Mortality Dependence and Longevity Bond Pricing:
A Dynamic Factor Copula Mortality Model with the GAS Structure

Hua Chen, Richard D. MacMinn and Tao Sun

April 30, 2014

Hua Chen*
Temple University
Email: hchen@temple.edu

Richard D. MacMinn
Illinois State University
Email: richard.macminn@ilstu.edu

Tao Sun
Temple University
Email: tao.sun@temple.edu

*Please address correspondence to:

Hua Chen
Department of Risk, Insurance, and Health Management
Temple University
1801 Liacouras Walk, 625 Alter Hall
Philadelphia, PA 19122

Mortality Dependence and Longevity Bond Pricing:

A Dynamic Factor Copula Mortality Model with the GAS Structure

Abstract

Modeling mortality dependence for multiple populations has significant implications for mortality/longevity risk management. A natural way to assess multivariate dependence is to use copula models. The application of copula models in the multi-population mortality analysis, however, is still in its infancy. Only two studies, i.e., Chen at al. (2013) and Wang et al. (2013), develop multi-population copula models. Extending their work, we present a dynamic multi-population mortality model based on a two-factor copula and capture the time-varying dependence using the generalized autoregressive score (GAS) framework. Our model is simple and flexible in terms of model specification and is widely applicable to high dimension data. Using the Swiss Re Kortis longevity trend bond as an example, we use our proposed model to predict the probability of principal reduction and compare our results with those reported by Risk Management Solutions (RMS) and Hunt and Blake (2013). Due to the similarity in the structure and design of CAT bonds and mortality/longevity bonds, we borrow CAT bond pricing techniques for mortality/longevity bond pricing. We find that our pricing model generates par spreads that are close to the actual spreads of previously issued mortality/longevity bonds.

Key Words: Multi-population mortality model, factor copulas, generalized autoregressive score models, the Kortis bond.

1. Introduction

Mortality/longevity risk management is fundamental to the life insurance and pension industries. Traditionally reinsurance, and more recently, securitization, renders a means of transferring or hedging mortality/longevity risks. Ever since the issuance of the first pure mortality bonds in December 2003, Swiss Re has been a major player in this market and has transferred over $2.2 billion of mortality risk to the capital market. These mortality bonds are principal-at-risk bonds and their payoffs are usually contingent on mortality experience of multiple populations (male and female, different age groups and different countries). Modelling mortality dependency among the indexed populations is essential; failing to do so would overestimate market prices of risk, thus discouraging risk transfer by (re)insurers (Lin et al. 2013). In addition to mortality risk in its books of life business, Swiss Re is also exposed to significant longevity risk in its books of pension business. These two risks can partially offset each other in a natural hedge, but basis risk exists when the mortality experiences of the reference populations in these two lines of business are not exactly the same. In order to hedge the residual longevity risk, Swiss Re issued the first longevity trend bond via Kortis Capital Ltd. in December 2010; the payoff is based on the difference in the morality improvements in the US male 55-65 and the UK male 75-85. Capturing mortality dependence between these two populations is of significance to Swiss Re for pricing and reserving.

There has been a growing literature on mortality models for multiple populations in recent years (see, for example, Li and Lee, 2005; Li and Hardy 2011; Dowd et al. 2011; Jarner and Kryger 2011; Cairns et al., 2011; Yang and Wang, 2013; Zhou et al. 2013a, 2013b). These models are typically structured assuming that the forecasted mortality experiences of two or more related populations are linked together and do not diverge over the long run. This assumption might be justified by the long-term mortality co-movements. It seems, however, too strong to model the short-term mortality dependence. From the pricing perspective, extant research often uses examples such as the previously issued mortality bonds or the EIB longevity bond which failed to launch. Though the issuance of the Kortis bond has attracted lots of attention as the first successful longevity trend bond, Hunt and Blake (2013) are the first to formally analyze the Kortis bond; they use a two-population mortality model to predict the bond payoff probability but they do not price the bond issue. We develop a multi-population mortality model based on a two-factor copula and introduce a time-varying dependence structure using the generalized autoregressive score (GAS) model. We then use our proposed mortality model to price the Kortis longevity bond by using existing CAT bond pricing techniques.

Copulas have been studied in both actuarial science and finance to examine dependencies among risks (Frees and Valdez, 1998; Embrechts et al. 2003). Particularly, in mortality studies copulas have been applied to model the bivariate survival function of the two lives of couples (see, e.g., Frees et al. 1996; Carriere 2000; Shemyakin and Youn 2006; Youn and Shemyakin 1999, 2001; Denuit et al. 2001). Surprisingly, the application of copula models in the multi-population mortality analysis is still in its infancy stage. To our the best of our knowledge, only two papers, Chen et al. (2013) and Wang et al. (2013), fall into this category.

A common feature in these two papers is that both build a multi-population mortality model based on a two-stage procedure. In the first stage, the mortality dynamics of each population is modeled using a time series analysis. In the second stage, the mortality dependence of the residuals from the first stage is captured using a copula model. The two-stage procedure permits the separate development of the marginal distributions and the copula model. Hence, this method allows the use of any existing model for the univariate analysis with the subsequent focus on the copula model. In spite of the similarity, their models differ in several ways. First, Chen et al. (2013) choose the best ARMA-GARCH model for mortality data of each population, whereas Wang et al. (2013) use an ARIMA (0,1,0) to fit the mortality index estimated from the Lee-Carter model for each country, ignoring the population-specific condition mean, not to mention the conditional variance model. Second, Chen et al. (2013) use a one-factor copula model which is widely applicable to high dimension data and very flexible in terms of model specification, while Wang et al. (2013) use copulas in the Elliptical or Archimedean family, which usually have only one or two parameters to characterize the dependence between all variables, and are thus quite restrictive when the number of variables increases. Third, Chen et al. (2013) find that the best one-factor model has a common factor with heavy tails and positive skewness, suggesting that mortality dependence is stronger during mortality deteriorations than during mortality improvements. This conforms to the conjecture that mortality jumps due, for example, to rare pandemic events are more likely to occur simultaneously. In contrast, Wang et al. (2013) note that mortality dependence may be asymmetric, they use four copulas in their study (Gaussian, Student-t copula, Gumbel and Clayton copula), among which Student-t copula provides the best fit. Hence they fail to capture the asymmetric tail dependence of mortality rates across countries. Finally, Wang et al. (2013) adopt the dynamic conditional correlation (DCC) approach in the copula model to allow correlations to change over time, but Chen et al. (2013) only consider a static approach.

In this paper, we adopt the factor copula to model mortality dependence because of its simplicity and flexibility. The factor copula is based on a simple linear, additive factor structure for the copula and is particularly attractive for high dimension applications. Next, the factor copula permits more flexibility for the number of variables and available data. We extend the one-factor copula model used in Chen et al. (2013) to a two-factor copula model, with one common factor representing the market force which drives mortality dynamics for all countries and one country-specific factor capturing mortality dependence across different age groups in each country.

We also incorporate the time-varying structure into the factor model by using the Generalized Autoregressive Score (GAS) framework. The GAS model was developed by Creal et al. (2013) who argue that the score function is an effective choice for introducing a driving mechanism for time-varying parameters. Oh and Patton (2013) use the lagged score of the copula model as the forcing variable to drive the factor loadings of conditional factor copulas over time and use this dynamic factor copula model to study Credit Default Swap (CDS) spreads on 100 US firms. As GAS models are observation-driven time-varying parameter models, their copula models do not require the additional integration over the path of the latent variables in order to calculate the likelihood. We adopt a similar approach in this paper to capture the time-varying mortality dependence for multiple populations.

Mortality/longevity securities involve significant valuation problems. Financial pricing approaches emphasize the creation of replicating portfolios (Black and Scholes, 1973; Merton, 1973) or the existence of a unique risk-neutral measure (Cox and Ross, 1976; Harrison and Kreps, 1979; Harrison and Pliska, 1981) in a complete market. In an incomplete market, the non-arbitrage condition ensures the existence of at least one risk-neutral measure. We, however, do not have clues regarding the appropriate choice of the risk-neutral measure with sparse market price data. This difficulty drives us to choose alternative insurance pricing approaches in CAT bond markets, which determines CAT bond premiums based on expected loss plus a risk loading. The risk loading can be proportional to expected loss itself, or proportional to the variance or standard deviation of the loss, or is determined by a risk cubic model based on two other risk factors, i.e., probability of first loss (PFL) and conditional expected loss (CEL), to capture the trade-off between the frequency and severity of the loss (Lane and Movchan, 1999; Lane, 2000). Mortality/longevity bonds are similar to CAT bonds in structure and design, so CAT bond pricing methodologies can be equally applied to mortality/longevity bonds. Lane (2000) uses the risk cubic model to price the Vita I and II mortality bonds. Chen and Cummins (2010) find that the risk cubic model can exactly replicate the risk premium of the Vita I mortality bond. They therefore employ the cubic model to explicitly derive risk premia for a proposed longevity bond. We follow the same approach but include some bond-specific characteristics, such as size, maturity, trigger, peril, and region, into our pricing model to explain CAT bond premiums, in addition to risk measures of the loss distribution (i.e., EL, PFL and CEL).

Our paper contributes to the existing literature in several ways. First, we are among the first to develop a copula-based mortality model to capture morality correlations among multiple populations. Copulas are powerful tools to model the dependence for tail events or extreme risks, therefore our model has significant applications to mortality risk pricing and hedging for correlated risks. Second, we extend the one-factor copula model to two-factor copula model, so our model can capture not only the mortality dependence across countries but also mortality dependence across age groups in each country. Third, we incorporate the time-varying dependence structure in our factor copula model, leading to a flexible yet parsimonious dynamic model for high dimension conditional distributions. Finally, we apply the CAT bond pricing model to mortality/longevity bonds. We find our model generates prices that are close to those observed in these transactions.

The remaining of this paper is organized as follows. In Section 2, we introduce the structure design of the Kortis longevity bond. In Section 3, we propose our multi-population mortality model with dynamic dependence based on a factor copula approach and the generalized autoregressive score model. In Section 4, we investigate mortality data for US male 55-65 and UK male 75-85 and estimate the parameters in our mortality model. In Section 5, we discuss the CAT bond pricing methodologies and apply it to price the Kortis longevity bond. Concluding remarks are given in Section 6.

2. The Kortis Longevity Bond

Swiss Re has a strong track record of developing the capital markets for insurance perils, initially with natural catastrophe bonds and more recently by periodically securitizing its life risks. To hedge that risk, it has sold about $2.2 billion in mortality bonds to date. Such bonds earn a healthy yield, so long as there’s no big rise in mortality. If there is, then the bondholders lose some or all of their principal, and it’s used instead to make those unexpected life-insurance payouts. We summarize some characteristics of mortality/longevity bonds issued to date and report them in Table 1.

Table 1: Mortality/Longvity Bonds Issued to Date

	Vita I	Vita II	Tartan	Osiris	Vita III	Nathan
Sponsor	Swiss Re	Swiss Re	Scottish Re	AXA	Swiss Re	Munich Re
Arranger	Swiss Re	Swiss Re	Goldman Sachs	Swiss Re	Swiss Re	Munich Re
Modelling Firm	Milliman	Milliman	Milliman	Milliman	Milliman	Milliman
SPV domicile	Cayman Islands	Cayman Islands	Cayman Islands	Ireland	Cayman Islands	Cayman Islands
Size	$400M	$362M	$155M	€345M	$705M	$100M
No. of Tranches	1	3	2	3	2	1
Issue date	December 2003	April 2005	May 2006	November 2006	January 2007	February 2008
Maturity	4 years	5 years	3 years	4 years	4 & 5 years	5 years
Index	US, UK, France, Italy, Switzerland	US, UK, Germany, Japan, Canada	US	France, Japan, US	US, UK, Germany, Japan, Canada	US, UK, Canada, Germany

	Vita IV	Kortis	Vita IV	Vita V	Atlas IX
Sponsor	Swiss Re	Swiss Re	Swiss Re	Swiss Re	SCOR Re
Arranger	Swiss Re	Swiss Re	Swiss Re	Swiss Re	Aon, BNP Paribas, Natixis
Modelling Firm	RMS	RMS	RMS	RMS	RMS
SPV domicile	Cayman Islands	Cayman Islands	Cayman Islands	Cayman Islands	Ireland
Size	$300M	$50M	$180M	$275M	$180M
No. of Tranches	4	1	2	2	2
Issue date	I: November 2009 II: May 2010 III: October 2010 IV: October 2010	December 2010	July 2011	July 2012	September 2013
Maturity	4 & 5 years	6 years	5 years	5 years	5 years
Index	I :US, UK II: US/UK III: US/Japan IV: Germany/Canada	US, UK	IV: Canada/German V:Canada/Germany/UK/US	D-1: Australia, Canada E-1: Australia, Canada, US	US

On the other hand, as ageing baby boomers put pension fund balances under increasing pressure, global exposure to longevity risk involves about $20.7 trillion of pension assets (Swiss Re, 2012). Heavyweight reinsurers are renewing their efforts to provide capacity for longevity risk. As an example, Swiss Re announced in 2009 that it would insure around £1 billion of pensioner liabilities for the Royal County of Berkshire Pension Fund. Insuring longevity risk, then, is a great deal for Swiss Re: not only should it be profitable, if it's priced correctly, but it also helps to naturally hedge the company's mortality risk.

Such natural hedge is perfect - basis risk can arise from two sources. First, trends in mortality improvement can be different in the US and UK, owing to the different lifestyles and medical systems operating on either side of the Atlantic. Second, annuity policyholders or pensioners are typically older than life assurance policyholders and rates of longevity improvement differ across ages. If the reference population in the pension pool experiences a much greater mortality improvement than those with the life pool then the natural hedge is not sufficient protection.

In order to transfer the residual longevity risk between its books of pension and life insurance business, Swiss Re sold $50 million of the first "longevity trend" bonds to capital market investors via a Cayman Island domiciled, off-balance-sheet vehicle called Kortis Capital Ltd., in December 2010. The bond pays quarterly coupons at a spread of 5.0% above the three-month LIBOR and matures in January 2017 (extendable until July 2019). The principal repayment is indexed on a Longevity Divergence Index Value (LDIV), defined as the divergence in the mortality improvements experienced between male lives aged 75-85 in England & Wales, which can be seen as representing Swiss Re's pension exposures, and male lives aged 55-65 in the US, which can be seen as representing Swiss Re's life insurance exposure (Swiss Re, 2012).

The LDIV is constructed in several steps. First, the mortality improvement in each population for men at different ages is averaged over *n* years, i.e.,

$$\Delta m_x^j\left(t-n,t\right)=1-\left(\frac{m_{x,t}^j}{m_{x,t-n}^j}\right)^{1/n}$$

where $m_{x,t}^j$ is the crude mortality rate observed at age *x* in year *t* for population *j* and *n* is the averaging period of the bond. An averaging period of eight years is used to calculate the observed improvement, so that the Kortis bond is indexed to changes in mortality rates over the period 1 January 2009 to 31 December 2016.

Second, the mortality improvement is then averaged across age groups (ages x_1 to x_2) for each year and country, i.e.,

$$\Delta m_t^j\left(x_1,x_2\right)=\frac{1}{1+x_2-x_1}\sum_{x=x_1}^{x_2}\Delta m_x^j\left(t-8,t\right).$$

To take into account the fact that pensioners are typically older than life insurance policyholders, $\Delta m_t^{\mathrm{EW}}\left(75,85\right)$ is calculated using ages between ages 75 and 85 for England & Wales, while $\Delta m_t^{\mathrm{US}}\left(55,65\right)$ is calculated using ages between 55 and 65 in the US.

Finally, the longevity divergence index value (LDIV) is calculated for year *t* as

$$\mathrm{LDIV}_t=\Delta m_t^{\mathrm{EW}}\left(75,85\right)-\Delta m_t^{\mathrm{US}}\left(55,65\right).$$

The "principal reduction factor" (PRF) is indexed on the LDIV and structured as a call option spread, similar to the Vita programs, i.e.,

$$\mathrm{PRF}=\frac{\left[\mathrm{LDIV}_{2016}-3.4\%\right]_+-\left[\mathrm{LDIV}_{2016}-3.9\%\right]_+}{3.9\%-3.4\%},$$

where 3.4% is the attachment point and 3.9% is the detachment point. In other words, the bond is structured so that if the rate of mortality improvement in England & Wales between 2008 and 2016 is significantly higher than that observed in the US (i.e., LDIV is greater than 3.4%), the principal of the bond is reduced linearly until full exhaustion of the principal if the LDIV is greater than 3.9%.

When the Kortis bond was issued, Standard and Poor's rated the single tranche Series 2010-I Class E Notes as BB+. This was based in part on the modelling work of Risk Management Solutions (RMS). Table 2 gives the distribution of the PRF as calculated by RMS and given in Standard and Poor's (2010).

Table 2: Distribution of PRF Calculated by RMS

LDIV ≥	PRF ≥	Probability
3.4%	0	0.88%
3.5%	20%	0.72%
3.6%	40%	0.58%
3.7%	60%	0.47%
3.8%	80%	0.38%
3.9%	100%	0.30%

3. The Dynamic Factor Copula Model for Multi-populations

Copula-based multivariate models allow researchers to specify the models for the marginal distributions separately from the dependence structure that links these distributions to form the joint distribution. This enriches the class of multivariate distributions and permits a much greater degree of flexibility in model specification. If the parameters of the marginal distributions are separable from those for the copula, we may estimate those parameters separately in two stages, which greatly simplifies the estimation procedure and facilitates the study of high-dimension multivariate problems. [1]

2.1. The First Stage: The Conditional Marginal Distribution

Suppose we need to model the dependence structure of time series data, y_{it}, $i = 1, 2, ..., N$, $t = 1, 2, ..., T$. We first model the conditional marginal distribution of y_{it}. We allow each time series to have a time-varying conditional mean and conditional variance, each governed by parametric models,

$$y_{it} = \mu_i\left(Z_{t-1}\right) + \sigma_i\left(Z_{t-1}\right)\eta_{it} \quad \text{for } i = 1, 2, ..., N \text{ and } Z_{t-1} \in F_{t-1}, \tag{1}$$

where $\mu_i\left(Z_{t-1}\right)$ and $\sigma_i\left(Z_{t-1}\right)$ denote the functional forms for the conditional mean and conditional volatility, F_{t-1} is the sigma-field containing information generated by $\{y_{t-1}, y_{t-2}, \cdots\}$.

The estimate of standardized residuals can be specified as

$$\hat{\eta}_{it} = \frac{y_{it} - \mu_i\left(Z_{t-1}; \hat{\alpha}\right)}{\sigma_i\left(Z_{t-1}; \hat{\alpha}\right)}, \tag{2}$$

where $\mu_i\left(Z_{t-1}; \hat{\alpha}\right)$ and $\sigma_i\left(Z_{t-1}; \hat{\alpha}\right)$ are the estimated conditional mean and conditional volatility with the vector of estimated parameters $\hat{\alpha}$.

We then estimate the distribution of the standardized residuals using the empirical distribution function (EDF), denoted as $\hat{F}_i$ for variable *i*,

$$\hat{F}_i\left(\eta\right) \equiv \frac{1}{T+1}\sum_{t=1}^{T}\mathbf{1}\{\hat{\eta}_{it} \le \eta\}. \tag{3}$$

---

[1] Clearly, two- or multiple-stage estimation is asymptotically less efficient than one-stage estimation. However, simulation studies in Joe (2005) and Patton (2006) indicate that this loss is not great in many cases.

where $\mathbf{1}\{\hat{\eta}_{it} \le \eta\}$ is an indicator function which is equal to 1 when $\hat{\eta}_{it} \le \eta$ and 0 otherwise.

The use of the EDF allows us to non-parametrically capture skewness and excess kurtosis in the residuals, if present, and allows these characteristics to differ across the mortality time series of each population.

2.2. The Second Stage: A Factor Copula Approach

In the mortality/longevity securitization, we intend to capture the joint distribution of tail risks, which relies on the dependence structure of the innovation process. A straightforward approach is to use a copula model to describe the joint distribution of innovations. With the EDF, the copula model for the standardized residuals can be structured as

$$\eta_t \equiv \left[\eta_{1t}, \cdots, \eta_{Nt}\right]^T \sim iid\ F_\eta = C\left(F_{\eta_1}, \cdots, F_{\eta_N}; \theta\right). \tag{4}$$

The copula is parameterized by a vector of parameters, θ, which can be estimated using the simulation-based estimation approach proposed by Oh and Patton (2012). Their method is close to Simulated Method of Moments (SMM), but is not exactly the same as SMM, because the “moments” that are used in estimation are functions of rank statistics. The SMM copula estimator is based on simulation from some joint distribution of a vector of latent variables X, $F_X\left(\theta\right)$, with implied copula $C\left(\theta\right)$. Briefly speaking, we estimate the vector of parameters, θ, by minimizing the sum squared errors between moment conditions (i.e., dependence measures) computed using simulations from $F_X\left(\theta\right)$ and those computed from standardized residuals $\{\hat{\eta}_t\}_{t=1}^T$. The details of the simulated-based method of moments are given in Appendix A.

In this study, we consider a two-factor model proposed by Oh and Patton (2012), which is generated by the following structure

$$\begin{aligned}&X_i = \lambda_i^0 Z_0 + \lambda_i^c Z_c + \varepsilon_i\\&Z_c \sim iid\ F_{Z_c}, c = \text{US or UK};\ Z_c \perp Z_0\ \ \forall c\\&\varepsilon_i \sim iid\ F_\varepsilon, i = 1,\dots,N;\ \varepsilon_i \perp Z_j\ \ \forall i, j\\&X \equiv \left[X_1,\cdots,X_N\right]' \sim F_X = C\left(F_{X_1},\cdots,F_{X_N}\right)\end{aligned}, \quad (5)$$

where X_i are latent variables, Z_0 is the common, market-wide, factor that affects both US and UK mortality rates, Z_c is the country factor that affect US or UK mortality rates only, and ε_i are idiosyncratic factors. It is noteworthy that we impose the same dependence structure, or the copula model, for the latent variables X and the original variables η_i but their marginal distributions might be different.

This class of factor copula models has a simple form but a very flexible dependence structure. For example, by allowing for a fat-tailed common factor the model captures the possibility of correlated crashes in different markets or correlated mortality jumps in different countries. By assuming a skewed distribution on the common factor the model allows for an asymmetric tail dependence structure, which is consistent with the fact that the stock returns are more correlated in crashes than in booms or mortality dependence is stronger during mortality deteriorations than during mortality improvements. If we impose the same loading on the common factor we have an equidependence structure, whereas different loadings on the common factor enable us to model heterogeneous pairs of variables at the cost of a more difficult estimation problem. We can also simplify the problem by assuming sub-sets of variables (for example, mortality rates for countries in the same continent or those for different age groups in the same country) have the same loading on the common factor. Furthermore, the one-factor model can be extended to multifactor models to capture heterogeneous pair-wise dependence within the overall multivariate copula.

2.3. Dynamic Dependence with GAS

The static factor copula model specified in equation (4) and (5) can be extended to a dynamic factor copula model that can capture time-varying dependence in high dimensions. For instance, we can allow the factor loadings, β_i, γ_i, to be time-varying, or let the "shape" parameters, θ, of the factors Z_0, Z_c to be time dependent. As a balance between the model flexibility and feasibility, we assume that the factor loading parameters are time-varying and the "shape" parameters of the factors Z_0, Z_c to be fixed through time, which are similar to Oh and Patton (2013).

$$\begin{aligned} X_{it} &= \lambda_{i,t}^{0} Z_0 + \lambda_{i,t}^{c} Z_c + \varepsilon_i, i = 1, ..., N \\ X_t &\equiv \left[X_{1t}, \cdots, X_{Nt} \right]' \sim F_{X_t} = C\left(F_{X_{1t}}(\lambda_{1t}(\theta)), \cdots, F_{X_{Nt}}(\lambda_{1t}(\theta)) \right), t = 1, ..., T , \quad (6) \\ \lambda_t &= \left[\lambda_{1,t}^{0}, ... \lambda_{N,t}^{0}, \lambda_{i,t}^{c}, ..., \lambda_{i,t}^{c} \right]' \end{aligned}$$

An important feature of any dynamic model is the specification for how the parameters evolve through time. We adopt the generalized autoregressive score (GAS) model of Creal et al. (2013) to model the dynamics of factor loading parameters. These authors propose using the lagged score of the density model (copula model, in our application) as the forcing variable. Specifically, for a copula with time-varying parameter δ_t, governed by fixed parameter θ, we have:

Let $U_t \left| F_{t-1} \sim C\left(\delta_t \left(\theta \right) \right)\right.$

Then $\delta_t = w + B\delta_{t-1} + As_{t-1}$, (7)

where $s_{t-1} = S_{t-1} \nabla_{t-1}$

$$\nabla_{t-1} = \frac{\partial \log c\left(u_{t-1}; \delta_{t-1}\right)}{\partial \delta_{t-1}}$$

and S_t is a scaling matrix (e.g., the inverse Hessian or its square root). This specification has several advantages: firstly, compared with modeling the time-vary parameters as a latent time series, such as stochastic volatility models (see, e.g., Shephard 2005) and related stochastic copula models (Hafner and Manner 2011; Manner and Segers 2011), modeling the varying parameter as some functions of observable avoids the need to "integrate out" the innovation terms driving the latent time series processes, which is very important for the tractability of the model in high dimensions. Secondly, the GAS model exploits the complete density structure to update the time-varying parameter based on density score rather than means and higher moment only. Thirdly, the recursion using the density score as the forcing variable can be justified as the steepest ascent direction for improving the model's fit, in terms of the likelihood, given the current value of the model parameter δ_t.

In the general GAS framework in equation (7), the $2N$ time-varying factor loadings in equation (6) would require the estimation of $N + 2(2N)^2$ parameters governing their evolution, which represents an infeasibly large number for even moderate values of N. To keep the model parsimonious, we impose that the coefficient matrices (B and A) are diagonal with a common parameter on the diagonal, as in the DCC model of Engle (2002). To avoid the estimation of $N \times N$ scaling matrix we set $S_t = I$. This simplifies the evolution model for time-varying parameters to be (in logs):

$$\log \lambda_{i,t}^{j} = \omega_i + \beta \log \lambda_{i,t-1}^{j} + \alpha s_{i,t-1},\ j \in \{0, c\} \tag{8}$$

where $s_{it} = \partial \log c(u_t; \boldsymbol{\lambda}_t, \theta_{z_0}, \theta_{z_c}, \theta_\varepsilon) / \partial \lambda_{it}^j$ an $\lambda_t = \left[\lambda_{1,t}^0, \ldots \lambda_{N,t}^0, \lambda_{1,t}^c, \ldots, \lambda_{N,t}^c\right]'$. Lastly, we adopt the "variance target" method proposed in Oh and Patton (2013) to further simplify the estimation of ω_i in equation (8). Using the result from Creal, et al. (2013) that $E_{t-1}\left[s_{it}\right] = 0$, we have $E\left[\log \lambda_{it}^j\right] = \omega_i / (1-\beta)$ equation (8) can be rewrite as

$$\log \lambda_{it}^j = E\left[\log \lambda_{it}^j\right](1-\beta) + \beta \log \lambda_{i,t-1}^j + \alpha s_{i,t-1} \tag{9}$$

where $E\left[\log \lambda_t\right]$ can be pre-estimated using sample rank correlation (see Proposition 1 in Oh and Patton, 2013). After the above reductions, we only need to estimate the following parameters in the dynamic copula model implied by equations (6) and (9): two parameters for the GAS dynamics (i.e. β, α) and the shape parameters for the common, country and idiosyncratic factors.

4. Model Estimation

4.1. Data

We collect mortality data for US and UK (England & Wales) from Human Mortality Database over the period 1933-2010.[2] This sample period is determined by US mortality data, which is available only after year 1933. To be consistent with the Swiss Re Kortis bond, we choose UK males with ages 75 to 85 and US males with ages 55 to 65.

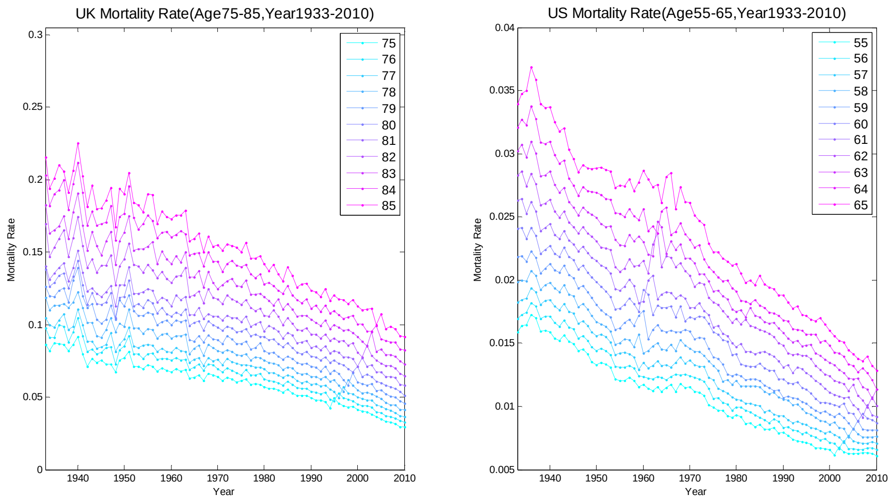
4.2. Conditional Marginal Distribution

Denote $m_{x,t}^j$ the crude mortality rate at age *x* in year *t* for country *j*. Figure 1 plots the crude mortality rates for UK male aged 75-85 and US male aged 55-65 from 1933 to 2010. We observe a clear trend of mortality improvements for all age groups, and a very

[2] Data source: Human Mortality Database, available at www.mortality.org

strong intra-group dependence of mortality dynamics in each country. We also find the evidence of the cohort effect. For example, a small hump in the mortality rate for UK male aged 75 in 1994 was passed on to subsequent years when this age group grew older. The US age groups exhibit even stronger cohort effect than the UK age groups. Clearly these mortality rates are not stationary. So we compute the first difference of the logged mortality rates, denoted by $y^{j}_{x,t} \equiv \ln m^{j}_{x,t} - \ln m^{j}_{x,t-1}$, to obtain the mortality rate changes over time, as can be seen in Figure 2.[3]

Figure 1: Mortality Rates for US male 55-65 and UK male 75-85



[3] A positive $y_t^{(i)}$ indicates mortality deterioration and a negative $y_t^{(i)}$ suggests morality improvement.

Figure 2: Mortality Rate Changes for US Male 55-65 and UK Male 75-85

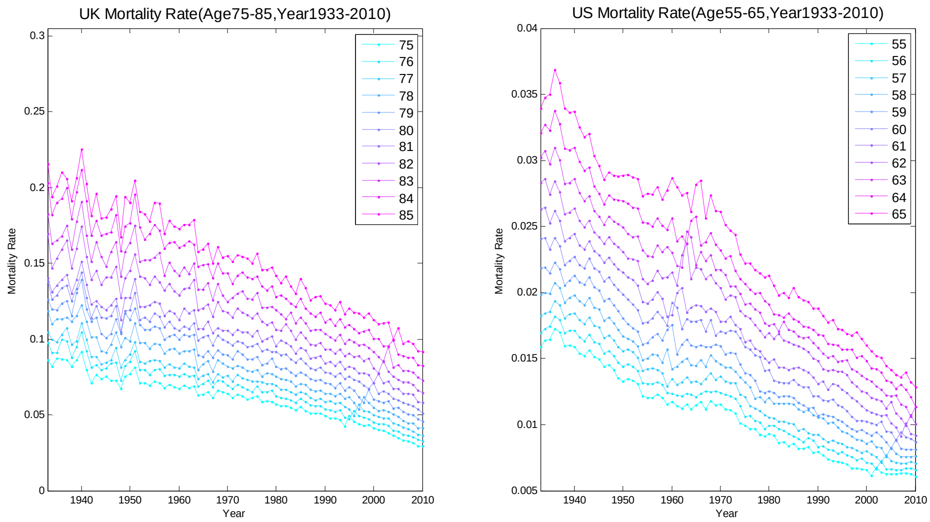


Table 3 reports the summary statistics for the mortality rate changes for the selected age groups in the US and UK. The mean and median of mortality rate changes are negative for all twenty-two age groups. For UK, mortality improvement is slightly more pronounced at younger ages, while the mortality improvement rates exhibit little variation across age groups in the US. The summary statistics also suggest that UK groups have larger standard deviations than the US groups. In the meanwhile, US groups seem to have fatter (upper) tails because ten out of eleven US groups have positive skewness and relatively larger kurtosis than UK groups. We perform the Phillips-Perron test to examine whether each series of mortality improvement is stationary. We are able to reject the null hypothesis of a unit root at the 1% significant level for all age groups in the US and UK.

Table 3: Summary Statistics for mortality improvements

Age	No. of Obs	Mean	Std. Dev	Median	Min	Max	Skewness	Kurtosis
UK75	77	-0.0141	0.0499	-0.0160	-0.1398	0.1538	0.3582	4.5500
UK76	77	-0.0143	0.0543	-0.0179	-0.1541	0.1265	0.0068	3.6428
UK77	77	-0.0137	0.0523	-0.0167	-0.1457	0.1352	0.0631	3.8443
UK78	77	-0.0137	0.0508	-0.0102	-0.1516	0.1400	0.1781	3.9124
UK79	77	-0.0133	0.0550	-0.0159	-0.1524	0.1396	0.2660	3.6663
UK80	77	-0.0129	0.0515	-0.0163	-0.1501	0.1262	-0.2525	3.5902
UK81	77	-0.0115	0.0512	-0.0111	-0.1415	0.1222	-0.1110	3.0261
UK82	77	-0.0125	0.0553	-0.0159	-0.1527	0.0958	-0.3692	3.1232
UK83	77	-0.0120	0.0590	-0.0152	-0.1535	0.1108	-0.3055	3.0358
UK84	77	-0.0117	0.0517	-0.0053	-0.1396	0.1029	-0.2580	3.0568
UK85	77	-0.0111	0.0572	-0.0078	-0.1501	0.1476	-0.1128	2.9797
US55	77	-0.0125	0.0301	-0.0144	-0.0702	0.0954	0.4832	4.0686
US56	77	-0.0122	0.0279	-0.0116	-0.0862	0.0858	0.1399	4.3162
US57	77	-0.0123	0.0278	-0.0150	-0.0737	0.0782	0.2449	3.6865
US58	77	-0.0124	0.0315	-0.0164	-0.0988	0.0861	0.2061	4.2160
US59	77	-0.0128	0.0313	-0.0125	-0.1383	0.0773	-0.4744	5.6521
US60	77	-0.0133	0.0319	-0.0132	-0.0912	0.0950	0.4094	4.2988
US61	77	-0.0136	0.0322	-0.0152	-0.0936	0.1053	0.6199	5.3628
US62	77	-0.0135	0.0337	-0.0148	-0.1214	0.1067	0.4639	5.4813
US63	77	-0.0127	0.0302	-0.0170	-0.0911	0.1173	1.1282	6.8815
US64	77	-0.0135	0.0260	-0.0126	-0.0862	0.0704	0.1689	4.0406
US65	77	-0.0126	0.0288	-0.0134	-0.1047	0.0761	0.2808	4.5029

Based on the Box-Jenkins (1976) approach, we fit the mortality rate changes with a conditional mean model ARMA (r, m)

$$y_t = c + \sum_{i=1}^{r} \varphi_i y_{t-i} + \sum_{j=1}^{m} \theta_j \varepsilon_{t-j} + \varepsilon_t , \tag{6}$$

and a conditional variance model GARCH (*p*, *q*) [4]

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j}^2 , \tag{7}$$

where $\varepsilon_t = \sigma_t z_t$ and z_t is an independent and identically distributed sequence of standardized random variables. We consider both Gaussian innovations, i.e., $z_t \sim N(0,1)$, and student-*t* innovations, i.e., $z_t \sim t(v)$ where v is the degree of freedom.[5]

We use the Bayesian Information Criterion (BIC) to determine the appropriate lags in the conditional mean and conditional variance models. Lag orders up to five are tested for the above criteria. Table 4 reports the optimal lag orders for each group and compares the BICs for each model assuming that the error terms follow either Gaussian or Student-*t* distributions. We can see that there is not too much difference in the model BIC by assuming Student-*t* or Gaussian innovations. In fact, the BIC is slightly lower for Gaussian innovations for nineteen out of twenty-two groups. We hence choose the best ARMA-GARCH model with Gaussian innovations. Overall, the conditional mean of mortality improvement can be explained by the MA(1) process consistently across groups; only four age groups in UK need AR(1) or AR(2). As to the conditional variance model, all but six UK age groups exhibit constant variance over time; the other six groups require ARCH(1) to remove the ARCH effect.

[4] We also estimated the ARMA-GJR-GARCH model. However, we find no "leverage" effect in the mortality improvement rate for each country.
[5] We also consider skewed-*t* innovations. We find, however, that neither of the skewness parameter estimates for all countries is significant.

Table 4: Conditional Mean and Variance Model selection

	Student t Innovation					Gaussian Innovation				
Group	AR	MA	ARCH	GARCH	BIC	AR	MA	ARCH	GARCH	BIC
UK75	0	1	1	0	-255.42	0	1	1	0	-259.74
UK76	1	3	0	0	-228.86	0	1	0	0	-230.26
UK77	0	1	0	0	-237.85	0	1	0	0	-238.47
UK78	0	1	0	0	-241.38	0	1	0	0	-242.02
UK79	0	1	0	0	-227.12	0	1	0	0	-230.22
UK80	1	2	0	0	-250.37	0	1	0	0	-250.88
UK81	0	1	0	0	-247.15	0	1	0	0	-250.12
UK82	0	1	0	0	-236.01	0	1	0	0	-238.05
UK83	0	1	1	0	-241.91	0	1	1	0	-246.37
UK84	0	1	0	0	-250.76	0	1	0	0	-249.53
UK85	0	1	0	0	-240.71	0	1	0	0	-243.81
US55	0	1	0	0	-315.22	0	1	0	0	-318.37
US56	1	0	0	0	-325.69	1	0	0	0	-328.23
US57	1	0	0	0	-328.59	1	0	0	0	-329.99
US58	0	1	0	0	-315.25	1	0	0	0	-317.01
US59	4	0	1	0	-310.82	2	0	1	0	-315.12
US60	0	1	0	0	-311.98	0	1	0	0	-313.81
US61	0	1	1	0	-310.11	0	1	1	0	-311.15
US62	2	0	0	0	-310.03	0	1	0	0	-305.54
US63	1	2	1	0	-314.15	0	1	1	0	-313.56
US64	0	1	0	0	-336.46	0	1	0	0	-337.21
US65	2	2	1	0	-321.72	0	1	1	0	-325.98

After the ARMA-GARCH fitting, we perform the Ljung-Box-Pierce Q tests on the standardized residuals (the innovations divided by the conditional standard deviations) and the squared standardized residuals. The results are reported in Table 5. Almost all test statistics are insignificant up to lag 10, which indicates that our selected model for each group sufficiently accounts for serial correlation and conditional heteroskedasticity.

Table 5: Ljung-Box-Pierce Q tests on standardized residuals and squared standardized residuals

Group	Q Test on the Standardized Innovations			Q Test on the Squared Standardized Innovations		
	Lag1	**Lag5**	**Lag10**	**Lag1**	**Lag5**	**Lag10**
UK75	0.4462 (0.5042)	5.6052 (0.3466)	23.8771 (0.0079)	0.6339 (0.4259)	3.4443 (0.6318)	8.3656 (0.5932)
UK76	0.6511 (0.4197)	8.2136 (0.1448)	21.2038 (0.0197)	1.6446 (0.1997)	19.1480 (0.0018)	33.5358 (0.0002)
UK77	0.1164 (0.7329)	6.7092 (0.2432)	12.0049 (0.2847)	0.8462 (0.3576)	9.2531 (0.0994)	14.5891 (0.1478)
UK78	0.0464 (0.8295)	2.2922 (0.8074)	10.2409 (0.4196)	0.0791 (0.7785)	6.7596 (0.2391)	9.9134 (0.4481)
UK79	0.1109 (0.7391)	13.8121 (0.0168)	19.5255 (0.0341)	3.6274 (0.0568)	15.4603 (0.0086)	21.1747 (0.0199)
UK80	0.0274 (0.8684)	6.0453 (0.3018)	12.8458 (0.2324)	2.3037 (0.1291)	13.8355 (0.0167)	31.4549 (0.0005)
UK81	0.0086 (0.9260)	2.8072 (0.7297)	12.9824 (0.2247)	1.2865 (0.2567)	10.1163 (0.0720)	19.4482 (0.0349)
UK82	0.3295 (0.5660)	5.2009 (0.3919)	11.1723 (0.3443)	0.5974 (0.4396)	5.6165 (0.3453)	13.9813 (0.1738)
UK83	1.6642 (0.1970)	2.6454 (0.7545)	20.2634 (0.0269)	0.4149 (0.5195)	5.5713 (0.3502)	9.2573 (0.5079)
UK84	0.5009 (0.4791)	8.6194 (0.1252)	13.7325 (0.1855)	0.1007 (0.7510)	7.4420 (0.1898)	17.2635 (0.0687)
UK85	0.0134 (0.9079)	5.8968 (0.3164)	12.0782 (0.2799)	0.1513 (0.6973)	8.5491 (0.1285)	23.0191 (0.0107)
US55	0.0517 (0.8201)	0.9429 (0.9670)	3.2236 (0.9757)	0.1266 (0.7220)	2.1691 (0.8253)	3.8196 (0.9551)
US56	0.0108 (0.9172)	3.5869 (0.6103)	7.5994 (0.6679)	1.5692 (0.2103)	3.5802 (0.6113)	6.7639 (0.7475)
US57	0.0068 (0.9343)	5.4294 (0.3658)	10.3394 (0.4112)	0.3592 (0.5489)	3.1517 (0.6766)	5.7708 (0.8341)
US58	0.0824 (0.7740)	4.8556 (0.4338)	6.7142 (0.7521)	0.2377 (0.6259)	1.7370 (0.8842)	6.1982 (0.7983)
US59	0.7177 (0.3969)	10.6906 (0.0579)	12.8255 (0.2336)	0.1062 (0.7446)	6.7354 (0.2411)	11.0004 (0.3575)
US60	0.0295 (0.8636)	0.6214 (0.9870)	6.9299 (0.7320)	0.0162 (0.8987)	3.1569 (0.6758)	6.0062 (0.8147)
US61	1.2016 (0.2730)	10.4631 (0.0631)	10.8365 (0.3704)	0.4179 (0.5180)	3.0864 (0.6867)	9.3094 (0.5030)
US62	0.3294 (0.5660)	3.2886 (0.6556)	9.9178 (0.4477)	0.1089 (0.7414)	3.0076 (0.6988)	4.7357 (0.9081)
US63	0.0428 (0.8360)	3.5451 (0.6166)	10.3685 (0.4088)	0.9906 (0.3196)	2.8423 (0.7243)	4.2740 (0.9341)
US64	0.0353 (0.8511)	2.3059 (0.8054)	6.3811 (0.7823)	1.8525 (0.1735)	11.6445 (0.0400)	12.1313 (0.2764)
US65	0.3890 (0.5329)	2.9149 (0.7131)	5.8990 (0.8237)	0.3630 (0.5468)	1.8131 (0.8744)	3.7619 (0.9574)

The summary statistics for standardized residuals are reported in Table 6. The mean is close to zero and the variance is about one for each group. In addition, we have very small skewness and reasonable kurtosis numbers. All of these observations provide evidence that Gaussian innovations in the ARMA-GARCH model is a valid assumption.

Table 6: Summary Statistics for Standardized Residuals

Group	No. of Obs	Mean	Std Dev	Median	Min	Max	Skewness	Kurtosis
UK75	77	-0.0365	1.0059	-0.1038	-3.1116	1.8940	-0.0848	3.0609
UK76	77	-0.0047	1.0065	-0.1274	-2.6796	2.8051	0.3805	3.7236
UK77	77	-0.0034	1.0065	0.0218	-2.5167	2.9362	0.5854	3.9953
UK78	77	-0.0062	1.0065	-0.0005	-2.6398	3.0046	0.2890	4.0395
UK79	77	-0.0040	1.0065	-0.1538	-2.1980	2.6826	0.5620	3.4419
UK80	77	-0.0107	1.0065	-0.0905	-2.8497	3.2326	0.3609	4.2594
UK81	77	-0.0037	1.0066	0.0774	-2.2761	3.2594	0.5403	3.7864
UK82	77	-0.0223	1.0063	-0.0642	-2.8195	2.8292	0.1795	3.8767
UK83	77	-0.0027	1.0066	-0.1750	-2.2317	2.2632	0.2317	2.4017
UK84	77	-0.0158	1.0064	-0.0520	-2.6640	3.5607	0.5670	4.8633
UK85	77	-0.0163	1.0064	0.0400	-2.1232	2.9289	0.2442	3.5872
US55	77	0.0092	1.0065	0.0023	-2.1761	3.1860	0.4014	3.6188
US56	77	0.0000	1.0066	-0.0135	-2.6135	3.1041	0.0381	3.7244
US57	77	0.0000	1.0066	-0.1011	-2.4630	2.8015	0.1064	3.7129
US58	77	0.0000	1.0066	-0.0969	-2.6777	2.3411	0.1093	3.4469
US59	77	-0.0075	1.0065	0.1162	-2.6654	2.5952	-0.1233	3.0720
US60	77	0.0028	1.0066	-0.0702	-2.1547	3.4938	0.3497	4.0414
US61	77	-0.0066	1.0066	-0.0130	-2.5206	3.7093	0.4542	4.4881
US62	77	-0.0561	1.0050	-0.1249	-2.1874	4.0170	0.8119	5.2592
US63	77	0.0072	1.0065	-0.1496	-2.2099	4.6148	1.3338	7.5363
US64	77	0.0070	1.0065	0.0639	-2.4140	3.2976	0.4130	4.0859
US65	77	0.0690	1.0042	-0.0555	-2.3199	2.5927	0.3396	3.0321

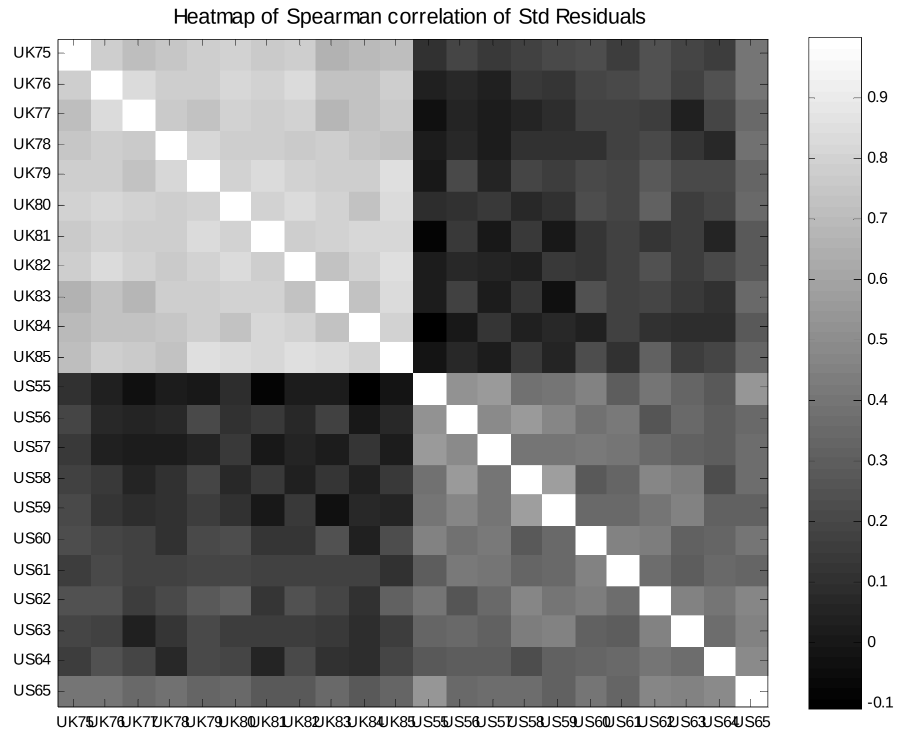
4.3. A Two-Factor Copula model

In this subsection, we use the factor model with varying loadings on the common factor. Based on the distribution assumptions for the factors, we consider 4 models: Student *t* - Normal, Student *t* - Student *t*, Skewed *t* - Normal and Skewed *t* - Student *t*. For example, in the Skewed *t* - Normal factor copula model, we assume that the common factor Z_0 follows a skewed *t* distribution with two shape parameters: a skewness parameter, $\lambda \in (-1,1)$, which controls for the degree of asymmetry, and a degree of freedom parameter, $\nu \in (2,\infty)$, which controls the thickness of the tails. We assume that the country factor Z_c and the idiosyncratic factor ε follow a normal distribution with mean 0 and variance σ^2. We further assume that they are all independent.

To implement the SMM estimator of the copula models, we have to choose dependence measures as our "moments". We use pair-wise Spearman's rho and quantile dependence with *q* = [0.25, 0.75]. Those measures are "pure" measures of dependence, meaning that they are solely affected by the changes in the copula, not by the changes in the marginal distribution. For a detailed discussion of such "pure" dependence measures, see Joe (1997) and Nelsen (2006).

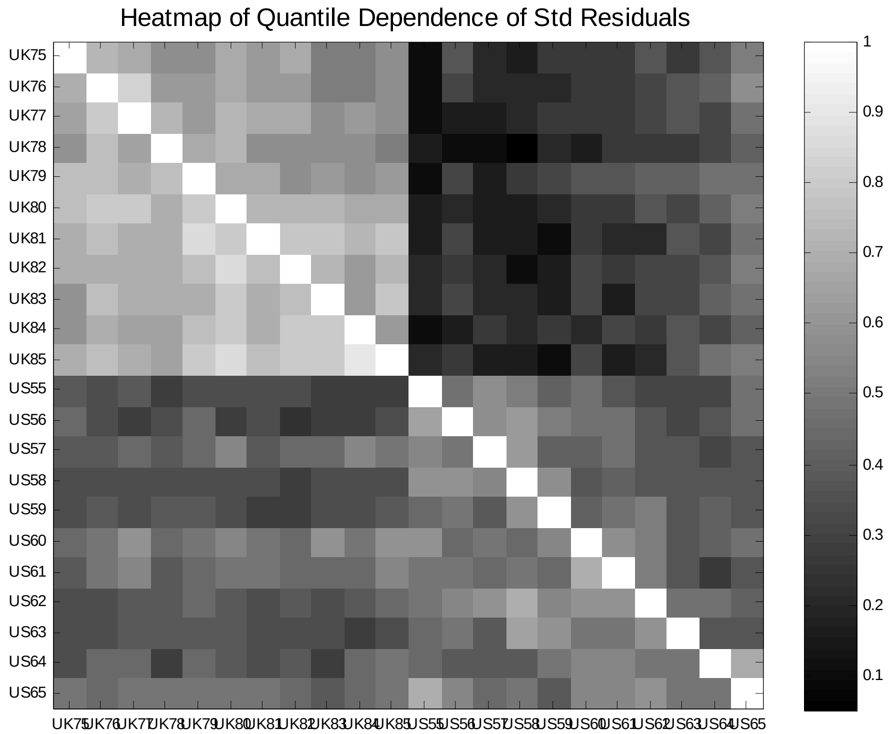
Figure 3 depicts the heat map of the Spearman's rho rank correlation matrix for standardized residuals obtained from the optimal ARMA-GARCH models. A lighter color in the heat map indicates a higher Spearman's rho. The upper left sub-matrix shows the intra-dependence between UK age groups and the lower right one for the intra-dependence between US groups. The upper right and lower left sub-matrices demonstrate the inter-group dependence. As we can see, most of the Spearman's rho are positive for both intra-group and inter-group measures with only a few exceptions. The result also suggests that the UK groups have stronger intra-group dependence (Spearman' rhos are close to 0.9) than the US groups. The inter-group dependence between UK and US groups are the weakest. We calculate the p-value for the Spearman's rho. We find that all intra-group dependence measures are highly significant at the 5% level while the inter-group dependence measures are almost insignificant.

Figure 3: Heatmap for Spearman rank correlation matrix



We also plot the heat map for the pair-wise quantile dependence measures at the 25th and 75th percentiles in Figure 4. The upper triangle matrix reports the 25th percentile dependence, and the lower triangle matrix reports the 75th percentile dependence. The heat map suggests an asymmetric dependence structure among the standardize residuals. That is, the upper tail dependence is larger than the lower tail dependence, especially for the inter-group tail dependence. This observation is consistent with our conjecture that mortality dependence is stronger when the market faces extreme mortality deterioration than it is at the time of graduate mortality improvement.

Figure 4: Heatmap for the 25th and 75th percentile dependence



Factor Copula Model Estimation (To be completed)

5. Pricing Methodology

Longevity/mortality bonds pay out periodical coupon payments at a rate of LIBOR plus a risk premium. Theoretically, risk premiums can be determined either by financial pricing approaches (creating a risk-neutral measure) or by insurance pricing approaches (adding a risk loading to expected losses). Embrechts (2000) provides an overview and comparison of these approaches.

Financial pricing approaches emphasize the creation of replicating portfolios (Black and Scholes, 1973; Merton, 1973) or the existence of a unique risk-neutral measure (Cox and Ross, 1976; Harrison and Kreps, 1979; Harrison and Pliska, 1981) in a complete market. In an incomplete market (e.g., the mortality/longevity market), with sparse market price data some prevalent pricing methodologies, such as the arbitrage free pricing method (Cairns et al. 2006, Bauer et al. 2010), the Wang transform (Dowd et al. 2006, Denuit et al. 2007, Lin and Cox 2008, Chen and Cox 2009), or the Esscher transform (Chen et al. 2010, Li et al. 2010), often require the user to make one or more subjective assumptions to derive the market prices of risk. The pricing process becomes even more difficult when multiple risks are involved.

The alternative insurance pricing approaches usually assume investors are risk averse, thereby adding a risk loading to the expected loss to compensate investors for the risk of the insurance contract. The risk loading is either proportional to the expected loss (EL), or to the variance or standard deviation of the loss, or determined by a Cobb–Douglas function on the probability of first loss (PFL) and the conditional expected loss (CEL) (Lane 2000). Extending the previous work, Lane and Beckwith (2008) and Lane and Mahul (2008) suggest a multiple linear model using the expected loss and a factor covering cycle effects to decide risk premiums. Other multiple linear approaches have been established by Berge (2005) and Dieckmann (2008). Both analyses identify Cat-bond-specific factors, such as peril, trigger, size, and rating, to capture the risk loading. Loglinear models have also been used in Major and Kreps (2003) and Dieckmann (2008).

The model that we propose to use in this paper can be written as follows

$$\ln\left(\text{Premium}\right) = \alpha + \beta \cdot \ln\left(PFL\right) + \gamma \cdot \ln\left(CEL\right) + \lambda \cdot Z + \varepsilon$$

We use the following explanatory variables in our model.

Risk Measures. In a perfect, riskless market, the risk premium should be equal to the expected loss, provided that insurance risk is uncorrelated with the movement of financial markets. However, insurance markets are not riskless and are far from perfect. Therefore, investors normally require a risk loading to compensate for risk bearing in the market for catastrophe securities (Lane, 2000). The expected loss principle sets the risk loading proportional to the expected loss. It requires only the first moment of the loss distribution and thus can be easily implemented. It, however, is not risk sensitive. The variance/standard deviation principle takes into account the variation of the risk and works well for symmetric random outcomes. However, insurance risk is often skewed: most of the probability mass is centered at zero loss, while there is a small probability of potentially large negative returns. Lane (2000) use the conditional expected loss (CEL) to measure the loss severity (and capture the asymmetry of insurance risk) and the probability of first loss (PFL) to measure the loss frequency in a risk cubic pricing model. We include EL, CEL and PFL in our model.

Trigger. The payout of a CAT bond depends on trigger mechanisms. Basically, there are five different trigger mechanisms: indemnity, modelled loss, industry loss, parametric or hybrid triggers. Cummins and Weiss (2009) and Dubinsky and Laster (2003) argue that CAT bond prices with an indemnity trigger might be higher than those with other triggers due to moral hazard. They also state that transaction costs for indemnity triggered CAT bonds are very high, because more documentation is needed compared to nonindemnity trigger mechanisms. For this purpose, we include a dummy variable *Indemnity* whose value is equal to one if the CAT bond is indemnity triggered and zero otherwise.

Rating. Investors usually require credit risk premiums for corporate bonds because of bearing credit risk. For CAT bonds, a rating has been assigned by at least one of the three rating agencies S&P, Moodys, or Fitch. To test whether the CAT bond rating plays a role in its premium pricing, we include a dummy variable *BBB+* to indicate a CAT bond with a BBB or A rating, and a dummy variable *BB* for a CAT bond with a BB rating. CAT bonds with a rating lower than BB are omitted.

Perils. The perils covered in this data set include hurricanes or more general windstorms, earthquakes, mortality risk as well as liability risk. We use a dummy variable *Wind* that equals one in case hurricane or windstorm risk is part of the underlying structure and zero otherwise. Dummy variables *Earthquake* and *Mortality* are constructed in a similar way. A catastrophe bond can be a single-peril or multi-peril construction. We include a dummy variable *Multi-peril* that equals one in case the bond is a multi-peril construction, and zero otherwise.

Region. The geographic regions in this data set include the United States and subregions, the Northern Atlantic, Europe, and Japan. A catastrophe bond can be a single-region or multi-region contract. We use a dummy variable *US* that equals one in case U.S. soil is part of the stipulated region, and zero otherwise. In addition, we construct a dummy variable *Multi-region* that equals one in case the bond is a multi-region structure, and zero otherwise.

Maturity. There are different maturities of CAT bonds in the markets. It could occur that sponsors prefer CAT bonds with longer maturities to avoid price changes on the reinsurance market. This assumption is analyzed by including dummy variables *12-24M* and *25-48M* to denote CAT bonds with maturities between 12-24 months and 25-48 months respectively. CAT bonds with maturities longer than 48 months are omitted.

Size. The amount of each bond is given by the face value expressed in million USD, two bonds in this data set are denoted in EUR currency.

Our data is compiled from two sources. The main source are publications and trade notes by Lane Financial L.L.C., which reports EL, PFL, CEL, Size, and Maturity of CAT bonds. Other information, such as Peril, Trigger, Region and Rating, comes from Aon's quarterly review on insurance-linked security market. Observations with a ratio of spread over expected losses larger than 20 and the spread itself higher than 20% are excluded from the dataset, as we consider them as outliers. After data cleaning, we have 387 bonds in our sample.

We try various model specifications and report the regression results in Table 7. As can be seen, the adjusted R-squared ranges from 75% to 82%, indicating good model fitness. The two risk measures, CEL and PFL, explain about 75% of the variation of the CAT bond spreads and their impacts are significantly positive. Size can be viewed as an indicator of market liquidity (Edward et al. 2007). Its effect on spreads is significantly positive at the 10% level. The impact of multiple perils is positive but not significant, so does Indemnity. We find that the CAT bond premium is significantly higher if U.S. soil is a part of the stipulated region. Mortality bonds usually are priced lower than CAT bonds with other perils. This is particularly true for mortality bonds covering extreme mortality risks in the US.

Figure 5 and 6 illustrate the goodness of in-sample fitting. We compare the observed spreads with the fitted values for all CAT bonds in our sample in Figure 5. The graph shows that our model fits the historical data quite well. We do the same for mortality bonds in Figure 6 and find that our pricing model can generate par spreads close to the actual spreads for previously issued mortality bonds. This provides us with additional evidence that CAT bond pricing methodologies can be equally applied to mortality/longevity bond pricing.

Table 7: Regression Results for CAT Bond Pricing

VARIABLES	Model (1)	Model (2)	Model (3)	Model (4)	Model (5)	Model (6)
Log(CEL)	0.178***	0.187***	0.174***	0.172***	0.163***	0.159***
	(0.0363)	(0.0435)	(0.0385)	(0.0414)	(0.0447)	(0.0483)
Log(PFL)	0.628***	0.605***	0.573***	0.569***	0.557***	0.553***
	(0.0327)	(0.0345)	(0.0242)	(0.0229)	(0.0252)	(0.0240)
Size		0.000432**	0.000368*	0.000331*	0.000364*	0.000328*
		(0.000189)	(0.000191)	(0.000187)	(0.000187)	(0.000184)
Maturity		-0.00214	-0.00118	-0.000189	-0.00146	-0.000463
		(0.00132)	(0.00138)	(0.00130)	(0.00139)	(0.00132)
Multi-peril		0.121***	0.127***	0.117***	0.0689	0.0557
		(0.0303)	(0.0306)	(0.0297)	(0.0540)	(0.0520)
Indemnity		0.0600*	0.0419	0.0280	0.0245	0.0107
		(0.0348)	(0.0337)	(0.0331)	(0.0331)	(0.0327)
US		0.249***	0.244***	0.306***	0.237***	0.324***
		(0.0452)	(0.0455)	(0.0385)	(0.0452)	(0.0860)
Wind					0.156***	0.180**
					(0.0574)	(0.0763)
Earthquake					0.0310	0.0502
					(0.0490)	(0.0694)
Mortality			-0.311*	0.117	-0.228	0.232
			(0.160)	(0.242)	(0.179)	(0.264)
US*Wind						-0.0214
						(0.0835)
US*Earthquake						-0.0208
						(0.0742)
US*Mortality				-0.770**		-0.806**
				(0.311)		(0.325)
Constant	-0.148	-0.464***	-0.606***	-0.694***	-0.764***	-0.877***
	(0.127)	(0.146)	(0.114)	(0.102)	(0.132)	(0.118)
Observations	387	387	387	387	387	387
R-squared	0.749	0.798	0.805	0.818	0.810	0.824
Adj R-squared	0.748	0.794	0.801	0.814	0.804	0.818

Note: The dependent variable is log(Premium). Robust standard errors are reported in parentheses. *** (** or *) denotes significance at the 1% (5% or 10%) level.

Figure 5: Observed and Predicted Spreads for All CAT Bonds in Our Sample

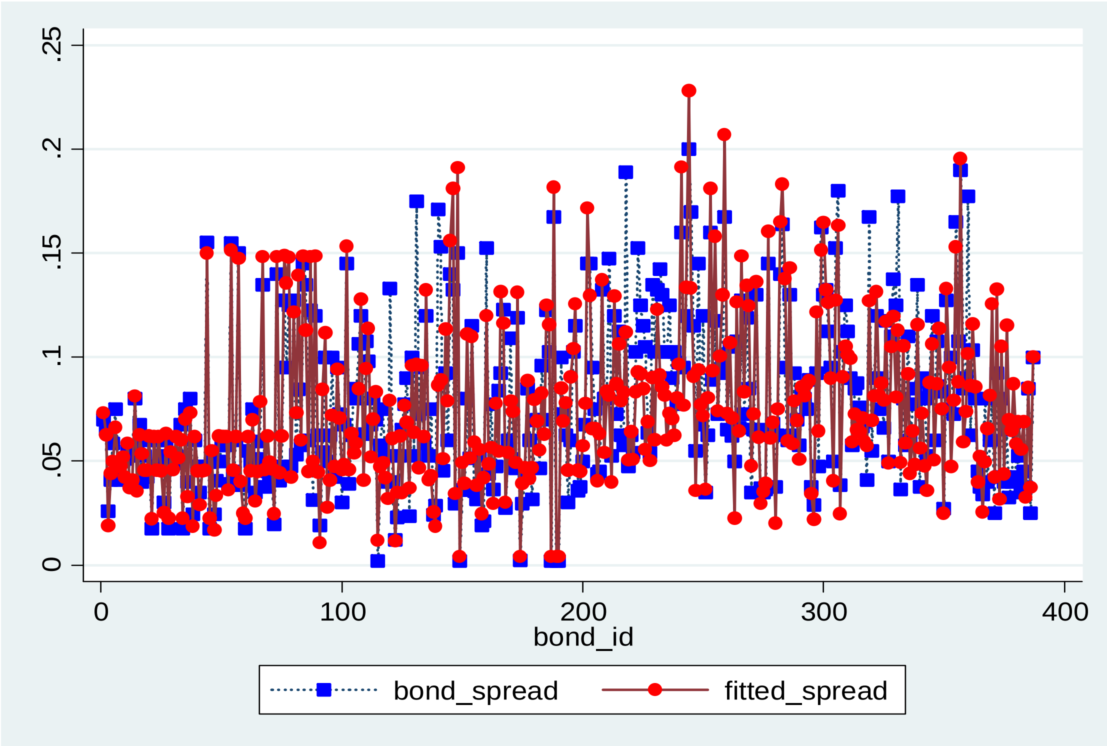
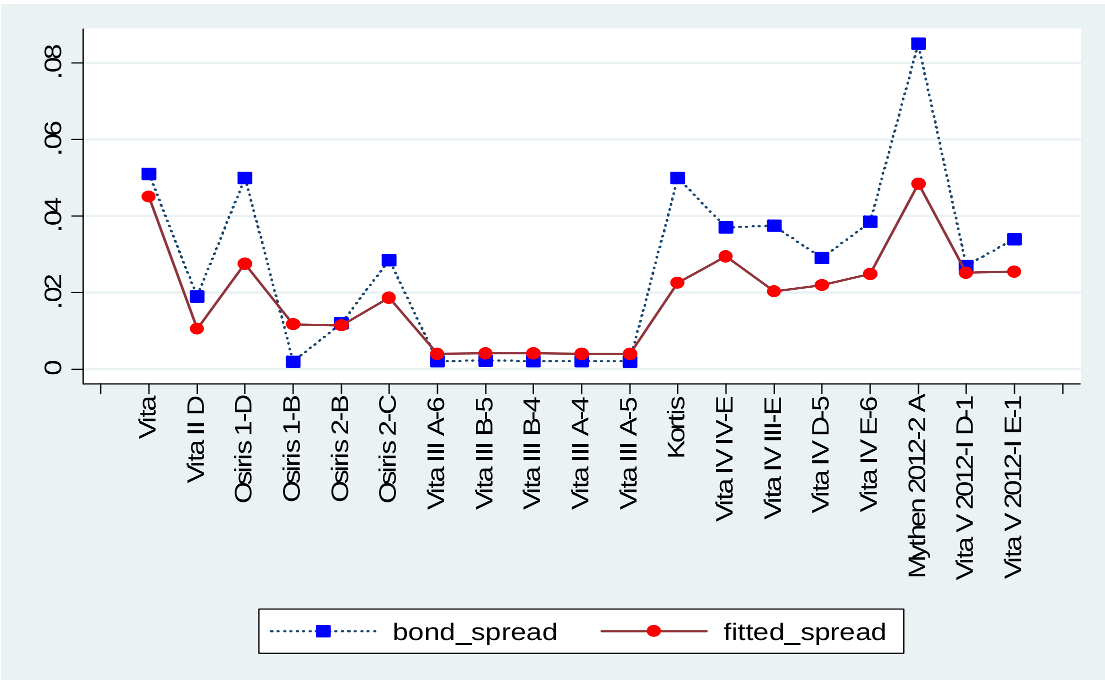


Figure 6: Observed and Predicted Spreads for Mortality Bonds



6. Conclusions

Many life insurers and reinsurers operate internationally and pool policies across countries. It is, therefore, increasingly important for them to understand the joint mortality dynamics for multiple populations and assess mortality/longevity risk in their own books of business. To hedge the risk, they actively participate in mortality/longevity securitizations and engage in issuing or trading mortality-linked securities. The reference populations associated with these hedging instruments usually exhibit mortality improvement rates that are different from those of the hedger's population and this introduces basis risk that the hedger must manage. In addition, almost all mortality securitizations, albeit their difference in structure, have payoffs contingent on a weighted mortality index of multiple populations. Therefore, developing mortality models for multiple populations and understanding the correlated mortality risks are crucial to life insurers and pension planners.

Although there is a rich literature for mortality modeling in a single population, only a few papers (see, e.g., the reference cited in introduction) examine two-population mortality models. These models are based on a critical assumption that mortality rates of two populations do not diverge in the long run, which seems too strong for short-term mortality forecasts. In spite of the extensive use of copula models in finance and economics, only two studies (i.e., Chen et al. 2013 and Wang et al. 2013) has explored its application to multi-population mortality analysis.

In this paper, we propose a dynamic multivariate mortality model based on a factor copula and the GAS structure. The factor copula model that we use has a simple (additive) form and is flexible in model specification according to data availability and the number of variables that need to be specified. It is particularly useful to high dimension application.

This characteristic is important when the number of populations that we need to model is large. We extend the one-factor model in Chen et al. (2013) to a two-factor copula model, with one factor representing the common, market force and the other having a country-specific mortality effect. We also introduce dynamic dependence among multiple populations using the GAS structure.

Finally, we illustrate how to apply our model to modeling risk modelling and pricing. Using the Kortis bond as one example, we fit the mortality model for US male 55-65 and UL male 75-85 and project the probability of the bond principal reduction. We apply the CAT bond empirical pricing methodology to mortality/longevity bond pricing. Our regression model can explain about 80% of the premium variations for CAT bonds. When examining mortality/longevity bonds only, we find the fitted par spreads are very close to the premiums observed on the primary market.

Reference:

Bauer, D., M. Boerger, and J. Russ (2010). On the pricing of longevity-linked securities, *Insurance: Mathematics and Economics* 46(1): 139-149.

Berge, T. (2005). *Katastrophenanleihen. Anwendung, Bewertung, estaltungsempfehlungen* (Lohmar: EUL Verlag).

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities, Journal of Political Economy 81 (3): 637–654.

Box, G. E. P., and G. M. Jenkins (1976). *Time Series Analysis: Forecasting and Control*, 2nd ed. San Francisco: Holden-Day.

Cairns, A. J. G., D. Blake, and K. Dowd (2006). A Two-factor model for stochastic mortality with parameter uncertainty: Theory and calibration, *Journal of Risk and Insurance* 73(4): 687-718.

Cairns, A. J. G., Blake, D., Dowd, K., Coughlan, G. D., and Khalaf-Allah, M. (2011) Bayesian Stochastic Mortality Modelling for Two Populations. *ASTIN Bulletin* 41: 29-59.

Carriere, J. F. (2000). Bivariate survival models for coupled lives. *Scandinavian Actuarial Journal* 1: 17–31.

Chen, H., and S. H. Cox (2009). Modeling mortality with jumps: Applications to mortality securitization, *Journal of Risk and Insurance* 76(3): 727-751.

Chen, H., S. H. Cox, and S. S. Wang (2010). Is the Home Equity Conversion Program in the United States sustainable? Evidence from pricing mortgage insurance premiums and Non-Recourse Provisions using the conditional Esscher transform, *Insurance: Mathematics and Economics* 46(2): 371-384.

Chen, H. and D. J. Cummins (2010). Longevity bond premiums: the extreme value approach and risk cubic pricing, Insurance: Mathematics and Economics 46(1): 150-161.

Chen, H., R. MacMinn and T. Sun (2013). Multi-population Mortality Models: A Factor Copula Approach, working paper, Temple University.

Cox, C. J. and S. A. Ross (1976). The valuation of options for alternative stochastic processes,Journal of Financial Economics, 3(1-2):145-166.

Creal, D.D., S.J. Koopman, and A. Lucas (2013). Generalized autoregressive score Models with Applications, Journal of Applied Econometrics 28(5):777-795.

Cummins, J. D. and M. A. Weiss (2009). Convergence of insurance and financial markets: hybrid and securitized risk transfer solutions, Journal of Risk and Insurance 76(3) 493-545.

Denuit, M., P. Devolder, and A. C. Goderniaux (2007). Securitization of longevity risk: Pricing survivor bonds with Wang Transform in the Lee–Carter framework, *Journal of Risk and Insurance* 74(1): 87-113.

Denuit, M., J. Dhaene, C., Le Bailly de Tilleghem, and S. Teghem (2001). Measuring the impact of dependence among insured lifelengths. *Belgian Actuarial Bulletin*, 1(1): 18–39.

Dieckmann, S. (2008). By force of nature: explaining the yield spread on catastrophe bonds, Working Paper, Wharton School, University of Pennsylvania, Philadelphia, PA.

Dowd, K., D. Blake, A. J. G. Cairns, and P. Dawson (2006). Survivor swaps, *Journal of Risk and Insurance* 73(1): 1-17.

Dowd, K., A. J. G. Cairns, D. Blake, G. D. Coughlan and M. Khalaf-Allah (2011). A gravity model of mortality rates for two related populations. *North American Actuarial Journal* 15(2): 334-356.

Dubinsky, W., and D. Laster (2003). Insurance-Linked Securities, Swiss Re Capital Markets Corporation.

Embrechts, P. (2000). Actuarial vs financial pricing of insurance, unpublished manuscript.

Embrechts, P., F. Lindskog, and A. McNeil (2003). Modeling dependence with copulas and applications to risk management In: *Handbook of Heavy Tailed Distributions in Finance*, ed. S. Rachev, Elsevier, Chapter 8, 329-384.

Engle, R.F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models, *Journal of Business & Economic Statistics* 20(3): 339-350.

Frees, E. W., J. F. Carriere, and E. A. Valdez (1996). Annuity valuation with dependent mortality. *Journal of Risk and Insurance* 63 (2), 229–261.

Frees, E. W., and E. A. Valdez (1998). Understanding relationships using copula, *North American Actuarial Journal* 2(1): 1-25.

Hafner, C.M. and H. Manner (2012). Dynamic stochastic copula models: estimation, inference and applications, *Journal of Applied Econometrics* 27(2):269-295.

Harrison, J. M. and D. M. Kreps (1979). Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20(3): 381-408.

Harrison, J.M. and S. R. Pliska (1981). Martingales and Stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11 (3): 215–260.

Hunt,A., and D. Blake (2013). Modelling longevity bonds: Analysing the Swiss Re Kortis Bond, working paper, Pension Institute.Jarner, S. F. and E. M. Kryger (2011). Modelling adult mortality in small populations: the SAINT model. *ASTIN Bulletin* 41: 377-418.

Joe, H. (1997). Multivariate models and dependence concepts, Monographs in Statistics and Probability 73, Chapman and Hall, London.

Joe, H. (2005). Asymptotic efficiency of the two-stage estimation method for copula-based models, *Journal of Multivariate Analysis* 94: 401-419

Lane, M. N. and O. Y. Movchan (1999). Risk cubes or price risk and rating (part II), *Journal of Risk Finance* 1(1): 71-86.

Lane, M. N.(2000). Pricing risk transfer transactions, *ASTIN Bulletin* 30(2): 259-293.

Lane, M. N., and R. G. Beckwith (2008). The 2008 Review of ILS Transactions—What Price ILS? A Work in Progress. Lane Financial LLC Trade Notes, March 31.

Lane, M. N., and O. Mahul (2008). Catastrophe Risk Pricing: An Empirical Analysis, Policy Research Working Paper 4765, The World Bank.

Lee, R.D., and L. R. Carter (1992). Modeling and forecasting U.S. mortality. *Journal of the American Statistical Association* 87: 659-671.

Li, N., and R. Lee (2005). Coherent mortality forecasts for a group of population: An extension to the classical Lee-Carter Approach. *Demography* 42: 575-594.

Li, J. S.-H. (2010). Pricing longevity risk with the parametric bootstrap: A maximum entropy approach, *Insurance: Mathematics and Economics* 47(2):176-186.

Li, J. S.-H. and M. R. Hardy (2011). Measuring basis risk in longevity hedges, *North American Actuarial Journal* 15(2): 177-200.

Li, J. S.-H., M. R. Hardy, and K. S. Tan (2010). Pricing and hedging the no-negative-equity-guarantee in equity release mechanisms, *Journal of Risk and Insurance* 77(2):499-522.

Lin, Y., and S. H. Cox (2008). Securitization of catastrophe mortality risks, *Insurance: Mathematics and Economics* 42(2): 628-637.

Lin, Y., S. Liu and J. Yu (2013). Pricing mortality securities with correlated mortality indexes, *Journal of Risk and Insurance* 80 (4): 921-948.

Manner, H. and J. Segers (2011). Tails of correlation mixtures of elliptical copulas, *Insurance: Mathematics and Economics* 48(1): 153-160.

Major, J., and R. Kreps (2003). Catastrophe risk pricing in the traditional market, in: M. Lane, ed., *Alternative Risk Strategies* (London: Risk Books).

Merton, C. R., (1973). Theory of rational option pricing, Bell Journal of Economics and Management Science 4(1): 141-183.

Nelsen, R. B. (2006). *An Introduction to Copulas*, Second Edition, Springer, U.S.A

Patton, A. J. (2006). Estimation of multivariate models for time series of possibly different lengths, *Journal of Applied Econometrics* 21(2): 147-173.

Oh, D. H., and A. J. Patton (2012). Modelling dependence in high dimensions with factor copulas, working paper, Duke University.

Oh, D. H., and A. J. Patton (2013). Time-varying systemic risk: evidence from a dynamic copula model of CDS spreads, working paper, Duke University.

Shephard, N., 2005, *Stochastic Volatility: Selected Readings*, Oxford University Press, Oxford.

Shemyakin, A. and H. Youn (2006). Copula models of joint last survivor analysis. *Applied Stochastic Models in Business and Industry*, 22: 211–224.

Swiss Re (2012). A mature market: building a capital market for longevity risk. Swiss Re Europe S.A., UK branch.

Wang, C. W., S. S. Yang, and H. C. Huang (2013). Modelling multi-coungry mortality dependence and its application in pricing survivor index swaps-a dynamic copula approach, working paper.

Yang, S. S. and C.-W Wang (2013). Pricing and securitization of multi-country longevity risk with mortality dependence, *Insurance: Mathematics and Economics* 52(2):157-169.

Youn, H. and A. Shemyakin (1999). Statistical aspects of joint life insurance pricing. *1999 Proceedings of the Business and Statistics Section of the American Statistical Association*, 34–38.

Youn, H. and A. Shemyakin (2001). Pricing practices for joint last survivor insurance. *Actuarial Research Clearing House*, 2001.1.

Zhou, R., J. S.-H. Li and K. S. Tan (2013a) Pricing standardized mortality securitizations: A two-population model with transitory jump effects, *Journal of Risk and Insurance* 80(3): 733-774.

Zhou, R., Y. Wang, K, Kaufhold, J. S.-H. Li and K. S. Tan (2013b). Modeling mortality of multiple populations with vector error correction models: Applications to solvency II, working paper.