Semiparametric detection of a change in long range dependence

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Today's presentation

1. The literature on breaks in the stochastic dynamics
2. Local Whittle estimate for a process subject to a break in the stochastic term
   - robust estimation of the memory parameter in presence of instability in the short term dynamics
   - a test to detect a change in persistence
   - estimation of the location of the break
3. Application to persistence of inflation in the Euro-area.

The Model:

Consider two stationary and invertible processes ξ_{1t} and ξ_{2t} such that

$$f_{\xi 1}(\lambda) \sim G_{\xi 1}\lambda^{-2\delta_1}, \quad f_{\xi 2}(\lambda) \sim G_{\xi 2}\lambda^{-2\delta_2} \text{ as } \lambda \to 0^+,$$

i.e., $\xi_{1t} \in I(\delta_1)$, $\xi_{2t} \in I(\delta_2)$, $-1/2 < \delta_1, \delta_2 < 1/2$.

ξ_{1t} and ξ_{2t} cannot be observed: we can only observe, at $t = 1, \ldots, n$,

$$x_t = \xi_{1t} 1(t \leq [\tau_0 n]) + \xi_{2t} 1(t > [\tau_0 n])$$

i.e.

$$x_t = \begin{cases} \xi_{1t} \text{ if } t \leq [\tau_0 n] \\ \xi_{2t} \text{ if } t > [\tau_0 n] \end{cases}$$

for a constant $\tau_0 \in (0, 1)$.

The break could be

- $G_{\xi_1} \neq G_{\xi 2}$ (change in the short term dynamics)
- $\delta_1 \neq \delta_2$ (change in persistence)
- No change, as it is for example when $\xi_{1t} = \xi_{2t}$.

The literature on testing procedures

- $\delta_1 = 0$ & $\delta_2 = 0$: testing instability in the short term dynamics:
  Andrews (1993) discussed a Chow test to detect the presence of a break, and Bai and Perron (1998) discussed the estimation of its location

- $\delta_1 = 0$ & $\delta_2 = 1$: testing extreme changes in persistence:
  Kim (2000), Kim, Belaire-Franch and Badilli-Amador (2002), Busetti and Taylor (2004), Harvey, Leybourne and Taylor (2004), Cavaliere and Taylor (2006)
- Dickey and Fuller type of tests are also sensitive to changes in the short term dynamics: Hamori and Tokihisa (1997)
- For non integer δ, Beran and Terrin (1996), Horvath and Shao (1999)

Local Whittle estimation

For the Fourier frequencies $\lambda_j = \frac{2\pi j}{n}$ (j integer and $0 < j < n$) the periodogram is

$$I_x(\lambda_j) = \frac{1}{2\pi} \sum_{|k|<n} c_x(k) \cos(\lambda_j k)$$

where $c_x(k)$ is the k^{th} sample covariance

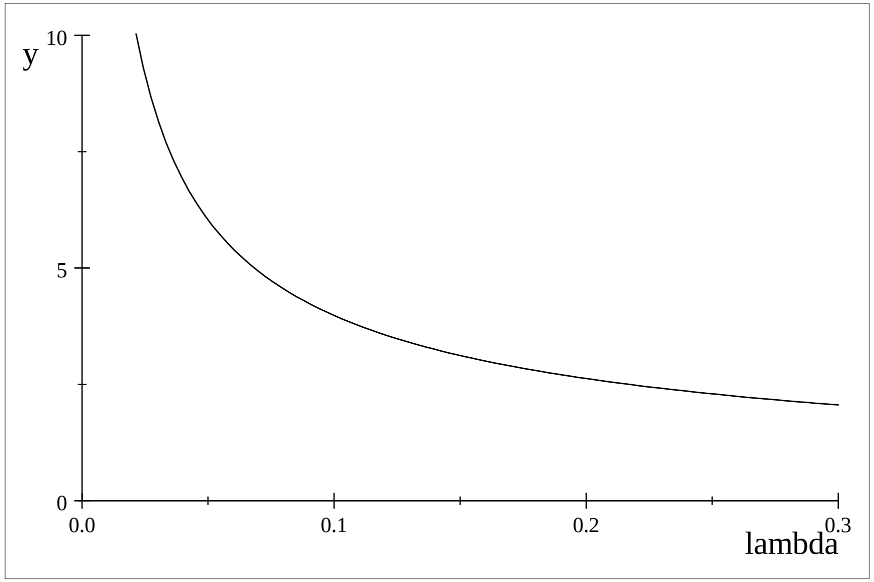
Property of the periodogram

For an $I(\delta)$ process ξ_t, for $j > 0$, as $j/n \to 0$,

$$E(I_\xi(\lambda_j)) = f_\xi(\lambda_j) + O(f_\xi(\lambda_j) j^{-1} \ln j)$$

Example (spectral density at low frequencies):

$\delta = 0.3$, then $\lambda^{-2\delta}$ is



The local Whittle estimate of δ

$$\hat{\delta} = \arg\min_{d} \ln\left\{ \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j) \right\} - 2d \frac{1}{m} \sum_{j=1}^{m} \ln(\lambda_j)$$

The expected value of a periodogram in case of a break:

If x_t contains a break, then for $j > 0$, as $j/n \to 0$,

$$
\begin{aligned}
&E(I_x(\lambda_j)) \\
&= \tau_0 f_{\xi 1}(\lambda_j) + (1-\tau_0) f_{\xi 2}(\lambda_j) \\
&\quad + O((f_{\xi 1}(\lambda_j) + f_{\xi 2}(\lambda_j))(j^{-1} \ln j))
\end{aligned}
$$

- If, $\delta_1 = \delta_2 = \delta$ (break in the short term dynamics)

$$E(I_x(\lambda_j)) = (\tau_0 G_{\xi 1} + (1 - \tau_0) G_{\xi_2}) \lambda_j^{-2\delta} + O\left(\lambda_j^{-2\delta} \frac{\ln j}{j} + \lambda_j^{-2\delta} \left(\frac{j}{n} \right)^{\alpha} \right)$$

x_t looks like a process of order δ:

the break in the short term dynamics is irrelevant

- If $\delta_1 > \delta_2$ (break in δ)

$$E(I_x(\lambda_j)) = \tau_0 G_{\xi 1} \lambda_j^{-2\delta_1} + O\left(\lambda_j^{-2\delta_1} \left(\frac{\ln j}{j} + \left(\frac{j}{n} \right)^{\alpha} \right) + \lambda_j^{-2\delta_2} \right)$$

x_t looks like a signal of order δ_1 with added noise of order δ_2:

δ_1 *(only) is consistently estimated*

Limit distribution / Lower order bias

When $\delta_1 \neq \delta_2$, let

$$\vartheta = |\delta_1 - \delta_2|$$

- if $\delta_1 > \delta_2$, $\frac{m^{2\vartheta+1/2}}{n^{2\vartheta}} \to 0$, as $n \to \infty$

$$2\sqrt{m\tau_0}\left(\widehat{\delta} - \delta_1\right) \to_d N(0,1)$$

- if $\delta_1 > \delta_2$, $\frac{m^{2\vartheta+1/2}}{n^{2\vartheta}} \to \infty$, as $n \to \infty$

$$\left(\frac{n}{m}\right)^{2\vartheta}\left(\widehat{\delta} - \delta_1\right) \to_p -\frac{1}{2}\frac{1-\tau_0}{\tau_0}\frac{G_{\xi 2}}{G_{\xi 1}}(2\pi)^{2\vartheta}\frac{2\vartheta}{(1+2\vartheta)^2}$$

(lower order bias)

Remarks

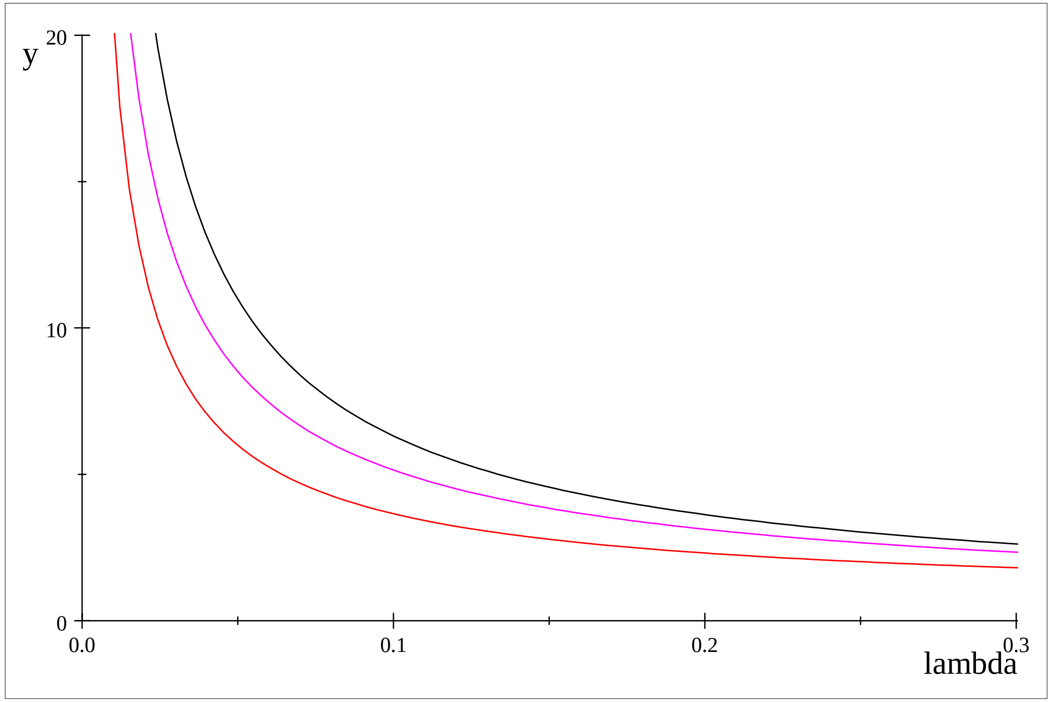
- the bigger the gap ϑ, the smaller the lower order bias *(cfr Andrews!!)*
- when there is a lower order bias, it is negative
- when $m = c_\kappa n^{0.8-\varepsilon}$, the lower order bias is present for any admissible combination δ_2, δ_1
- the smaller the gap ϑ, the smaller has to be the order of magnitude of the bandwidth m in order to avoid the lower order bias

Explanation of the lower order bias: an example

$\lambda^{-2\delta}$, with $\delta = 0.4$: black;

$\lambda^{-2\delta_1}/2 + \lambda^{-2\delta_2}/2$, with $\delta_1 = 0$, $\delta_2 = 0.4$: red;

$\lambda^{-2\delta_1}/2 + \lambda^{-2\delta_2}/2$, with $\delta_1 = 0.2$, $\delta_2 = 0.4$: magenta



A recursive version of the Chow test to detect a break:

- $\widehat{\delta}_1(\tau)$: estimate using the observations $x_1, \ldots, x_{[\tau n]}, 0, \ldots, 0$
- $\widehat{\delta}_2(\tau)$: estimate using the observations $0, \ldots, 0, x_{[\tau n]+1}, \ldots, x_n$

Let

$$\widehat{t}(\tau) = 2\sqrt{\tau(1-\tau)m}\left(\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right),$$

if $\delta_1 = \delta_2$, as $n \to \infty$, then

$$\widehat{t}(\tau) \to N(0,1)$$

introduce

$$\hat{t}^2 = \sup_{\tau \in [\tau_l, \tau_h] \subset (0,1)} \hat{t}(\tau)^2$$

then

$$\begin{cases} \hat{t}^2 \Rightarrow \sup_\tau \frac{(B(\tau) - \tau B(1))^2}{4\tau(1-\tau)} \text{ if } \vartheta = 0 \\ \hat{t}(\tau)^2 \to \infty \text{ if } \vartheta > 0 \end{cases}$$

where $\sup_\tau \frac{(B(\tau)-\tau B(1))^2}{4\tau(1-\tau)}$ is the supremum on $[\tau_l, \tau_h]$ of the square of a standardised tied down Bessel process *(cfr Andrews)*

Consistent estimation

of the location of the breakpoint:

$$\widehat{\tau} = \arg \min_{\tau \in [\tau_l, \tau_h] \subset (0,1)} \widehat{Q}_n(\tau) = \tau \widehat{\delta}_1(\tau) + (1 - \tau) \widehat{\delta}_2(\tau)$$

(choose τ_l, τ_h so that $\tau_0 \in [\tau_l, \tau_h]$)

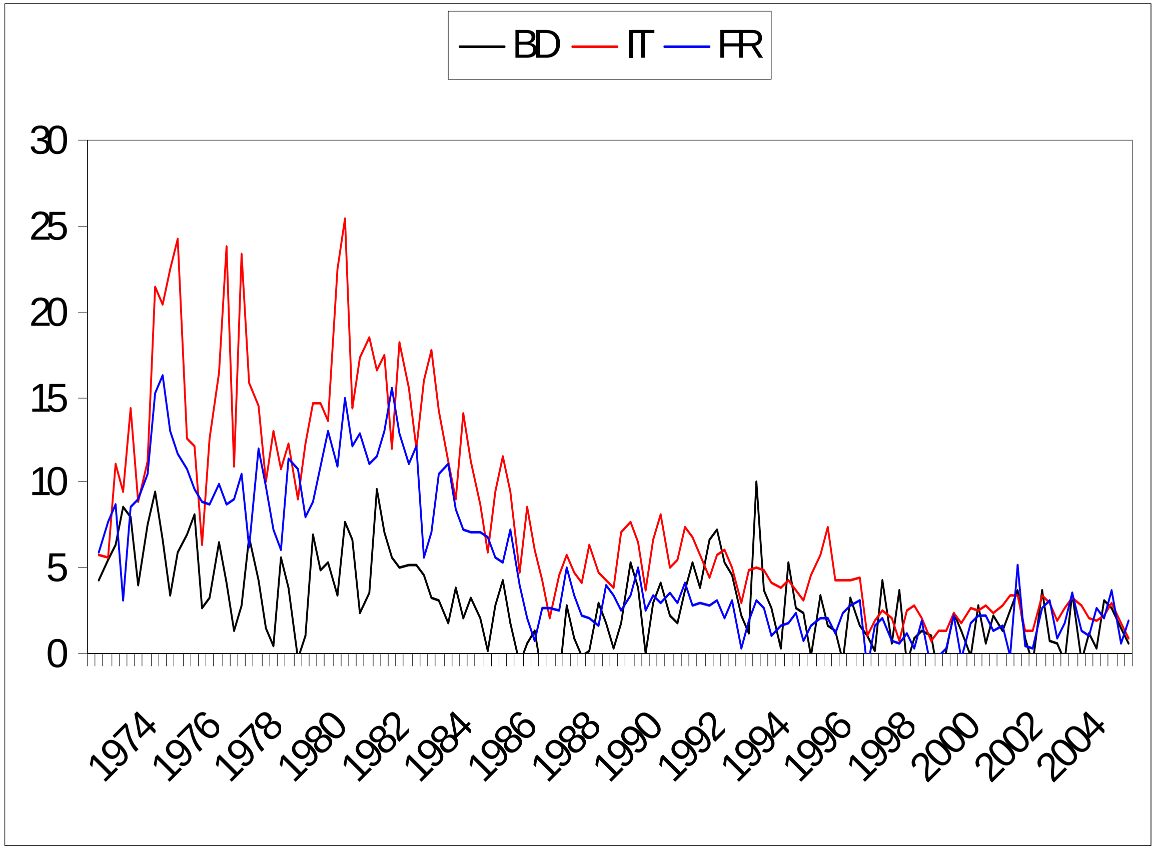
Monte Carlo evidence / 1

- A break in the short term dynamics has no effects on the estimation of δ (neither consistency nor the limit distribution are affected in a visible way)
- A large break in δ ($\vartheta = 0.4$) does not generate visible lower order bias (a residual bias appeared in in small samples, if the fraction of observations with high memory was little)
- A small break in δ ($\vartheta = 0.2$) does generate lower order bias

Monte Carlo evidence / 2

- $\widehat{t}^2$ has good size properties
- $\widehat{t}^2$ has good power
- $\widehat{\tau}$ estimates the location of the break with satisfactory precision even in moderate samples

Practical Problem:
Did the introduction of the ECB reduce the persistence of inflationary shocks?



Inflation stabilisation , 1973-2005

		$\widehat{\delta}$	$\widehat{\delta}_1(\tau_{99})$	$\widehat{\delta}_2(\tau_{99})$	$\widehat{t}^2(\tau_{99})$	$\widehat{t}^2$	$\widehat{\tau}n$
IT	lev.	0.83	0.87	0.71	0.39	2.47	124
	f. diff.	-0.12	-0.13	-0.59	3.56	13.60	122
FR	lev.	0.80	0.84	0.66	0.53	5.51	154
	f. diff.	-0.26	-0.27	-0.76	4.09	7.87	153
IR	lev.	0.57	0.61	0.70	0.16	4.25	117
	f. diff.	-0.42	-0.36	-0.40	0.02	8.43	142
BG	lev.	0.67	0.75	0.43	1.76	8.81	153
	f. diff.	-0.52	-0.41	-0.27	0.32	1.28	83

(the breaks are estimates to be between 1982 and 1984; no breaks are detected on the sample 1985-2005)

		$\widehat{\delta}$	$\widehat{\delta}_1(\tau_{99})$	$\widehat{\delta}_2(\tau_{99})$	$\widehat{t}^2(\tau_{99})$	$\widehat{t}^2$	$\widehat{\tau}n$
PT	lev.	0.53	0.55	0.86	1.64	3.91	258
	f. diff.	-0.48	-0.50	-0.10	2.66	6.58	49
ES	lev.	0.62	0.66	0.72	0.05	1.35	170
	f. diff.	-0.39	-0.32	-0.29	0.01	5.03	60
GR	lev.	0.47	0.49	0.43	0.05	0.93	340
	f. diff.	-0.48	-0.52	-0.64	0.28	6.86	67

		$\widehat{\delta}$	$\widehat{\delta}_1(\tau_{99})$	$\widehat{\delta}_2(\tau_{99})$	$\widehat{t}^2(\tau_{99})$	$\widehat{t}^2$	$\widehat{\tau}n$
BD	lev.	0.48	0.47	0.46	0.00	2.09	
	f. diff.	-0.34	-0.70	-0.20	4.27	9.79	
NL	lev.	0.79	0.84	0.51	1.92	6.74	
	f. diff.	-0.29	-0.32	-0.17	0.35	3.17	
FN	lev.	0.83	0.80	0.53	1.27	7.26	
	f. diff.	-0.30	-0.30	-0.53	0.84	7.07	
OE	lev.	0.54	0.60	0.71	0.20	2.92	
	f. diff.	-0.54	-0.66	-0.23	3.13	7.12	

Empirical Size of $\widehat{\delta}$ and of $\widehat{t}^{2}$ (theoretical size 5%)

Model	n	$\widehat{\delta}$	$\widehat{t}^{2}$
$x_t \in I(0.4)$	64	14.6	14.1
	128	13.4	13.9
	256	9.6	10.4
	512	8.0	7.6
	1024	7.4	7.7
$x_t \in I(0.4)$, $G_{\xi 2} = 2G_{\xi 1}$, $\tau_0 = 1/2$	64	16.6	13.8
	128	14.1	14.7
	256	11.0	11.9
	512	10.0	8.8
	1024	9.0	8.7

Empirical power of $\widehat{t}^2$ and properties of $\widehat{\tau}$

Model	n	$\widehat{t}^2$	$\widehat{\tau}$, bias	$\widehat{\tau}$, st. dev.	$\widehat{\tau}$, low 5%	$\widehat{\tau}$, up 5%
$\xi_{1t} \in I(0.4)$,	64	45.3	-0.0259	0.1908	0.1406	0.8438
$\xi_{2t} \in I(0)$,	128	58.0	-0.0294	0.1548	0.1875	0.7813
$\tau_0 = 1/2$	256	78.7	-0.0283	0.1173	0.2461	0.6875
	1024	99.7	-0.0145	0.0445	0.4150	0.5293
$\xi_{1t} \in I(0.4)$,	64	42.4	0.0733	0.2107	0.1406	0.8438
$\xi_{2t} \in I(0)$,	128	58.0	0.0354	0.1725	0.1484	0.7969
$\tau_0 = 1/3$	256	74.6	-0.0008	0.1238	0.1641	0.6016
	1024	99.8	-0.0103	0.0420	0.2607	0.3682
$\xi_{1t} \in I(0.2)$,	64	23.1	-0.0209	0.2318	0.1250	0.8594
$\xi_{2t} \in I(0)$,	128	25.2	-0.0320	0.2138	0.1406	0.8438
$\tau_0 = 1/2$	256	32.0	-0.0403	0.2080	0.1484	0.8398
	1024	61.4	-0.0319	0.1316	0.2246	0.7314