CVA Models for Linking Default & Exposure

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Overview

This talk will cover models for looking at right-way/wrong-way risk

• Some will be familiar, some (hopefully) less so

Different models appropriate in different circumstances and to answer different questions

• What is my credit-market cross gamma bleed?
• What is my post-default exposure likely to be?
• How wrong can my model be?

Look at a few examples of wrong-way risk and models for investigating them

• Stochastic Hazard Rates
• Copulas
• Jump at default modelling
• Direct default modelling and limits on CVA
Linking Exposure & Default

General expression for (unilateral) CVA

\[
CVA^U = (1 - R_C) E \left[ \frac{N_0}{N_{\tau_C}} V_{\tau_C}^+ 1_{\tau_C \leq T} \mid F_0 \right] \]

\(N_t\): Numeraire
\(\tau_C\): Time of counterparty default
\(V_t\): Portfolio MTM

If we assume that time of default and MTM are independent, CVA follows as integration over Expected Exposure

\[
CVA^U = (1 - R_C) \int_0^T dP_{\tau_C} (s) E \left[ \frac{N_0}{N_s} V_s^+ \right] \\
\approx (1 - R_C) \sum_{n=1}^N E \left[ \frac{N_0}{N_{T_n}} V_{T_n}^+ \right] P(T_{n-1} < \tau_C < T_n)
\]

\(N_t\): Numeraire
\(\tau_C\): Time of counterparty default
\(V_t\): Portfolio MTM
Modelling Implications

Natural to challenge independence assumption on general and specific grounds

• Credit market is not independent of other market factors
• Trade MTM may be systemically linked to counterparty creditworthiness
  – Effect of oil price on a producer and their swap hedges
• Default event may impact the price at which we can close out
  – FX rate could move following default of a sovereign-backed name
Modelling Implications

Link between exposure and default is a major contributor to P/L of a CVA desk

- If no default, realised credit-market cross gamma gives a continuous hedging bleed (or carry)
- At default, the P/L is driven by the exposure that has been hedged

Important to have models that incorporate this link

- CVA desks are the experts on the dependence between credit and market and need to be able to price and manage the risk
- Accurate pricing provides appropriate incentives for the firm as a whole: a key reason for having a dedicated CVA desk
No Perfect Answer

Of the available models none is perfect

- Different models capture different aspects of the default and exposure linkage
- All models require extra parameters for specification and/or calibration. These can be hard to observe
  - CDSwaption vols, Credit vs. Market correlations, Credit vs Credit correlations
- Hedging strategy implied by models may be impractical
  - Frequent rebalance of credit hedge to capture cross-gamma is expensive

Having a range of models gives a suite of tools to understand the effect of the link between exposure and default

Some models may be more used for pricing; others give insight into hedging
Stochastic Hazard Rate Modelling

Natural first step is a dynamic hazard rate framework

Default driven by stochastic intensity

\[ P(\tau_C > t) = E\left[ \exp(-\int_0^t \lambda_s \, ds) \right] \]

Intensity can be correlated with other market factors

Conditioned on a given path for the intensity, default time and exposure are independent

Partitioning across default times, CVA formula becomes

\[ CVA^U = (1 - R_C) \sum_{n=1}^{N} E\left[ \left( e^{-\int_0^{T_n-1} \lambda_s \, ds} - e^{-\int_0^{T_n} \lambda_s \, ds} \right) \frac{N_0}{N_{T_n}} V_{T_n}^+ \right] \]
Stochastic Hazard Rate Modelling

Write this as

\[ CVA^U = (1 - R_C) \sum_{n=1}^{N} E \left[ \frac{N_0}{N_{T_n}} \left( e^{-\int_0^{T_{n-1}} \lambda_s \, ds} - e^{-\int_0^{T_n} \lambda_s \, ds} \right) \frac{V_{T_n}^+}{P(\tau_C \in [T_{n-1}, T_n])} \right] \times P(\tau_C \in [T_{n-1}, T_n]) \]

This is interpreted as exposure given default

Ratio of stochastic default probability to deterministic default probability is acting as a path-reweighting/measure correction

Same formalism is applicable to bilateral calculation

- Two hazard rates (correlated with each other and with market)
- Conditioned on hazard rate trajectories, default times are independent
Hazard Rate Example

10 Year Receive fixed IRS

Strong negative correlation between rates and credit

Wrong-way is clear from exposure profile
Hazard Rate Dynamics

Several choices for the hazard rate models:

**Gaussian**
\[ d\lambda_t = \kappa_t (m_t - \lambda_t) dt + \sigma_t \, dW_t \]
- Straightforward calibration and simulation
- Negative hazard rates: negative default densities

**CIR**
\[ d\lambda_t = \kappa_t (m_t - \lambda_t) dt + \sigma_t \sqrt{\lambda_t} \, dW_t \]
- Tractable calibration
- Usual issues of simulating a square-root process

**Black-Karasinski**
\[ d\ln \lambda_t = \kappa_t (m_t - \ln \lambda_t) dt + \sigma_t \, dW_t \]
- Simulation straightforward
- Calibration tricky
Hazard Rate Models

Hazard rate models give a good starting point for correlating market and credit factors

Notwithstanding observability, parameters are amenable to calibration

- Calibrate vols to CDSwaption
- Historic correlations of credit spreads with market factors

Setup can be extended to bilateral calculation

However, they have a significant limitation

- All we are doing is correlating the probabilities of default rather than time of default itself.
- Consequently link between default times of us, counterparty and exposure is rather weak
Using Copulas

Copulas give a framework for imposing more a more direct linkage between default times

How can these be applied to CVA?

• Bilateral CVA (with discrete partitioning) involves terms of the form

\[
E\left[V(T_n) | \tau_c \in [T_{n-1}, T_n]\right] P\left(T_{n-1} \leq \tau_c \leq T_n, \tau_B > T_n\right)
\]

• We can use a copula to calculate the bucket default probability

• This provides a direct link between default time of counterparty and us

• We can use a dynamic hazard rate model to get conditional expectations and path-dependent survival probabilities to link this with exposure
Using Copulas

Copula joins marginal default probabilities

\[ p_n = P(\tau_C \leq T_n), \quad q_m = P(\tau_B \leq T_m) \]

\[ P(\tau_C \leq T_n, \tau_B \leq T_m) = C(p_n, q_m) \]

Get bucket default probability from

\[ P(T_{n-1} \leq \tau_C \leq T_n, \tau_B > T_n) = p_n - p_{n-1} - C(p_n, q_m) + C(p_{n-1}, q_m) \]
Effect of Copulas

Copulas change the first-to-default density weights we apply to forward and reverse exposures

Example: Gaussian copula for different correlations

- B hazard rate 3% per year, C hazard rate 2% per year
Copula Example

Example Impact: 10y XCCY Swap
Jump at Default Modelling

A very different approach is to model a market jump at default

Popular choice in for Emerging Market trades:

- A fact of life in the quanto CDS market: implied FX devaluation makes quanto and regular CDS spreads very different
- When pricing EM trades under this model there will be a jump in PV at default
- What is the impact for CVA?
Jump at Default Modelling

Introduce a jump into a log-normal FX dynamic

\[
\frac{dX(t)}{X(t^-)} = \begin{cases} 
(r_i^d - r_i^f) dt + \sigma_i dW_t - J(t)[dU_t - \lambda_i dt] & t < \tau \\
(r_i^d - r_i^f) dt + \sigma_i dW & t > \tau 
\end{cases}
\]

\(U\) is a Poisson process with intensity \(\lambda\) (deterministic/stochastic)

Link counterparty default to first jump of this process

- Makes FX jump simultaneously with counterparty default
- Hazard rate can be calibrated to counterparty credit
- Jump risk can in principle be hedged with FX option or CDS
FX Jump Model Implications

Additional drift of $J(t) \lambda(t) \, dt$ to compensate for the jump risk.
Relative devaluation size at default is given by $J(t)$.
Distribution of $X(t)$ is no longer lognormal
Compared to BS model (no jumps), some volatility will be transferred to the jump $J$

- Typically spot vols are lower in this model

The distribution conditioned on the default time is lognormal.
The FX forward price is still a martingale (as it must be)
Modelling framework uses FX rate with additional drift correction

- This is the FX rate conditioned on no default

When we need the devalued rate we multiply by $(1 - J(t))$
CVA in Devaluation Model

We need to define what we mean by CVA

- We want counterparty credit risk to sit with the CVA desk, and FX jump risk to sit with the EM desk
- But FX jump risk and default jump risk are the same... How to split them?

One approach: CVA is everything that has recovery dependence

- Corresponds to pricing underlying fair value in Merton model (FX with Jumps), and CVA as a correction to this

$$CV^A_B = -(1 - R_C) \int_0^T g_C(s) E\left[ \frac{N_0}{N_t} \text{Post Default Model, Devalued FX} \right] ds$$

$$- (1 - R_B) \int_0^T g_B(s) E\left[ \frac{N_0}{N_t} V^{-}(s, X(s)) \right] ds$$

- Reverse exposure uses jumps in pricing model. Forward exposure does not
- In reality, FX will still probably have jump risk after default, but model does not capture this
Hedging & Calibration

In this model, any FX hedge will also jump on default

• Hedge ratios need to take this into account
• Again, details depend on split between EM and CVA trading

Several parameters require calibration

• Hazard rate model
• Term structure of jumps
• FX spot volatility

Hazard rates calibrated as before

Both quanto CDS and FX vanilla options depend on jumps and vol

• Typically an iterative approach works fine (repeatedly hold one fixed and calibrate the other)
• A direct joint calibration is also possible
Effect of FX devaluation

Example: Expected exposure for different jump sizes

- 10Y near-ATM FX forward; bank pays 1.3m USD, receives 1m EUR
- EUR devalues on counterparty default
- Trade MTM (from bank perspective) drops on default: right-way risk in CVA
Effect of FX devaluation

Same trade: Expected reverse exposure for different jump sizes

- Does not use the devalued FX rate: we default first
- FX drift compensates for the market risk of their default
  - The bigger J, the more the trade will tend to be ITM for us if they do not default
- Reverse exposure decreases as jump size increases
Effect of FX devaluation on CVA

Example trades:

- 10Y ATM FX forward; bank pays 1.3m USD, receives 1m EUR (blue)
- 10Y ITM FX forward; bank pays 0.5m USD, receives 1m EUR (red)

A large jump lowers both our exposure and theirs, so bilateral CVA shrinks
Direct Sampling of Default

So far we have separated the portfolio and the default density. Another approach is to sample them jointly:

\[ CVA^U = (1 - R_C)E \left[ \frac{N_0}{N_{\tau_C}} V_{\tau_C}^+ 1_{\tau_C \leq T} \mid F_0 \right] \]

Idea is to draw default time and exposure and link them directly.

How to do this efficiently?
Direct Sampling of Default

Exposure sampling ideas of Sokol\(^1\) form the basis of this approach. Fundamental idea is to separate:

- Simulation of MTM distribution (slow)
- Simulation of defaults (fast)

Instead of simulating the MTM at the default time, we sample from its previously calculated distribution.

Introducing correlation into this approach gives a framework capable of capturing strong right-way and wrong-way effects.

Here we consider unilateral CVA only, but bilateral is possible.

\(^1\) Alexander Sokol, Fast Monte Carlo CVA using exposure sampling method, Marcus Evans CVA conference, 2011
In Detail: MTM Sampling

We want to approximate the distribution (or inverse cdf) of $V(T_n)$

Specify a set of percentiles $0 < p_1 < \ldots < p_r < 1$

Corresponding MTM values $V_1 < \ldots < V_r$ are estimated empirically from Monte Carlo simulation of the portfolio (assume credit-independent)

Inverse CDF is built by interpolating (eg linearly) from points $(V_i, p_i)$
In Detail: Default time sampling

We want to approximate the distribution of $\tau_C$ to be able to sample from it

- Need to build inverse cumulative distribution function (cdf)

Tabulate default time values and associated survival probabilities (calculated from credit curve), from $T_0$ to 40 years typically.

- Recall $P(\tau_C > t) = \exp(-\int_0^t \lambda ds)$

- Therefore $\exp(-\int_0^{\tau_C} \lambda ds)$ has uniform distribution

To generate a default time value:

- Generate a uniform random number $U_1$

- Find the time interval $[t_n, t_{n+1}]$ it falls into

- Between $t_n$ and $t_{n+1}$, assume constant hazard rate $\lambda$, therefore:

$$p_{n+1} = p_ne^{-\lambda(t_{n+1}-t_n)} \quad U_1 = p_ne^{-\lambda(\tau_C-t_n)} \quad \Rightarrow \tau_C$$
Complete calculation

Use 2 random numbers to sample default time and MtM value:

• 1st step: find default time from random number U1 and find corresponding simulation date (choose endpoint)

• 2nd step: sample MtM value (floor at 0) from that cdf with random number U2

Average to get CVA

This simulation is fast so it can be done using many default time and MtM samples
Introducing correlation

In this framework, the first idea is to correlate directly the portfolio value and the default time.

Use copula to generate joint distribution of $U_1$ and $U_2$.

Correlation is controlled by single copula parameter $p$ (correlation is with the whole portfolio).

- $p > 0$: default is more likely when portfolio is ITM for the bank (wrong-way risk).
- $p < 0$: is the opposite (right-way risk).

$C_{\rho}(\tau) \sim \text{U}(0, 1)$

MtM value

$CDF_V(t_2)^{-1}$

$CDF_{\tau}^{-1}$
We compare the exposure sampling approach with a correlated hazard rate approach.

On a 2Y FX forward:

- Impact of correlation is very strong with the exposure sampling approach.
- CVA goes rapidly to 0 for right-way risk.
Validity of correlation structure

This is the kind of behaviour we are looking for

- However it is not desirable to have the CVA value drop to 0: this is effectively assuming that the counterparty is risk-free

There is also a more subtle issue

- Is the implied joint distribution a valid one?

What are the conditions for a valid structure?

- When we introduce correlation, marginal laws of default time and MtM must be satisfied:
- Default probabilities \( P(\tau_C \leq T_i) \), \( i=1, \ldots, N \) implied by the correlation structure must be the same as the ones implied by credit curve
- MtM full path probabilities \( P(V_{T_1} \leq v_1, \ldots, V_{T_N} \leq v_N) \) implied by the structure must be the same as the ones directly calculated from the Monte Carlo simulation
Validity of correlation structure

Here we are correlating two uniforms $U_1$ and $U_2$ with a copula

$U_1$ directly relates to the default time

• This guarantees that the marginal law of default is satisfied

But as $U_2$ relates only to the MtM distribution at the time of default, we cannot guarantee the condition for the full MtM distribution

• It would be satisfied if we had a copula formula for the full path:

$$P(\tau_C \leq T_{i+1}, V_{T_i} \leq v_1, \ldots, V_{T_N} \leq v_N) =$$

$$C_\rho(P(\tau_C \leq T_{i+1}), P(V_{T_i} \leq v_1, \ldots, V_{T_N} \leq v_N))$$

• But we can only write a copula formula for each date

Informally, the copula is allowing more than one default on each path
Consider a simple two period model

Worst case CVA has all T2 defaults on upper path

- So none can happen on upper path at T1

Extreme CVA does not come from maximising each bucket in turn

- Global optimisation (hard in general) is needed

Problem is that we have not respected conditional transition probabilities

T1: PD = 5%
T2: PD = 10%
Alternative approach for correlation

How to build a consistent framework to correlate the portfolio with default time?

We still want to simulate default time and MtM values but we need to look at the whole path distribution

We want to have a more realistic correlation structure

We will introduce a Markov framework that will make it easier to satisfy marginal conditions
Markov processes framework – Outline

Idea: view mark-to-market (MTM) and default process as discrete Markov processes

- Build empirical transition matrices for MTM from Monte Carlo simulation
- Calculate transition probabilities for default process from credit curve

Justification

- The different models (IR, FX, Commodities…) used for portfolio simulation are Markovian
- For a large number of trade, the value of the trade itself is Markov. Notable exceptions: Path-dependent options
- Default time is first jump of Poisson process which is Markov

If we can accurately estimate these Markov processes, we can generate paths for the joint process, with correlation

Calculate CVA from implied joint distribution
MTM Markov process

Evolution of the portfolio MTM (from Monte Carlo simulation):

- At each date, MTM distribution is divided into a number of buckets
- Bucket boundaries are calculated from a set of percentiles as in exposure sampling method

Implied transition probabilities: \( P(V_4 \in B_{4,2} \mid V_3 \in B_{t,3}) = \frac{1}{2} \)

We can build transition matrices:

\[
A_{k,k+1} = \begin{pmatrix}
A_{k,k+1}(i,j) = P(V_{k+1} \in B_{k+1,j} \mid V_k \in B_{k,i})
\end{pmatrix}
\]
Define a default process \( (\delta_k)_{k=0,\ldots,N} \) with 2 states:

- 0: Default has not happened at time \( T_k \)
- 1: Default has happened at time \( T_k \)

Transition probabilities are easily calculated from credit curve

Transition matrix:

\[
D_{k,k+1} = \begin{pmatrix}
0 & 1 - p_k \\
p_k & 1
\end{pmatrix}
\]

\[
p_k = P(\delta_{k+1} = 0|\delta_k = 0) = \exp\left(-\int_{T_k}^{T_{k+1}} \lambda_s \, ds\right)
\]
Joint paths simulation

We can create new paths from the estimated Markov processes

- We always start from state $(V_0, 0)$: initial MtM value is known, and default has not happened
- Evolve both processes simultaneously:

\[
A_{k-1,k} = \begin{pmatrix}
1 & 0
\end{pmatrix}
\]

\[
A_{k,k+1} = \begin{pmatrix}
i
\end{pmatrix}
\]

\[
U_1 \sim U(0,1)
\]

\[
U_1 < 0.8: \text{no default at time } k+1. \text{ Otherwise: default}
\]

This procedure is simply generating paths that are consistent with the Monte Carlo simulation and the credit curve
Correlating MtM and default

We can introduce right-way / wrong-way risk by correlating $U_1$ and $U_2$ with copula, as in marginal exposure sampling method

Here, it is easy to see that the marginal conditions are satisfied:

- $U_1$ and $U_2$ are still uniform, so transition probabilities of each process are not modified
- Because we assume that they are Markov, this is sufficient to imply that the full MTM distribution and the default time distribution are respected

Correlation structure is easy to understand: if $p>0$, then the portfolio value going up makes it more likely for the counterparty to default, and the opposite if $p<0$

- Perhaps closer to interpretation of instantaneous hazard rate/market correlation
Results

Set up

- Percentiles used for MtM buckets: 1%, 5%, 10%, 20%, …, 90%, 95%, 99%
- Typical interval between two simulation dates: 6 months for a 20Y portfolio

5Y swap (USD 20m)

Blue: Markov framework
Pink: Usual approach (correlation with hazard rate)

- CVA without correlation matches CVA calculated with the usual approach
- Impact of correlation is significantly stronger
- CVA does not go to 0 for right-way risk
Results

Set up

- Percentiles used for MtM buckets: 1%, 5%, 10%, 20%, ..., 90%, 95%, 99%
- Typical interval between two simulation dates: 6 months for a 20Y portfolio
- 30Y swap (USD 20m)

Blue: Markov framework
Pink: Usual approach (correlation with hazard rate)

- Difference between two methods is less visible on long-dated portfolios
Results

Impact of extreme correlation with Markov framework and usual method, for different maturities

The range of CVA is much wider when correlating MtM value with default time directly

- This is particularly true for short-dated trades and on wrong-way risk side

Behaviour is more reasonable at extreme correlations
Extending the Approach

Approach can be extended in several ways to introduce more realism or complexity

Bilateral framework
• Straightforward extension: model a second default process and look at first to default

Stochastic Credit Spreads
• Simulate bucket default probabilities at the same time as the MTM process
• Hazard rate is Markov, so these probabilities are also Markov
  – Calibrate another Markov transition matrix for this process
• Correlation between this and MTM process can be determined empirically
• Simulate MTM, default probability, and default indicator
  – Default indicators use stochastic default probabilities at each step
Extending the Approach

Multiple Portfolios

- Similarly, we can model multiple portfolios at once
- Each portfolio MTM has its own calibrated Markov process, with empirical correlations between them
- Joint simulation of portfolio MTM
  - Need to supply correlations between default indicators and MTM

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- Can look at loss distribution at book level
Other Uses

The idea that MTM is a Markov process can be exploited for other purposes

• Statistics such as RWA, CVA are calculated on a spot basis using today’s EE profile

• What is their distribution on a future date?

• To answer we need to calculate

\[
E \left[ V(t_n)^+ \mid F_{t_m} \right]
\]

\[
\begin{cases}
    t_n & \text{Exposure Calculation Date} \\
    t_m & \text{Statistic Calculation Date}
\end{cases}
\]

• In Markov framework we can simulate this directly

• Or we can use regression to approximate
Conclusions

Presented a range of ways to link default and exposure

- Dynamic hazard rates
- Copulas
- Jumps-at-default
- Direct default sampling

Interpretation and justification of model parameters is always hard

- Even when calibration is theoretically possible, relevant instruments are usually very illiquid

For this reason, a range of models provides a complementary set of ways to look at the problem

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