t} \\ s_{t+1|t} \end{pmatrix}}_{y_{t+1|t}} + \underbrace{\begin{pmatrix} -\gamma_b & 0 \\ -\varphi & -\rho \end{pmatrix}}_{A_{-1}} \underbrace{\begin{pmatrix} \pi_{t-1} \\ s_{t-1} \end{pmatrix}}_{y_{t-1}} = \underbrace{\begin{pmatrix} \epsilon_t \\ \epsilon_{2t} \end{pmatrix}}_{z_t = u_t}.$$

The corresponding `PcLiRE` code is given below.

-------------------------- Sample Code --------------------------

```
#include "PcLiRE1.ox"
main()
{
    decl cY, model, mA1, mA0, mAb1;
    cY = 2;                          // number of endogenous variables.
```

```
    model = new PcLiRE(cY);         // create the PcLiRE object.
    mA1 = <-0.591, 0; 0, 0>;        // specify coefficients.
    mA0 = <1, -0.05; 0, 1>;
    mAb1 = <-.378, 0; 0.1, -0.9>;
    model.SetYParameter({mA1, 1, 0}, {mAb1, -1, 0}, {mA0, 0, 0});
                                    // set LiRE equation parameters.
/*  Defaults:
    model.SetZParameter(unit(2), zeros(2,2), zeros(2,1));
    model.SetUParameter(unit(2));
*/
}
```

The function `SetYParameter(...)` takes an unspecified number of arguments, thus allowing the solution of models with arbitrary number of lags/leads.[4] Each argument is a three-dimensional array: the first argument is the coefficient matrix, the second is an integer specifying whether it refers to a lead (positive) or lag (negative). The third argument refers to the information set (zero for contemporaneous expectations $\mathrm{E}(\cdot\ |\mathcal{F}_t)$). The last two functions specify Q, Φ, c_z and Σ_u. They are redundant in this example, because their values coincide with the program defaults: $Q = I$, $\Phi = 0$ and $c_z = 0$ and $\Sigma_u = I$.

3.1 The LiRE model in canonical form

Equation (3) is a high-order expectational difference equation, but it can be written as a first-order system. This is referred to as the **canonical form** of model (3):

$$A_c\,\mathrm{E}(\mathbf{x}_{t+1}|\mathcal{F}_t) = B_c\,\mathbf{x}_t + Q_c\,z_t \tag{5}$$

This can be done in many ways. In our implementation, we have chosen to stack all the non-predetermined variables at the bottom of $\mathbf{x}_t$, as follows:

$$\underset{n(k+l)\times 1}{\mathbf{x}_t} = \begin{pmatrix} y_{t-k} \\ \vdots \\ y_t \\ \vdots \\ y_{t+l-1|t} \end{pmatrix} \qquad Q_c = \begin{pmatrix} Q \\ \mathbf{0}_{(k+l)(n-1)\times m} \end{pmatrix} \qquad A_c = \begin{pmatrix} A_{-k+1} & \dots & A_0 & \dots & A_l \\ I_n & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & I_n & 0 \end{pmatrix}$$

[4]However, computational problems arise when `cY` is large.

$$
\underset{pk\times pk}{B_c} = \begin{pmatrix} -A_{-k} & 0 & \dots & 0 \\ 0 & I_n & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I_n \end{pmatrix}
$$

Let $N = n(k+l)$ denote the dimension of canonical vector $\mathbf{x}_t$. This canonical form is derived in `PcLiRE` by the function `GetCanonicalForm()`, which casts A_c, B_c and Q_c into the class members `m_mA`, `m_mB` and `m_mQ`.

It is also noteworthy to point out that the non-zero latent roots of the LiRE matrix polynomial $A(\lambda) = \sum_{i=-k}^{l} A_i\,\lambda^i$ are the same as the generalized eigenvalues of the canonical matrix pair (A_c, B_c). Those latent roots are the λs that make $A(\lambda)$ singular, i.e., they satisfy the equation:

$$
\left(\sum_{i=-k}^{l} A_i\,\lambda^i\right) v = 0
$$

for some $v \neq 0$. For $\lambda \neq 0$, this can be re-written equivalently as

$$
\left(\sum_{i=-k}^{l} A_i\,\lambda^{i+k}\right) v = 0.
$$

Now, consider a (right) generalized eigenvector $v = (v_1', \dots, v_{k+l}')'$ of the pair (A_c, B_c) which satisfies the equation $\lambda\,A_c\,v = B_c\,v$, or

$$
\begin{aligned}
\sum_{i=-k+1}^{l} A_i\,\lambda\,v_{i+k} &= -A_{-k}\,v_1 \\
\lambda v_i &= v_{i+1}, \qquad i = 1, \dots, k+l-1.
\end{aligned} \tag{6}
$$

Thus, $v_{i+1} = \lambda^i\,v_1$ and inserting this in (6) we get:

$$
\sum_{i=-k+1}^{l} A_i\,\lambda^{i+k} v_1 + A_{-k}\,v_1 = \left(\sum_{i=0}^{k+l} A_{i-k}\lambda^i\right) v_1 = 0,
$$

thus proving that $\lambda \neq 0$ is also a latent root of $A(z)$.

The roots of $A(z)$ may be printed out in `PcLiRE`, by calling the function

```
model.PrintRoots(TRUE);   //default is FALSE.
```

see the examples below.

3.2 Solving the Canonical LiRE model

By construction, $\mathbf{x}_t$ is split into predetermined and non-predetermined variables:

$$\mathbf{x}_t = \begin{pmatrix} \mathbf{x}_t^p \\ \mathbf{x}_t^f \end{pmatrix} \quad \mathbf{x}_t^p = \begin{pmatrix} y_{t-k} \\ \vdots \\ y_{t-1} \end{pmatrix} \quad \mathbf{x}_t^f = \begin{pmatrix} y_t \\ \vdots \\ y_{t+l|t} \end{pmatrix} \tag{7}$$

Let $N_p = nk$ denote the number of predetermined variables in the system and $N_f = nl$ the number of non-predetermined ones. Since the predetermined variables are lags of y_t, which are known, the solution to the model will involve expressing y_t as a function of the predetermined variables $\mathbf{x}_t^p$ and the forcing variables z_t. Evidently, the only variables that need to be 'solved' are the non-predetermined ones (also referred to as forward-looking), $\mathbf{x}_t^f$.

We do this by computing the Generalized (real) Schur decomposition of the matrices A_c and B_c in (5). This involves solving the generalized eigenproblem $|A_c\, x - B_c| = 0$ (where A_c may be singular, otherwise the problem reduces to a simple one) and finding a block upper triangular matrix S and an upper triangular matrix T with associated orthogonal matrices U and V such that $A_c = U\,S\,V'$ and $B_c = U\,T\,V'$.[5] A generalized eigenvalue for a pair of matrices (A_c, B_c) is a scalar λ or a ratio $\alpha/\beta = \lambda$, such that $|\lambda\, A_c - B_c|$ is singular. It is usually represented as the pair (α, β), where α could be complex and β is real, as there is a reasonable interpretation for $\beta = 0$ ('infinite' eigenvalue) or both being zero (ill-conditioned case). Moreover, when λ_i is real, it is equal to t_{ii}/s_{ii}, whereas when it is complex, λ_i and its conjugate λ_{i+1} are the generalized eigenvalues of the (2×2) submatrix pair $\big(S_{[i:j][i:j]}, T_{[i:j][i:j]}\big)$, $j = i + 1$.

The above decomposition is non-unique. In particular, we may re-order the generalized eigenvalues λ_i and partition S and T conformably so that the blocks at the bottom S_{22} and T_{22} correspond to the explosive roots ($|\lambda_i| > 1$), see Klein (2000, Theorem 3.1):

$$\begin{aligned} A_c = USV' &= \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix} \\ B_c = UTV' &= \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix} \end{aligned} \tag{8}$$

[5] The diagonal of S contains (1×1) or (2×2) blocks, corresponding to real and complex eigenvalues respectively.

Similarly, we define

$$\mathbf{w}_t = V'\mathbf{x}_t = \begin{pmatrix} V_1'\mathbf{x}_t \\ \\ V_2'\mathbf{x}_t \end{pmatrix} = \begin{pmatrix} \mathbf{w}_t^s \\ \\ \mathbf{w}_t^e \end{pmatrix},$$

where $\mathbf{w}_t^s$ are the linear combinations of $\mathbf{x}_t$ corresponding to the non-explosive roots, whereas $\mathbf{w}_t^e$ correspond to the explosive roots.

Note that, unlike Klein (2000), we did not rule out the presence of unit roots, which is why we distinguished between non-explosive/explosive, as opposed to stable/unstable eigenvalues. Indeed, the presence of unit roots may generate some additional considerations, which we discuss later on. However, it is highly restrictive to rule them out at the outset, since they are common both in empirical and in theoretical macroeconomic models.

We note the dimensions of those two different sets, N_s and N_e, which are crucial for the existence and uniqueness of the solution. It is also useful to partition the columns of V conformably with the classification of eigenvalues and its rows conformably with the distinction into predetermined and non-predetermined variables as follows:

$$V = \begin{pmatrix} \underset{N_p \times N_s}{V_{ps}} & \underset{N_p \times N_e}{V_{pe}} \\ \\ \underset{N_f \times N_s}{V_{fs}} & \underset{N_f \times N_e}{V_{fe}} \end{pmatrix} \tag{9}$$

Hence, the following relationships hold:

$$\mathbf{w}_t^s = V_{ps}'\,\mathbf{x}_t^p + V_{fs}'\,\mathbf{x}_t^f \tag{10}$$

$$\mathbf{w}_t^e = V_{pe}'\,\mathbf{x}_t^p + V_{fe}'\,\mathbf{x}_t^f \tag{11}$$

$$\mathbf{x}_t^f = V_{fs}\,\mathbf{w}_t^s + V_{fe}\,\mathbf{w}_t^e \tag{12}$$

The canonical model (5) can then be equivalently re-written as the (quasi) triangular system:

$$\begin{pmatrix} S_{11} & S_{12} \\ \\ 0 & S_{22} \end{pmatrix} E\left[\begin{pmatrix} \mathbf{w}_{t+1}^s \\ \\ \mathbf{w}_{t+1}^e \end{pmatrix} \Bigg| \mathcal{F}_t\right] = \begin{pmatrix} T_{11} & T_{12} \\ \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \mathbf{w}_t^s \\ \\ \mathbf{w}_t^e \end{pmatrix} + \begin{pmatrix} U_1' \\ \\ U_2' \end{pmatrix} Q_c\, z_t. \tag{13}$$

For convenience, we gather the notation in the table below:

Mnemonic	Description
n	number of endogenous variables y_t.
m	number of forcing variables z_t.
k	number of lags in LiRE model (3).
l	number of leads in LiRE model (3).
N	dimension of canonical vector $\mathbf{x}_t$, $N = n(k+l)$.
N_p, N_f	number of pred./nonpred. variables in $\mathbf{x}_t$: $N_p + N_f = N$.
N_s, N_e	number of non-explosive/explosive eigenvalues in (5): $N_s + N_e = N$.

3.3 Conditions for existence and uniqueness of a solution

The following regularity conditions are needed.

Assumption 2 *The matrix polynomial $P(x) = A_c\,x - B_c$ is* regular*, i.e., there exists $x \in \mathbb{C}$ such that $|A_c\,x - B_c| \neq 0$.*

This condition rules out singularities in the matrix polynomial. Its necessity is exemplified by the discussion in Klein (2000, section 2.1). Suppose it doesn't hold, so that there exists a matrix polynomial $a(x)$ such that $a(x)'(A_c\,x - B_c) = 0$ for all $x \in \mathbb{C}$. Letting $x = L^{-1}$ be the forward shift operator, equation (5) would imply $a(L^{-1})'Q_c\,z_t = 0$, which is not true for a generic process z_t and matrix Q_c.

An additional condition required for our solution to work is the following.

Assumption 3 *The matrix of right generalized eigenvectors V in (9) is such that, whenever $N_f = N_e$, V_{fe} is of full rank N_f.*

We should emphasize that this condition is necessary for our proposed solution method to work, see below, but it is not clear whether it is necessary for the existence of a solution in general. As noted also in the discussion of Klein (2000, Section 5.3.1), the Schur decomposition of a matrix pair is non-unique, and we cannot rule out the possibility that the above condition would be satisfied by a different choice of basis V for the generalized right eigenspace of (A_c, B_c).

Next, drawing on Klein (2000, section 5.2), we find the solution to the transformed model (13). Since the generalized eigenvalues of the matrix pair (S_{22}, T_{22}) are all explosive, and assuming this eigenproblem is regular (ruling out $S_{ii} = T_{ii} = 0$), T_{22} is invertible (no zeros on the diagonal) and the infinite sum $\sum_{j=0}^{\infty}(T_{22}^{-1}S_{22})^j$ converges. The solution for the explosive part $\mathbf{w}_t^e$ then follows by forward substitution:

$$\begin{aligned}
\mathbf{w}_t^e &= -T_{22}^{-1}\sum_{j=0}^{\infty}\left(T_{22}^{-1}S_{22}\right)^j U_2'\,Q_c\mathrm{E}(z_{t+j}|\mathcal{F}_t) + \lim_{\tau\to\infty}\left(T_{22}^{-1}S_{22}\right)^\tau \mathrm{E}(\mathbf{w}_{t+\tau}^e|\mathcal{F}_t) \\
&= -T_{22}^{-1}\sum_{j=0}^{\infty}\left(T_{22}^{-1}S_{22}\right)^j U_2'\,Q_c\,\left[\Phi^j\left(z_t-\mu_z\right)+\mu_z\right] \\
&= M\,z_t + M_c\,c_z, \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
\mu_z &= (I-\Phi)^{-1}c_z \\
\mathrm{vec}\,(M) &= \left[(\Phi'\otimes S_{22})-(I_m\otimes T_{22})\right]^{-1}\mathrm{vec}\,(U_2'Q_c)\,, \quad \text{and} \\
M_c &= \left[-T_{22}^{-1}\left(I_{N_e}-T_{22}^{-1}\,S_{22}\right)^{-1}U_2'\,Q_c - M\right](I-\Phi)^{-1}.
\end{aligned}$$

The derivation of M follows from the fact that

$$\sum_{j=0}^{\infty}\mathrm{vec}\left(A^jBC^j\right)=\sum_{j=0}^{\infty}(C'\otimes A)^j\mathrm{vec}\,(B)=\left[I-(C'\otimes A)\right]^{-1}\mathrm{vec}\,(B)\,.$$

The only other condition required for the above result is a bound on the growth of $\mathrm{E}(z_{t+\tau}|\mathcal{F}_t)$ so that

$$\lim_{\tau\to\infty}\left(T_{22}^{-1}S_{22}\right)^\tau \mathrm{E}(\mathbf{w}_{t+\tau}^e|\mathcal{F}_t)=\lim_{\tau\to\infty}\left(T_{22}^{-1}S_{22}\right)^\tau\,M\,\mathrm{E}(z_{t+\tau}|\mathcal{F}_t)=0$$

This is clearly satisfied under assumption 1 even when the process z_t contains deterministic polynomial trends (but it is satisfied by some explosive processes as well).

On the other hand, the non-explosive part can be solved by backward substitution:

$$\begin{aligned}
\mathbf{w}_{t+1|t}^s &= S_{11}^{-1}\,T_{11}\,\mathbf{w}_t^s + S_{11}^{-1}\,T_{12}\,\mathbf{w}_t^e - S_{11}^{-1}\,S_{12}\,\mathbf{w}_{t+1|t}^e + S_{11}^{-1}\,U_1'\,Q_c \\
&= S_{11}^{-1}\,T_{11}\,\mathbf{w}_t^s + S_{11}^{-1}\,\left(T_{12}\,M\;-S_{12}\,M\,\Phi+U_1'\,Q_c\right)\,z_t
\end{aligned}$$

Using (10), defining the expectational error $\eta_{t+1}=\mathbf{x}_{t+1}^f-\mathbf{x}_{t+1|t}^f$ and noting that the $\mathbf{x}_{t+1}^p=\mathbf{x}_{t+1|t}^p$ by predeterminedness, the above yields:

$$\mathbf{w}_{t+1}^s = S_{11}^{-1}\,T_{11}\,\mathbf{w}_t^s + S_{11}^{-1}\,\left(T_{12}\,M\;-S_{12}\,M\,\Phi+U_1'\,Q_c\right)\,z_t + V_{fs}'\,\eta_{t+1} \qquad (15)$$

Now, we turn to the solution to the canonical LiRE model (5). There are three cases, which are outlined in the following result.

Proposition 3.1 *Under assumptions 1, 2 and 3, we distinguish three cases for the solution to the canonical LiRE model (3):*

1. *If $N_e > N_f$, there exists no solution.*

2. *If $N_e = N_f$, there exists a unique solution.*

3. *If $N_e < N_f$, there exist infinitely many solutions.*

Proof. Equation (11) implies:

$$V'_{fe}\,\mathbf{x}_t^f = -V'_{pe}\,\mathbf{x}_t^p + M\,z_t + M_c\,c_z. \tag{16}$$

Since $\mathbf{x}_t^p$ and z_t are given, these are N_e equations in N_f unknowns. When $N_f < N_e$ no solution exists for all t, unless the square N_e-dimensional matrix $\left(V'_{fe},\ -V'_{pe}\,\mathbf{x}_t^p + M\,z_t\right)$ is of reduced rank N_f for all t, which implies that $N_e - N_f$ equations are redundant. But this in turn requires that $\mathbf{x}_t^p, z_t$ are constant across time, which is not true.

When $N_f = N_e$, there exists a unique solution, which, by assumption 3, is:

$$\mathbf{x}_t^f = -(V'_{fe})^{-1}\,V'_{pe}\,\mathbf{x}_t^p + (V'_{fe})^{-1}\,M\,z_t + (V'_{fe})^{-1}\,M_c\,c_z. \tag{17}$$

When $N_f > N_e$, there are infinite solutions, since there are more unknowns than equations in (16). This multiplicity of solutions can be characterized by $N_f - N_e$ arbitrary martingale difference sequences (MDS) ξ_t, and a $(N_f - N_e) \times m$ matrix of free parameters Γ, as follows. Consider the system of solutions (15) and (14):

$$\begin{pmatrix} \mathbf{w}_t^s \\ \\ \mathbf{w}_t^e \end{pmatrix} = V'\mathbf{x}_t = \begin{pmatrix} B & 0 \\ \\ 0 & 0 \end{pmatrix} V'\,\mathbf{x}_{t-1} + \begin{pmatrix} D \\ \\ M\Phi \end{pmatrix} z_{t-1} + \begin{pmatrix} V'_{fe} \\ \\ 0 \end{pmatrix} \eta_t + \begin{pmatrix} 0 \\ \\ M \end{pmatrix} \underbrace{(z_t - z_{t|t-1})}_{u_t},$$

where $B = S_{11}^{-1}\,T_{11}$ and $D = (T_{12}\,M\, - S_{12}\,M\,\Phi + U'_1\,Q_c)$. Solving for the non-predetermined variables, x_t^f, yields:

$$\mathbf{x}_t^f = V_{fs}B\ V'_{ps}\mathbf{x}_{t-1}^p + V_{fs}B\ V'_{fs}\mathbf{x}_{t-1}^f + (V_{fs}D + V_{fe}M\Phi)z_{t-1} + V_{fs}V'_{fs}\eta_t + V_{fe}M\ u_t$$

By definition, η_t satisfies:

$$\eta_t = V_{fs}V'_{fs}\eta_t + V_{fe}M\ u_t \tag{18}$$

It is easy to see that, when $N_f = N_e$, (18) solves to $(V'_{fe})^{-1}M\ v_t$. At the other extreme, when $N_e = 0$, $V_{fs}V'_{fs} = I_{N_f}$ and $V_{fe} = 0$, so that η_t is completely indeterminate. In that case, it can be orthogonally decomposed to $\xi_t + \Gamma u_t$, where ξ is a MDS orthogonal to u_t, usually referred to as a 'sunspot' shock. In the intermediate cases $0 < N_e < N_f$, the matrix $I - V_{fs}V'_{fs}$ is of rank N_e, thus leaving $N_f - N_e$ elements in η_t indeterminate. ■

3.4 Solving the original model

Given a unique solution for $\mathbf{x}_t^f$ found above and using the definition of $\mathbf{x}_t$ in (7), we can infer the solution for the endogenous variables y_t as a function of its lags and z_t:

$$y_t = \sum_{i=1}^{k} \Pi_i\, y_{t-i} + Q_s\, z_t + c \tag{19}$$

where

$$\begin{pmatrix}\Pi_k & \dots & \Pi_1\end{pmatrix} = \left(-(V'_{fe})^{-1}\, V'_{pe}\right)_{[:n][\cdot]},\quad Q_s = \left((V'_{fe})^{-1}\, M\right)_{[:n][\cdot]} \quad \text{and} \quad c = \left((V'_{fe})^{-1}\, M_c\right)_{[:n][\cdot]}\, c_z.$$

The notation $(X)_{[:n][\cdot]}$ denotes selecting the first n rows of a matrix X. In `PcLiRE`, the solution is implemented by the function `PcLiRE::Solve()`, or `PcLiRE::Print()`, as exemplified below.

---------------------------- Sample Code ----------------------------

```
#include "PcLiRE1.ox"
main()
{
    decl cY, model, mA1, mA0, mAb1;
    cY = 2;                         // number of endogenous variables.
    model = new PcLiRE(cY);         // create the PcLiRE object.
    mA1 = <-0.591, 0; 0, 0>;        // specify coefficients.
    mA0 = <1, -0.05; 0, 1>;
    mAb1 = <-.378, 0; 0.1, -0.9>;
    model.SetYParameter({mA1, 1, 0}, {mAb1, -1, 0}, {mA0, 0, 0});
                                    // set LiRE equation parameters.
    model.Print();
}
```

---------------------------- Output ----------------------------

```
------------------- PcLiRE (1.0) --------------------
y is (2 x 1).

The structural model is:

A0 y[t] + A1 y[t+1|t] + Ab1 y[t-1] = z[t]

z[t] = u[t]

A0 =
       1.0000    -0.050000
      0.00000       1.0000
A1 =
     -0.59100      0.00000
      0.00000      0.00000
Ab1 =
     -0.37800      0.00000
      0.10000     -0.90000
u ~ N(0,1)

The solved model is a VAR(1):

y[t] = Pi1 y[t-1] + Qs z[t]
Pi1 =
      0.51169      0.27159
     -0.10000      0.90000
Qs =
       1.4335      0.30176
      0.00000       1.0000
```

3.5 Verifying the solution

The solution coefficients Π_i and Q_s, satisfy some restrictions, the validity of which could provide a verification of the proposed solution. In particular, abstracting from the deterministic term c, we can write (19) in companion form, using $Y_t = (y_t', \ldots, y_{t-k+1}')'$:

$$Y_t = \Pi\, Y_{t-1} + \begin{pmatrix} Q_s \\ \\ 0 \end{pmatrix} z_t,$$

where Π is the usual square companion matrix of dimension equal to nk. Define $\Pi^j_{i_1,i_2} = (\Pi^j)_{i_1,i_2}$, namely the (i_1, i_2) $n \times n$ block of the $nk \times nk$ matrix Π^j, where $i_1, i_2 = 1, \ldots, k$. Thus,

$$y_{t+j|t} = \sum_{i=1}^{k} \Pi^j_{1i}\, y_{t-i+1} + Q_s\, \Phi^j\, z_t$$

Substituting into the LiRE model (3) we have:

$$\sum_{i=0}^{k} A_{-i}\, y_{t-i} + \sum_{j=1}^{l} A_j \sum_{i=1}^{k} \Pi^j_{1i}\, y_{t-i+1} = \left(Q - \sum_{j=1}^{l} A_j\, Q_s\, \Phi^j \right) z_t.$$

By defining

$$
\begin{aligned}
B_0 &= A_0 + \textstyle\sum_{j=1}^{l} A_j \Pi_{11}^{j}, \\
B_{i-1} &= A_{-i+1} + \textstyle\sum_{j=1}^{l} A_j \Pi_{1i}^{j}, \quad i = 2, \ldots, k \\
B_k &= A_{-k}
\end{aligned}
$$

the above can be re-written as:

$$
\sum_{i=0}^{k} B_i \, y_{t-i} = \left(Q - \sum_{j=1}^{l} A_j \, Q_s \, \Phi^j \right) z_t \tag{20}
$$

Hence, the restrictions implied by Rational Expectations can be found by matching the coefficients in (19) and (20):

$$
\begin{aligned}
B_0 \, \Pi_i + B_i &= 0, \qquad i = 1, \ldots, k, \quad \text{and} \\
B_0 \, Q_s + \textstyle\sum_{j=1}^{l} A_j \, Q_s \, \Phi^j &= Q
\end{aligned}
$$

3.6 Other issues

3.6.1 Unit roots

The presence of unit roots in the LiRE polynomial $A(\lambda)$ (or equivalently, the canonical polynomial $A_c \, \lambda - B_c$) does not pose any problem for the unique solution given in (19) insofar as there are as many explosive roots as there are non-predetermined variables. Any additional unit roots in the non-explosive part will typically be present in the solution for y_t, which will be integrated. Although we give no proof of this conjecture, it can be easily checked in any given application by looking at the eigenvalues of the companion matrix Π. These can be reported by the program at the user's request, using the function `PcLiRE::CointAnalysis()`, as shown in the following example, whereby we also print out the roots of the LiRE polynomial $A(L)$.

-------------------------- Sample Code --------------------------

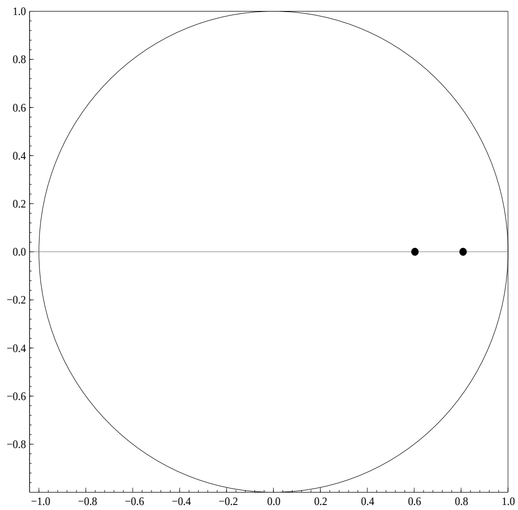
```
#include <oxstd.h>
#include "PcLiRE1.ox"

main()
{
    decl cY, model, mA1, mA0, mAb1;
    cY = 2;                          // number of endogenous variables.
    model = new PcLiRE(cY);          // create the PcLiRE object.
    mA1 = <-0.591, 0; 0, 0>;         // specify coefficients.
    mA0 = <1, -0.05; 0, 1>;
    mAb1 = <-.378, 0; 0.1, -0.9>;
    model.SetYParameter({mA1, 1, 0}, {mAb1, -1, 0},
        {mA0, 0, 0});   // set LiRE equation parameters.
```

```
    model.PrintRoots(TRUE);
    model.Solve();
    model.CointAnalysis();
}
```

---------------------------- Output ----------------------------

```
Roots of PcLiRE polynomial:
        alpha         beta      modulus
      0.62294       1.0327      0.60319
      0.87904       1.0872      0.80850
     -0.95351      0.00000  1.9965e+016
      0.65156      0.55200       1.1804
Eigenvalues of companion matrix Pi
      0.80850
      0.60319
```



Plot of the eigenvalues of the companion matrix.

Notice the role of the φ parameter in the example (4). When $\varphi = 0$, together with the restriction $\gamma_f + \gamma_b = 1, \gamma_f < \gamma_b$, the system (inflation) has a unit root.[6] In that case, the output of PcLiRE is:

---------------------------- Output ----------------------------

```
Roots of PcLiRE polynomial:
        alpha         beta      modulus
      0.82462      0.82462       1.0000
      0.94420       1.0491      0.90000
      0.72904      0.48603       1.5000
      0.95131      0.00000  1.2809e+016
Eigenvalues of companion matrix Pi
       1.0000
      0.90000
There are 1 unit roots. Long-run matrix
      0.00000      0.18750
      0.00000     -0.10000
alpha =
     -0.18750
```

[6] See Mavroeidis (2002, chapter 4).

```
      0.10000
beta' =
      0.00000      -1.0000
---------------------------------------------------------------
```

3.6.2 Back-dated expectations

When the LiRE model contains back-dated expectational terms, such as $\mathrm{E}(y_{t+1}|\mathcal{F}_{t-1})$, the solution provided by proposition 3.1 is no longer sufficient. The reason is that it is not adequate to express the non-predetermined variables $\mathbf{x}_t^f$ as a function of predetermined ones $\mathbf{x}_t^p$, since the latter will contain expectational terms such as $\mathrm{E}(y_{t+1}|\mathcal{F}_{t-1})$, which have not been specified.

As mentioned above, this complication can be dealt with by simple algebraic manipulations, and will be part of a future implementation of the program [to be completed!].

Preliminary notes on the treatment of back-dated expectations

Casting into PcLiRE form Transforming the general LiRE model (1) into the simplified form involving only contemporaneous expectations (3) is straightforward. Define the new variable $\mathbf{y}_t = (y_t', y_{t+1|t}', \ldots, y_{t+H|t}')'$, where $H = \max\{l-1, \max[j\ : M_{ij} \neq 0, i > 0]\}$. Then, the model in $\mathbf{y}_t$ admits the representation:

$$\sum_{i=0}^{k} A_{-i}\,\mathbf{y}_{t-i} + A_1\,\mathrm{E}(\mathbf{y}_{t+1}|\mathcal{F}_t) = \tilde{Q}\,z_t, \qquad t = 0, 1, \ldots \tag{21}$$

with A_1, $A_{-i}, i = 0, \ldots, k$ and $\tilde{Q}$ given by:

$$A_1 = \begin{pmatrix} 0 & M_{0,H+1} \\ -I_{nH} & 0 \end{pmatrix} \quad A_0 = \begin{pmatrix} M_{00} & \ldots & M_{0H} \\ 0 & I_{nH} & \end{pmatrix}, \quad A_{-i} = \begin{pmatrix} M_{i0} & \ldots & M_{iH} \\ & 0 & \end{pmatrix},$$

$i = 1, \ldots, k$ and $\tilde{Q} = (Q', 0)'$. Whenever a unique solution for $\mathbf{y}_t$ exists, it can be expressed in the form (19), namely

$$\mathbf{y}_t = \sum_{i=1}^{k} \Pi_i\,\mathbf{y}_{t-i} + Q_s\,z_t + c$$

The first n of the above $n(H+1)$ equations give y_t as a function of predetermined variables. However, since the terms $\mathbf{y}_{t-i}$ that lie on the RHS of the above equation contain variables like $y_{t+j-i|t-i}$, for $i > 0$, this is solution is still not cast explicitly in a VAR form. This can be done with a few manipulations. We illustrate in the following simple example.

Example Consider the partial adjustment model:

$$y_t + \beta\ y_{t+1|t-1} + \gamma\ y_{t-1} = z_t$$

Casting this into the form (21) involves using the variable $\mathbf{y}_t = (y_t, y_{t+1|t}, y_{t+2|t})'$, $H = 2$, and $k = 1$. The unique solution for $\mathbf{y}_t$, if it exists, is:

$$\begin{pmatrix} y_t \\ y_{t+1|t} \\ y_{t+2|t} \end{pmatrix} = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t|t-1} \\ y_{t+1|t-1} \end{pmatrix} = \begin{pmatrix} Q_s^1 \\ Q_s^2 \\ Q_s^3 \end{pmatrix} z_t \tag{22}$$

Taking expectations of the above conditional on $\mathcal{F}_{t-1}$ yields the equations:

$$\begin{pmatrix} y_{t|t-1} \\ y_{t+1|t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \Pi_{11} \\ \Pi_{21} \end{pmatrix}}_{\bar{\Pi}_1} y_{t-1} + \underbrace{\begin{pmatrix} \Pi_{12} & \Pi_{13} \\ \Pi_{22} & \Pi_{23} \end{pmatrix}}_{\bar{\Pi}} \begin{pmatrix} y_{t|t-1} \\ y_{t+1|t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} Q_s^1 \\ Q_s^2 \end{pmatrix}}_{\bar{Q}_s} \Phi\ z_{t-1}$$

If $I_2 - \bar{\Pi}$ is invertible, we can solve for $(y_{t|t-1}, y_{t+1|t-1})$ to get

$$\begin{pmatrix} y_{t|t-1} \\ y_{t+1|t-1} \end{pmatrix} = (I - \bar{\Pi})^{-1}\bar{\Pi}_1\ y_{t-1} + (I - \bar{\Pi})^{-1}\bar{Q}_s\Phi\ z_{t-1}$$

and substituting this back in the first equation of the solution (22) yields the final solution in VAR form.

3.7 Simulation and impulse responses

For simulation studies, we need to specify a distribution for the forcing variables z_t, and then draw samples from the resulting distribution of y_t. This can be done in `PcLiRE` directly using using the commands:

`PcLiRE::SetUDistribution(iDistType,vDf)`

`PcLiRE::SetUParameter(mSigma), and PcLiRE::Generate(cT).`

The first one determines the type of distribution of the innovations u_t. Currently, the program allows multivariate normal (`iDistType = MVNORMAL`) or Student's t (`iDistType = T_DIST` and `vDf` is a vector of positive integers, carrying the degrees of freedom for each of the variables in u_t). The second function is used to set the covariance matrix of the innovations. Importantly, this can be singular (can contain zeros on the diagonal) so that we can simulate models where the stochastic variation is of lower dimension than the system. The last function returns a matrix Y of dimension $T \times n$. Concerning the initial conditions, the program draws Z_0 randomly from a distribution that would make z_t stationary. We do not use the same approach in the case of Y_0, since it is unjustified when unit roots are present in the y_t equation. Following the approach in PcNaive, Y_0 is set to zero by default.

Alternative simulation using PcNaive The above simulations can be performed also using the `PcNaiveDgp` class, which comes with the standard Ox 3.2 release, see Doornik and Hendry (1998) for details. For that purpose, the user can recover the coefficients of the solved model using the function `PcLiRE::GetSolutionPar()` in order to use them in PcNaive. The example below shows how 100 observations can be drawn from the previous model directly using `PcLiRE`, or indirectly using `PcNaiveDgp`.

--------------------------- Sample Code ---------------------------

```
#include <oxstd.h>
#include "PcLiRE1.ox"
#import <pcnaive>

main()
{
    decl model, mA1, mA0, mAb1;
    model = new PcLiRE(2);                          // create the PcLiRE object.
    mA1 = <-0.591, 0; 0, 0>;                           // specify coefficients.
    mA0 = <1, -0.05; 0, 1>;
    mAb1 = <-.378, 0; 0.1, -0.9>;
    model.SetYParameter({mA1, 1, 0}, {mAb1, -1, 0},
        {mA0, 0, 0});                           // set LiRE equation parameters.
    model.Solve();

/*  direct simulation:  */
    model.SetUParameter(unit(2));                              // variance of u.
    model.SetUDistribution(MVNORMAL, 0);                   // distribution of u.
    decl mdata;
    mdata = model.Generate(100);

/*  indirect simulation: */
    decl mQ, mPi;
    [mQ, mPi] = model.GetSolutionPar();                      // in array {Q, Pi}.
    decl dgp = new PcNaiveDgp(2,0);                    //create PcNaiveDgp object.

    dgp.SetYParameter(zeros(2,2), mPi, 0, zeros(2,1));
    dgp.SetDistribution(U_DGP, MVNORMAL, zeros(2,1), mQ*mQ');
    mdata = dgp.Generate(100);
}
```

----

4 Work in progress

The following features are still in preparation:

1. FIML estimation. Requires treatment of the case where there exist multiple solutions. In that case, the model is no longer a finite order VAR, but rather a VARMA process, with a moving average component depending on the multiplicity of the solutions, à la Broze, Gouriéroux, and Szafarz (1995). Some discussion is given in Pesaran (1987, chapter 7), but this method is, to date, unpopular due to the multivariate moving average component.

2. Optimal control. Can be done with "brute force" methods, but a more efficient approach would involve the Ricatti equation.

3. Discussion of identification. Identification of RE models depends on the nature of the forcing variables and the type of solution, see Pesaran (1987, chapter 6). I am not aware of any generic rank condition for identification, since this also depends very much on the specifics of each model, i.e. which structural parameters are 'free'. However, once the reduced form is found, using PcLiRE, then the analysis of Mavroeidis (2003) shows how generic as well as empirical identification based on any information set can be checked, by computing the *concentration parameter*. I have a separate set of functions to do that, but the aim is to integrate them in PcLiRE.

References

Anderson, G. A. and G. R. Moore (1985). A linear algebraic procedure for solving perfect foresight models. *Economics Letters 17*, 247–252.

Binder, M. and H. M. Pesaran (1995). Multivariate rational expectations models: A review and some new results. In Pesaran and Wickens (Eds.), *Handbook of applied econometrics*, Volume Macroeconomics, pp. 139–187. Basil-Blackwell.

Blanchard, O. J. and C. M. Kahn (1980). The solution of linear difference models under rational expectations. *Econometrica 48*, 1305–11.

Broze, L., C. Gouriéroux, and A. Szafarz (1995). Solutions of multivariate rational expectations models. *Econometric Theory 11*(2), 229–257.

Doornik, J. A. and D. F. Hendry (1998). Monte Carlo simulation using PcNaive for Windows. Unpublished typescript, Nuffield College, University of Oxford.

Galí, J. and M. Gertler (1999). Inflation dynamics: a structural econometric analysis. *Journal of Monetary Economics 44*, 195–222.

Johansen, S. (1995). *Likelihood-based Inference in Cointegrated Vector Autoregressive Models.* Oxford: Oxford University Press.

Klein, P. (2000). Using the generalized Schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control 0*, 0–0.

Mavroeidis, S. (2002). *Econometric Issues in Forward-Looking Monetary Models.* DPhil thesis, Oxford University, Oxford.

Mavroeidis, S. (2003). Identification and mis-specification in forward-looking models. Working paper, University of Amsterdam, Amsterdam.

Pesaran, M. H. (1987). *The limits to Rational Expectations.* Oxford: Blackwell Publishers.