Forward mortality rates

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Agenda

- Why forward mortality rates?
- Defining forward mortality rates
- Market consistent measure
- Hedging
- Discussion
Why forward mortality rates?

- Valuing technical provisions and pricing longevity-linked securities requires consistent expectations of future mortality rates
  - C.f. forward interest rates embedded in yield curve for bond pricing
- Other approaches to forward mortality rates
  - Non-parametric – Zhu and Bauer (2011a, b, 2014)
  - Olivier-Smith model – Olivier and Jeffrey (2004), Smith (2005)
Defining forward mortality rates

- Hypothetical market in “longevity zeros” with price
  
  \[ \text{Price}(t, \tau) = B(\tau, \tau + t) \mathbb{E}_{\tau} t p_{x,\tau} \]

- Define
  
  \[ t P_{x,\tau}(\tau) = \mathbb{E}_{\tau} t p_{x,\tau} = \mathbb{E}_{\tau} \exp \left( - \sum_{u=0}^{t-1} \mu_{x+u,\tau+u} \right) \]

- Forward mortality rates in discrete time
  
  \[ \nu_{x,t}(\tau) = - \ln \left( \frac{t-\tau+1 P_{x-t+\tau,\tau}(\tau)}{t-\tau P_{x-t+\tau,\tau}(\tau)} \right) \]

  \[ t P_{x,\tau}(\tau) = \exp \left( - \sum_{u=0}^{t-1} \nu_{x+u,\tau+u}(\tau) \right) \]
Defining forward mortality rates

- We identify
  \[ \nu_{x,t}(\tau) = \mathbb{E}_{\tau} \mu_{x,t} \]
- Approximation due to Jensen’s inequality but tested numerically and reasonable (within 0.1%) across most ages and years
- Assume that short mortality rates are modelled by an age/period/cohort mortality model – Hunt and Blake (2014d)
  \[ \ln(\mu_{x,t}) = \eta_{x,t} = \alpha_x + \beta_x^T \kappa_t + \gamma_{t-x} \]
- Then
  \[ \nu_{x,t}(\tau) = \exp \left( \alpha_x + \beta_x^T \mathbb{E}_{\tau} \kappa_t + \frac{1}{2} \beta_x^T \text{Var}_{\tau}(\kappa_t) \beta_x + \mathbb{E}_{\tau} \gamma_{t-x} + \frac{1}{2} \text{Var}_{\tau}(\gamma_{t-x}) \right) \]
Defining forward mortality rates

- Assume random walk with drifts for the period functions
  \[ \kappa_t = \mu X_t + \kappa_{t-1} + \epsilon_t \]

- Deterministic functions may be included in drift, \( X_t \), for identifiability reasons – Hunt and Blake (2014b,c)

- Therefore

\[ \mathbb{E}_\tau \kappa_t = \kappa_\tau + \mu \sum_{s=\tau+1}^{t} X_s \]

\[ \text{Var}_\tau(\kappa_t) = (t - \tau) \Sigma \]
Defining forward mortality rates

- Use Bayesian approach to model and project the cohort parameters

- Fitted parameter estimates based on partial information

- Assume annual observations of each cohort providing new information

- Cohort parameter only known with certainty once observed over its entire life
Defining forward mortality rates

- Details get quite involved – see Hunt and Blake (2014a)

\[
E_{\tau} \gamma_y = M(y, \tau) = \sum_{s=0}^{\infty} \left[ \prod_{r=0}^{s-1} (1 - D_{\tau - y + r}) \right] \rho^s \left[ \gamma_{y-s}(\tau) + (1 - D_{\tau - y + s}) \beta(Y_{y-s} - \rho Y_{y-s-1}) \right]
\]

\[
Var_{\tau}(\gamma_y) \equiv V(y, \tau) = \sum_{s=0}^{\infty} \left[ \prod_{r=0}^{s-1} (1 - D_{\tau - y + r})^2 \right] (1 - D_{\tau - y + s}) \rho^s \sigma^2
\]

- However, this approach is necessary for measuring risk, as discussed later
Defining forward mortality rates

- Together, these give the forward mortality surface
- Difference < 0.1%, due to rounding errors in simulations
Market consistent measure

- In order to value liabilities or value securities, we need to convert the forward mortality surface from the historic to a market consistent measure.

- Use Esscher transform, see Gerber and Shiu (1994)

\[
\mathbb{E}^Q_{\exp}(\eta) = \frac{\mathbb{E}^P_{\exp}(Z\eta)}{\mathbb{E}^P_{\exp}(Z)}
\]

\[
Z_{x,t} = \beta_x^\top \Lambda \kappa_t + \lambda^\gamma \gamma_{t-x}
\]

\[
\Lambda = \begin{pmatrix}
\lambda^{(1)} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda^{(N)}
\end{pmatrix}
\]

\[
\nu^Q_{x,t}(\tau) = \exp \left( \beta_x^\top \Lambda V \text{ar}_\tau^P(\kappa_t) \beta_x + \lambda^\gamma V \text{ar}_\tau^P(\gamma_{t-x}) \right) \nu^P_{x,t}(\tau)
\]
Market consistent measure

- Values of market prices of longevity risk, $\chi^{(j)}$, found from:
  - prices of traded longevity securities (if they exist) or
  - deterministic projection of mortality (e.g., CMI Projection Model)
Market consistent measure

- Consistent prices for liabilities, e.g., annuity values, can now be found using the same forward mortality surface.
Market consistent measure

- Can also find prices for other longevity linked securities consistent with few market prices observed
- For example, premiums on index longevity swaps at different ages

![Graph showing longevity swap premiums across different ages for various models: LC, CBDX, APC, RP, and GP.](graph.png)
Market consistent measure

- Can also find prices for other longevity linked securities consistent with few market prices observed
- For example, of 10 year q-forwards
Hedging

- For many purposes, we need to know how the forward mortality surface updates
  - E.g., Value at Risk, hedging
- This depends upon how the period and cohort functions update with one year’s extra observations
- NB – by tower property of conditional expectations, have

\[
\nu_{x,t}(\tau) = \mathbb{E}_\tau \nu_{x,t}(\tau + 1)
\]

- Period functions are straightforward

\[
\mathbb{E}_{\tau+1} \kappa_t = \mathbb{E}_\tau \kappa_t + \varepsilon_{\tau+1} \\
Var_{\tau+1}(\kappa_t) = (t - \tau - 1)\Sigma
\]
Hedging

- Cohort functions, need to use Bayesian approach and assumed data generating process

\[
\gamma_y(\tau + 1) = \gamma_y(\tau) + d_{\tau+1-y} \gamma_{y}^{\tau+1-y}
\]

\[
\gamma_{y}^{\tau+1-y} | \mathcal{F}_{\tau, y}, \beta, \rho, \sigma^2 \sim N \left( \beta Y_y + \rho (M(y - 1, \tau) - \beta Y_{y-1}), V(y - 1, \tau) + \frac{\sigma^2}{d_{\tau+1-y}} \right)
\]
Hedging

- Using this framework, we can update the forward mortality surface by one year and recalculate liability values or securities prices
- Value at Risk
Hedging

- Because liabilities and securities prices are valued from the same forward mortality surface, they will be updated consistently with one another.

- Useful for investigating hedge effectiveness for different value hedging strategies:
  - Single hedging instrument to hedge annuity portfolio
  - Hedge ratio chosen to minimise variance
Hedging

- Empirical distribution of liability value for different hedging instruments
Hedging

<table>
<thead>
<tr>
<th>Risk measure (as % of liabilities)</th>
<th>Unhedged</th>
<th>Q-forward</th>
<th>LE-forward</th>
<th>Swap</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR(95%)</td>
<td>2.75%</td>
<td>0.79%</td>
<td>0.24%</td>
<td>0.05%</td>
</tr>
<tr>
<td>% reduction</td>
<td>-</td>
<td>71%</td>
<td>91%</td>
<td>98%</td>
</tr>
<tr>
<td>TVaR(95%)</td>
<td>3.56%</td>
<td>1.06%</td>
<td>0.33%</td>
<td>0.11%</td>
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<tr>
<td>% reduction</td>
<td>-</td>
<td>70%</td>
<td>91%</td>
<td>97%</td>
</tr>
</tbody>
</table>

- Relatively high reductions in risk for very simple (single instrument) hedging strategies
- Model dependent (though valuation will be mark-to-model for foreseeable future)
- No allowance for basis risk
Discussion

- Forward mortality rates provide a useful framework for many of the issues with the valuation / risk management of longevity risk
- We have introduced a discrete time forward mortality rate framework which:
  - Is consistent with models of the short mortality rate
  - Can be calibrated easily to available data
  - Can be used with a variety of individual short rate models
  - Can be extended for different processes governing period and (more difficult) cohort functions
Selected References


Questions?

- Thank you very much for your attention and your feedback
Addendum

<table>
<thead>
<tr>
<th></th>
<th>$\lambda^{(1)}$</th>
<th>$\lambda^{(2)}$</th>
<th>$\lambda^{(3)}$</th>
<th>$\lambda^{(y)}$</th>
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<tbody>
<tr>
<td>LC</td>
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<td></td>
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<tr>
<td>CBDX</td>
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<td>-32.6</td>
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<td>APC</td>
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<td>281.3</td>
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<td>RP</td>
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<td>-5.4</td>
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<tr>
<td>GP</td>
<td>-23.9</td>
<td>20.7</td>
<td>-155.2</td>
<td>165.5</td>
</tr>
</tbody>
</table>

- Market prices of risk are dimensionless and not directly comparable across models
Addendum

- Can also find prices for other longevity linked securities consistent with few market prices observed
- For example, index of mortality improvement rates used in construction of the Kortis bond
Addendum

![Graph showing a relationship between age and a measure labeled as $a_{65}$ for different measures: GP - Q Measure, GP - P Measure, and CMI 1.75% over the age range of 60 to 80 years.]
Addendum

- Solvency II SCR is the 99.5% VaR of the technical provisions
- Therefore, forward rate model can calculate SCR by repeated updates of forward mortality surface
  - Avoids nested sims for SCR
- Compare with Solvency II standard model - 20% shock to mortality to proxy for VaR
  - C.f., Börger (2010)
Addendum

\[ \text{Risk Margin} = CoC \times \sum_{s=0}^{\infty} SCR(s)(1 + r_s)^{-s} \]

- Calculation of risk margin suffers from calculation problems
- Short rate approach:
  - Needs simulations (to give liabilities at \( s+1 \)) within simulations (to give VaR at \( s \)) within simulations (to model the run off of liabilities to \( s \))
- Forward mortality rate approach
  - Needs simulations (to give VaR at \( s \)) within simulations (to model the run off of liabilities to \( s \))
  - Progress, but not the complete answer
Addendum

- EIOPA (2014) suggests projecting deterministically to time $t$ to avoid nested simulations
  - May distort estimation of VaR, especially in tails
- We propose alternative approach based on limited number of model points

**Algorithm 1** Approximate estimation of the risk margin

1: Perform $N$ simulations to obtain empirical distribution of $\mathcal{L}(\tau + 1)$ for estimation of $\text{SCR}(\tau)$;
2: Select $p$ sets of latent variables $\{k_{\tau+1}, \gamma_{\tau+1-x}\}$ corresponding to $p$ model points in the distribution of $\mathcal{L}(\tau + 1)$;
3: Perform $N$ simulations for each model point to obtain $p$ empirical distributions of $\mathcal{L}(\tau + 2)|\mathcal{L}(\tau + 1) = \mathcal{L}^{(i)}(\tau + 1)$;
4: Calculate $\text{SCR}^{(i)}(\tau + 1)$ for each model point, and $\text{SCR}(\tau + 1) = \sum_{i=1}^{p} w_i \text{SCR}^{(i)}(\tau + 1)$ where $w_i$ are a set of weights based on the relative probability of model point $i$;
5: Repeat steps 2 and 3 for each future year until the liabilities have run off;
6: Calculate the risk margin using $\text{CoC} \times \sum_{s=0}^{\infty} \text{SCR}(s)(1 + r_s)^{-s}$
• Fix $p \times N = 10,000$
  
• Trade off between high $p$ (distribution at each time) and high $N$ (robust estimate of 99.5% VaR)
Generally, low $p$ means lower uncertainty in estimate, but biased SCR.

If $p=10$, SCR(0) = 5.4% and Risk Margin = 4.0% of best estimate of liability value.