

TN 01 Basic algebra and financial mathematics

This teaching note was prepared by Georgio Questa with help from Dr. Enrico Biffis and Dr. Lorenzo Trapani

1. The summation operator (Σ)
2. Arithmetic and geometric series. Present value of an annuity & price of a coupon bond
3. Linear equations
4. Non-linear equations. Calculating the internal rate of return (IRR)
5. Mathematical induction ♦ Optional
6. The dividend discount model (DDM) ♦ Optional
7. List of Excel functions used in this Teaching Note TN01

Appendix: more on NPV and IRR ♦ Optional PDF on website

§1.1 –The summation operator (Σ)

- The (Σ) operator is widely used both in finance and statistics for finance applications.
- Therefore it makes sense to review it briefly in this teaching note.

Summation 1

- In the simplest case, (Σ) it is merely a compact way to denote a sum. Note that Excel has a worksheet function (SUM) to carry out a summation
- In the following equation we use a summation index (usually denoted with: i, j, k, \dots)
- This index is an integer. The lower value is indicated below the (Σ) sign, while the upper level is indicated above it
- Both the lower and the upper level are included in the summation

$$\sum_{j=1}^{10} a_j = a_1 + a_2 + \dots + a_{10}$$

Summation 2

- A very useful feature of the (Σ) operator is that we can use the index ($j = 1, 2, \dots, n$) both as a counter and as a parameter of the quantities we are adding up
- The simplest example is:

$$\sum_{j=1}^{10} j = 1 + 2 + \dots + 10 = 55$$

Summation 3

- A finance-specific use of the summation index is to denote the present value (PV) of a unit annuity (a unit annuity is simply a unit amount paid at regular intervals)
- The following equation indicates the present value (using compound yield) of a unit amount due for (n) years in arrears, y = annual yield, which is assumed to be fixed, irrespective of the time-horizon of the annuity

$$\begin{aligned} PV &= \frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \dots + \frac{1}{(1+y)^n} \\ &= (1+y)^{-1} + (1+y)^{-2} + \dots + (1+y)^{-n} \\ &= \sum_{j=1}^n (1+y)^{-j} \end{aligned}$$

Summation 4

- Another use of the indexed summation sign is to indicate a sum of products
- Excel has a spreadsheet function (appropriately named SUMPRODUCT) that will calculate a sum of products
- In TN 4 we'll see how to express this in terms on vector multiplication

$$\begin{aligned} & \sum_{j=1}^n (a_j \cdot b_j) \\ &= (a_1 \cdot b_1) + (a_2 \cdot b_2) + \dots + (a_n \cdot b_n) \end{aligned}$$

Summation 5

- One more application of the (Σ) operator is the nested sum of products
- The most common finance application of this notation is to indicate the variance of a portfolio (see TN 5 and TN 7)

$$\begin{aligned} M &= (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n) \\ &= (a_1 + a_2 + \dots + a_n)\left(\sum_{k=1}^n b_k\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n (a_j \cdot b_k) \end{aligned}$$

§1.2 - Arithmetic and geometric series.

Present value of an annuity & price of a coupon bond

In this teaching note we start with arithmetic series and then we proceed to geometric series, which have a number of important applications in finance (two examples are included in this note)

Arithmetic sequence

- An arithmetic progression, also known as an arithmetic sequence, is a sequence of numbers such that the differences between successive terms is a constant
- A generic arithmetic sequence comprising of (n) terms can be written as:

$$a_1 = a$$

$$a_2 = a + d$$

...

$$a_n = a_{n-1} + d$$

Arithmetic series 1

- An arithmetic series is simply the sum of an arithmetic progression
- A generic arithmetic series (SA) comprising of (n) terms can be written as follows:

$$S_A = a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$$

For example if $a=1$ and $d=2$ then

$$S_A = 1 + 3 + 5 + \dots + 2(n - 1)$$

Arithmetic series 2

- The value of an arithmetic series can be easily calculated by computer.
- However, there is a simple equation for the value of the series (which we are likely to find in finance documents)
- This formula is derived with one of the most commonly used algebraic manipulations: namely multiplying both sides of an equation by the same non-zero value (the number 2, in this case)

$$S_A = a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$$

$$2 \cdot S_A = 2 \cdot \{a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]\}$$

Arithmetic series 3

- The following algebraic manipulation shows that the value of an arithmetic series equals $(n/2)$ times the sum of the 1-st and n -th terms of the series
- To visualize this, we add the two series in inverse order:

$$\begin{array}{rcccc} S_A = & & a & + \dots + & [a + (n - 1)d] \\ +S_A = & & [a + (n - 1)d] & + \dots + & a \\ \hline \end{array}$$

$$\begin{aligned} 2 \cdot S_A &= 2a + (n - 1)d + \dots + 2a + (n - 1)d \\ &= n \cdot [2a + (n - 1)d] \end{aligned}$$

$$\therefore S_A = \frac{n}{2} \cdot [2a + (n - 1)d]$$

Arithmetic series 4 (numerical example)

- For example, the value of the arithmetic series $\{1+2+3+\dots+100\}$ can be computed as

$$\begin{array}{r} S_A = \quad 1 \quad + \dots + \quad 100 \\ +S_A = \quad 100 \quad + \dots + \quad 1 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \cdot S_A = \quad 101 \quad + \dots + \quad 101 \\ = \quad 100 \cdot 101 \end{array}$$

$$\therefore S_A = \frac{100}{2} \cdot 101 = 5050$$

Geometric Sequence

- An geometric progression, also known as a geometric sequence, is an ordered sequence in which the ratio of any two successive terms is a constant, known as common ratio. For example consider the sequence:

$$1, a, a^2, \dots, a^n$$
$$\therefore \frac{a_j}{a_{j-1}} = a$$

Value of a geometric series

- An intuitive way to find the sum of a geometric series is to multiply both sides of the equation by the same value, namely $(1 - a)$ in our case

$$\begin{aligned}(1 - a)S_g &= (1 - a) \cdot [1 + a + a^2 + \dots + a^n] \\ &= (1 - a) + (a - a^2) + (a^2 - a^3) + \dots + (a^{n-1} - a^n + a^n - a^{n+1}) \\ &= 1 + (-a + a) + (-a^2 + a^2) + \dots + (-a^n + a^n) - a^{n+1} \\ &= 1 - a^{n+1} \\ \therefore S_g &= \frac{1 - a^{n+1}}{1 - a}\end{aligned}$$

Present value of an annuity 1

- We have already examined the equation for the present value of an annuity when discussing the (Σ) operator
- The equation itself is a geometric series. Therefore, its value is:

$$\begin{aligned} \text{PV} &= (1+y)^{-1} + (1+y)^{-2} + \dots + (1+y)^{-n} \\ &= [(1+y)^{-1}] [1 + (1+y)^{-1} + \dots + (1+y)^{-(n-1)}] \\ &= (1+y)^{-1} \left[\frac{1-(1+y)^{-n}}{1-(1+y)^{-1}} \right] \\ &= \frac{1-(1+y)^{-n}}{(1+y)^{-1}} \\ &= \frac{1-(1+y)^{-n}}{y} \end{aligned}$$

Present value of a unit annuity 2

(bond pricing example)

- We now compute the present value of a 10-year bond, with principal = 100, annual coupon of 5, and yield = 6.00%
- The PV will equal the sum of the coupons annuity plus the PV of the principal, which will be paid at maturity (10-years)
- The bond is priced below par (92.64 < 100) because the yield is higher than the coupon rate

$$\begin{aligned} P &= 5 \left[\frac{1 - 1.06^{-10}}{0.06} \right] + 100 \cdot 1.06^{-10} \\ &= 36.8004 + 55.8395 = 92.6399 \\ &\quad \text{coupons} \qquad \qquad \text{principal} \end{aligned}$$

§1.3 -Linear equations

- Linear equations (and systems of linear equations) are very simple to deal with. Due to this, they are widely used in a wide range of applications in economics, finance and management
- In this section we'll also discuss systems of two linear equations in two unknowns. This will allow us to develop an intuitive understanding of the conditions under which a system of linear equations will have a meaningful solution

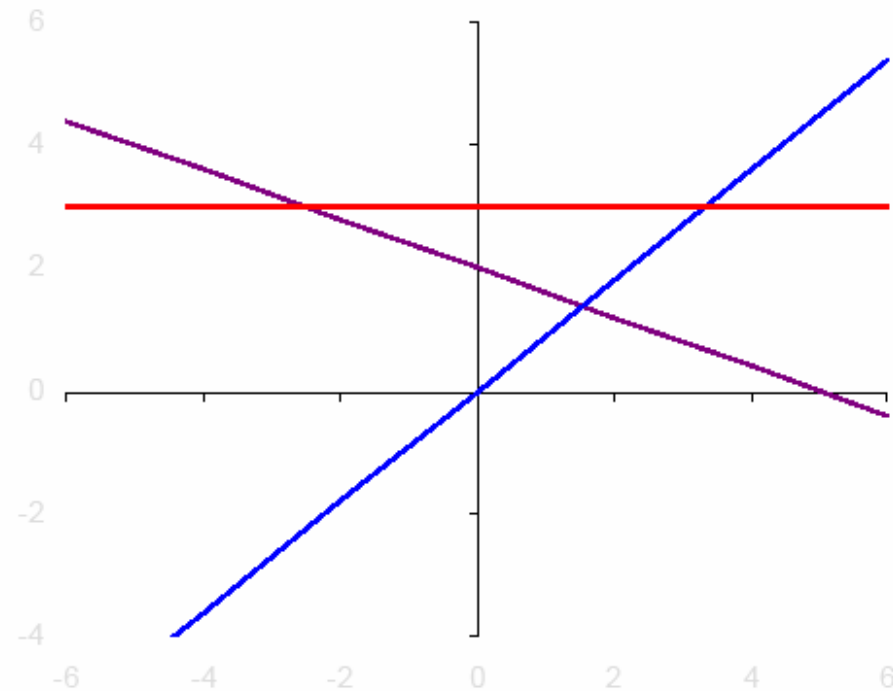
Linear functions

- A linear function (its graph is a straight line) is written in terms of two parameters (a, b).
- The parameter (a) determines where the straight line will intersect the y-axis (for $x = 0$) while the parameter (b) tells us the slope of the function
- If ($b > 0$) the line will be upwards sloping
- If ($b = 0$) the line will be parallel to the x-axis
- If ($b < 0$) the line will be downwards sloping

$$y = a + bx$$

Graphical Illustration

Exhibit 1 - Three linear functions with slope coefficients:
($b = 0$), ($b > 0$), ($b < 0$)



One linear equation in one variable

- Solving one linear equation in one variable means finding the value of (x) for which ($a + bx = 0$); that is to say where the linear function ($y = a + bx$) intersects the x-axis on the Cartesian plane
- If the linear equation is defined over the entire real line i.e. $(-\infty, \infty)$, then it will surely have a real solution, provided that the slope coefficient is not zero (see exhibit 1)
- The following equation & numerical example show the simple procedure to solve the equation

$$\begin{aligned}a + bx &= 0 \\x &= -\frac{a}{b} \\2 - 0.4x &= 0 \\x &= -\frac{2}{-0.4} \\\therefore x &= 5\end{aligned}$$

Two linear equations in two variables 1

- Let us consider the following system of two linear equations in two variables (including the numerical example)

$$\begin{cases} a_1 + b_1x + c_1y & = 0 \\ a_2 + b_2x + c_2y & = 0 \end{cases}$$

$$\begin{cases} -2 + 0.4x + y & = 0 \\ 0 - 1.8x + 2y & = 0 \end{cases}$$

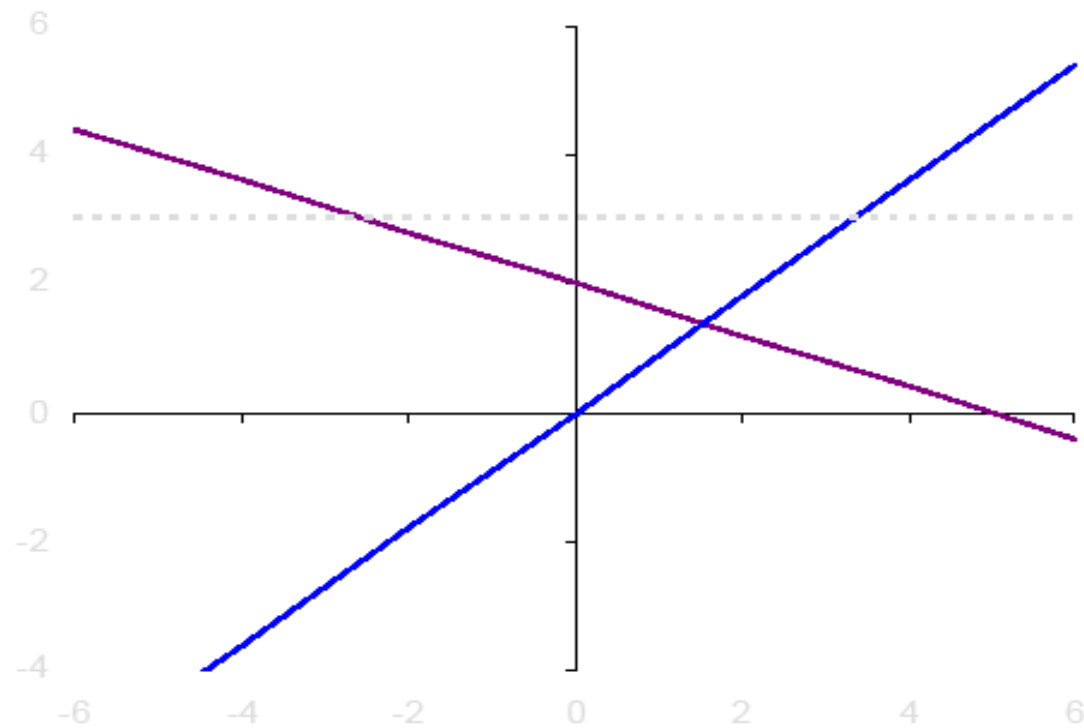
Two linear equations in two variables 2

- Let us consider the following system of two linear equations in two variables (including the numerical example)
- Each of the two equations can be transformed in a linear function of (x) on the Cartesian plane
- Thus, solving a system of two linear equations in two variables (x, y) means finding the two values (x*, y*) for which the two functions will intersect (see exhibit 2, based on the numerical example)

$$\left\{ \begin{array}{l} y = \frac{a_1}{c_1} - \frac{b_1}{c_1}x \\ y = \frac{a_2}{c_2} - \frac{b_2}{c_2}x \end{array} \right. \quad \left| \quad \begin{array}{l} y = 2 - 0.4x \\ y = 0 + 0.9x \end{array}$$

Two linear equations in two variables 2

Exhibit 2 – graphical representation of a system of two linear equations in two variables (based on the numerical example)



Two linear equations in two variables 4

- The standard way of solving a system of two linear equations is to multiply one of the equations (we choose the second one) by a constant (λ) and subtract it from the other equation (the first in our case).
- The constant is chosen in such a way that it will eliminate one of the variables (y in our case) in the first equation that we can now solve for (x) as a standard one-unknown linear equation
- Once we know (x), we substitute its value in the second equation and we solve for (y)
- In the next slide we show the necessary steps with reference to the numerical example

Two linear equations in two variables 5

- Solving by elimination two linear equations in two unknowns

$$\left. \begin{array}{r} -2 + 0.4x + y = 0 \\ 0 - 1.8x + 2y = 0 \end{array} \right\} \lambda = \frac{1}{2}$$

$$-2 + 0.4x + y = 0$$

$$0 - 0.9x + y = 0$$

$$-2 + 1.3x + 0 = 0$$

$$x = \frac{2}{1.3} = 1.5385$$

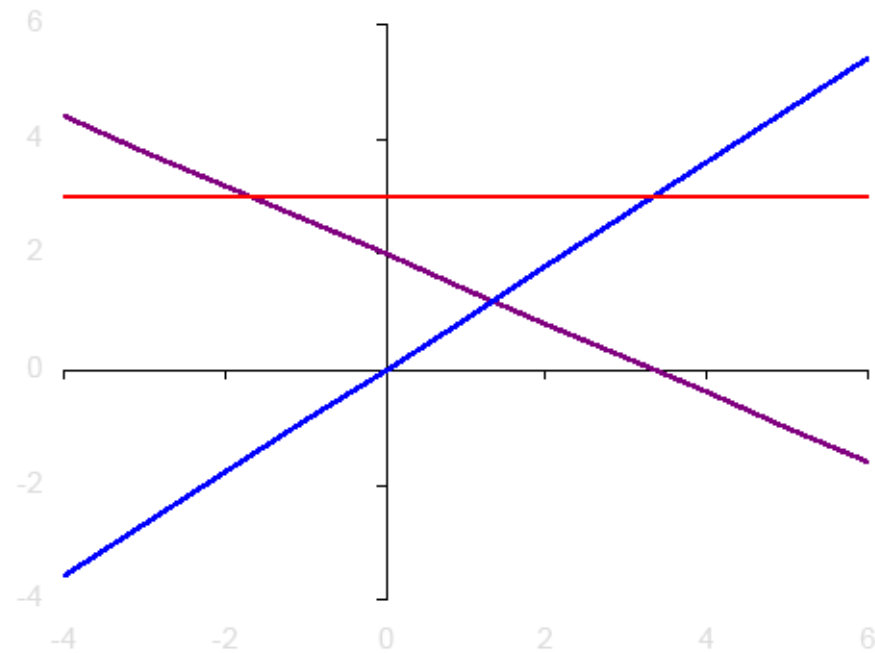
$$\therefore y = 0.9 \cdot 1.5385 = 1.3846$$

Counting equations and unknowns 1

- A system of linear equations will have a solution if the number of equations equals the number of unknowns (for example 2 equations in 2 unknowns)
- If we have more equations than unknown (for example, 3 equations in 2 unknowns) the system will not have a solution. In exhibit 1b the three lines have zero probability of intersecting in the same point

Graphical Illustration

Exhibit 1b - If we have more equations than unknowns (for example, 3 equations in 2 unknowns) the system will not have a solution. In the figure the three lines have zero probability of intersecting in the same point



Counting equations and unknowns 3

- If we have less equations than unknown (for example, 1 equation in 2 unknowns) the system will be indeterminate and have infinitely many solutions
- This follows from the fact that one equation in two unknowns describes a single real line on the Cartesian plane. Therefore, all the points on the line are valid solutions to the equation

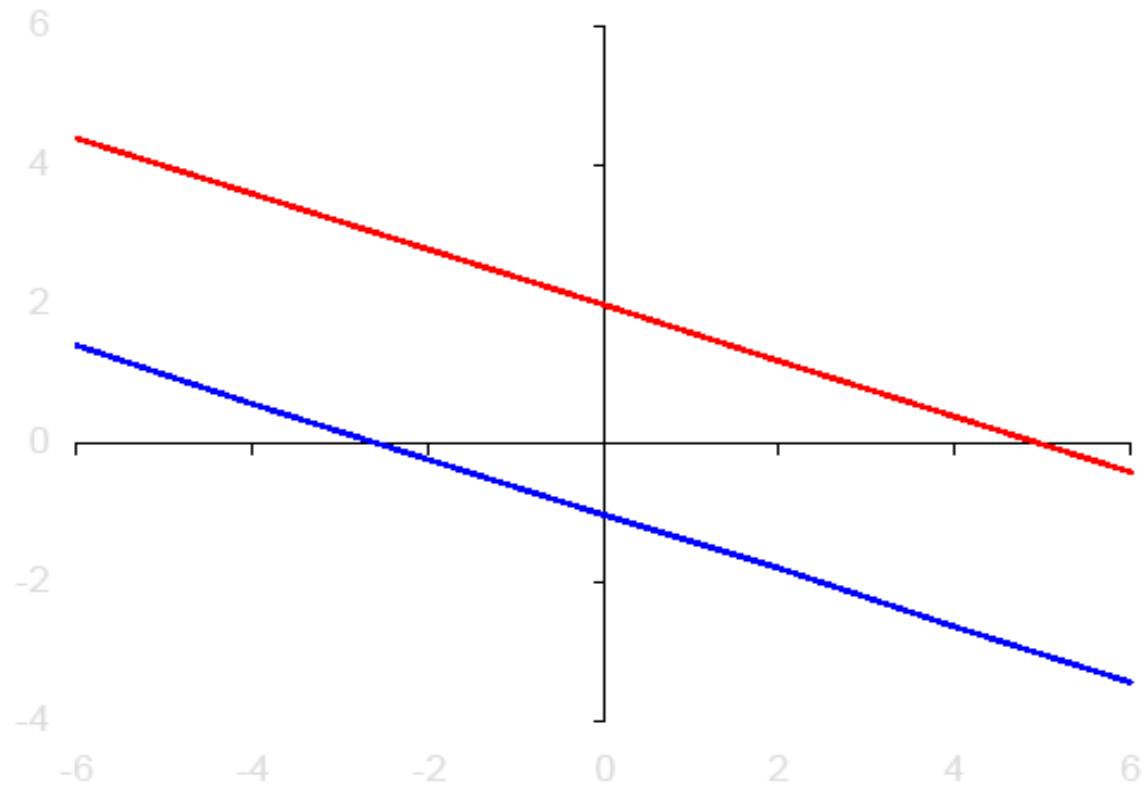
$$\begin{aligned}a + bx + cy &= 0 \\ \therefore y &= -\frac{a}{c} - \frac{b}{c}x\end{aligned}$$

Counting equations and unknowns 4

- ❑ One last point. The system will have a solution only if the equations are linearly independent
- ❑ In our two equations example this entails that the two lines are not parallel

Counting equations and unknowns 5

Exhibit 3 - Two linear equations in two variables. Note that this system does not have a solution because the two slope coefficients are equal



§1.4 -Non-linear equations. Calculating the internal rate of return (IRR)

- ❑ Linear equations and systems of linear equations are relatively simple to solve (see also TN 4 for solving multi-equation systems using matrices)
- ❑ This is not the case with non-linear equations that, in most instances, must be solved with numerical methods (they have no closed-form solution)
- ❑ Fortunately for us computers are great at using numerical methods, and some of the necessary software is also available in Excel

Finding the IRR of a cash flow 1

- We have seen that the present value of a cash flow (with equally spaced payments) is given by the following equation where:
- C_j = Payment at the end of year- j
- If we know the discount rate (y) finding PV is very easy, and the calculations can be carried out in Excel, with PV or NPV spreadsheet functions
- However, if we know PV determining (y) requires numerical methods. In Excel we can use the IRR function or SOLVER

$$PV = C_1(1 + y)^{-1} + C_2(1 + y)^{-2} + \dots + C_n(1 + y)^{-n}$$
$$= \sum_{j=1}^n C_j(1 + y)^{-j}$$

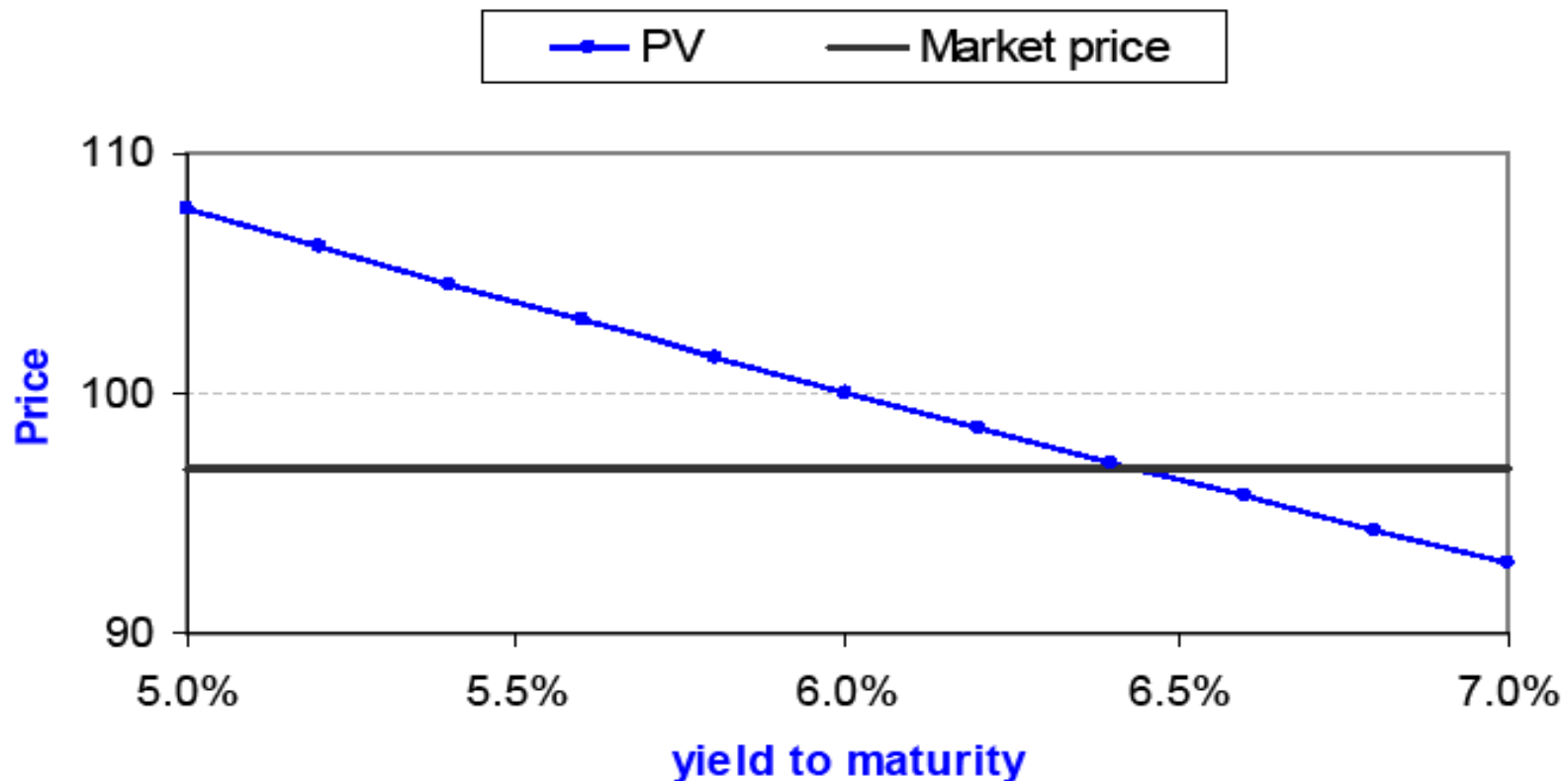
Finding the yield to maturity of a bond

- Let us consider the case in which we know both the price of a bond (P) and its cash flow: coupons (C) and maturity (n)
- If we want to determine the bond's IRR (called yield to maturity or redemption yield) we have to use numerical methods to find the level of (y) that will satisfy the following equation
- In the numerical example we consider a 10 year 6% annual coupon bond, priced at 96.80. The numerical solution, obtained using the IRR Excel spreadsheet function, is (y = 6.4440%)

$$\begin{aligned} PV &= C \sum_{j=1}^n (1+y)^{-j} + 100(1+y)^{-n} \\ 96.8 &= 6 \sum_{j=1}^{10} (1+y)^{-j} + 100(1+y)^{-10} \end{aligned}$$

Graphical Illustration

Exhibit 4 - graphical visualization of solving for the YTM of a 10 year 6% annual coupon bond, priced at 96.80. The numerical solution, obtained using the IRR Excel function is ($y = 6.4440\%$)



§5 –Mathematical induction (optional)

- One of the ways natural numbers (1, 2, ..., n) are used in applied mathematics (and therefore in finance) is as the base for proofs by mathematical induction. This kind of proof is clearly useful when we must prove that a given assertion is true for all natural numbers.
- Mathematical induction is used in a number of finance applications. For example:
 - The dividend discount model equation (see §6 of this TN)
 - Binomial trees for option pricing
 - Proof that a floating rate note is priced at par at the start of coupon accrual periods

Proof by mathematical induction 1

- Take a sequence of statements indexed by consecutive natural numbers

$$A_k, A_{k+1}, A_{k+2}, A_{k+3}, \dots$$

Proof by mathematical induction 2

- We can prove that the statements are true for all values of (k) if we can:
- First, prove that statement number (k) is true (this is also known as the inductive base)
- Second, establish that assertion $(k + j + 1)$ is true whenever assertion $(k + j)$ is true (this is also known as the inductive step)

$$A_k = True$$

$$A_{k+j} \Rightarrow A_{k+j+1} \quad (j = 0, 1, 2, \dots)$$

Proof by mathematical induction 3

(arithmetic progression example)

- We now prove the formula for the sum of the arithmetic progression of natural numbers using mathematical induction.
- Based on our knowledge of arithmetic series this formula is $(n/2)(1+n)$
- The inductive base of the proof is the self-evident identity:

$$S_A = 1 + 2 \equiv \frac{2(1+2)}{2}$$

Proof by mathematical induction 4

(arithmetic progression example)

- We now proceed to prove the inductive step

$$\begin{aligned}1 + \dots + (n + 1) &= \{1 + \dots + n\} + (n + 1) \\ &= \left\{ \frac{n(n+1)}{2} \right\} + (n + 1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

$$\therefore 1 + \dots + n = \frac{n(n+1)}{2}$$

$$\Rightarrow 1 + \dots + (n + 1) = \frac{(n+1)(n+2)}{2}$$

§1.6 -The Dividend Discount Model

(Optional)

- The Dividend Discount Model (DDM) is a standard tool in the valuation of shares and of companies. In this teaching note we will concentrate mainly on its quantitative analysis

Dividend Discount Model 1

- The classic way in which this model is presented is to consider a one year (or one quarter) holding period of one share of a public traded company for which stock prices are available
- We shall use the following symbols:
- P_0 = our valuation of the share price at time 0
- P_1 = our forecast of the expected share price at time 1 (year end)
- D_1 = our forecast of the expected dividend payment at time 1 (year end)
- k = the corporation's risk-adjusted rate (hurdle rate, cost of capital)

Dividend Discount Model 2

- Let us assume that the expected dividend and the expected share price at year end are known
- Clearly, the current share price can be easily calculated with a simple discount equation
- However, this also entails that we have a reliable estimate of the risk-adjusted discount rate (k = cost of capital)

$$P_0 = \frac{D_1 + P_1}{1 + k}$$

$$D_1 = 3, P_1 = 110, k = 10\%$$

$$P_0 = \frac{3 + 110}{1.10} = 102.7273$$

Dividend Discount Model 3

- How can we determine (P_1), which clearly accounts for most of the value of the estimated price at time 0 ?
- The answer is to replicate the procedure for one more year
- If we substitute result in the first equation we obtain the following value for (P_0):

$$\begin{aligned} P_1 &= \frac{D_2 + P_2}{1+k} \\ P_0 &= \frac{D_1}{1+k} + \frac{P_1}{1+k} \\ &= \frac{D_1}{1+k} + \left[\frac{D_2}{(1+k)^2} + \frac{P_2}{(1+k)^2} \right] \end{aligned}$$

Dividend Discount Model 4

- The process can now be repeated (forward induction)
- For example, If we consider n-years we can write the following equation (k is assumed to remain constant):

$$P_0 = \frac{D_1}{(1+k)} + \frac{D_2}{(1+k)^2} + \dots + \frac{D_n}{(1+k)^n} + \frac{P_n}{(1+k)^n}$$

Dividend Discount Model 5

- When analysts use the DDM they try to forecast expected dividends for a few years in the future. However, these forecasts become less and less reliable as the forecasting horizon increases
- Therefore, it is common practice to use a constant growth rate after a few years, as a simplifying assumption

$$D_t = (1 + g)D_{t-1}$$
$$P_0 = \frac{D_1}{1+k} + \frac{D_2}{(1+k)^2} + \dots + \frac{D_5}{(1+k)^5} + \frac{(1+g)D_5}{(1+k)^6} + \dots + \frac{(1+g)^{n-5}D_5 + P_n}{(1+k)^n}$$

Dividend Discount Model 6

- In order to assign a value to (P_n) we usually assume that the corporation will go on forever, or, at least, until the present value of (P_n) becomes extremely small
- To obtain uncluttered mathematics, in the following equations we assume that the constant growth phase begins immediately
- In the following equation ($k > g$) in order not to obtain the paradoxical result of having an infinite price when $n \rightarrow \infty$

$$P_0 = \frac{D_1}{1+k} + \frac{(1+g)D_1}{(1+k)^2} + \dots + \frac{(1+g)^{n-1}D_1}{(1+k)^n} + \frac{P_n}{(1+k)^n}$$

Dividend Discount Model 7

- If ($k > g$) and ($n \rightarrow \infty$), we can prove (see appendix) that the constant growth equation can be expressed with the following compact equation
- Note however, that if ($k < g$) we cannot derive the equation. (P_0) would be infinite (and not negative as one could think reading the equation)

$$P_0 = \frac{D_1}{k-g} (k > g)$$

$$D_1 = 3, k = 10\%, g = 6\%$$

$$P_0 = \frac{3}{0.1-0.06} = 75$$

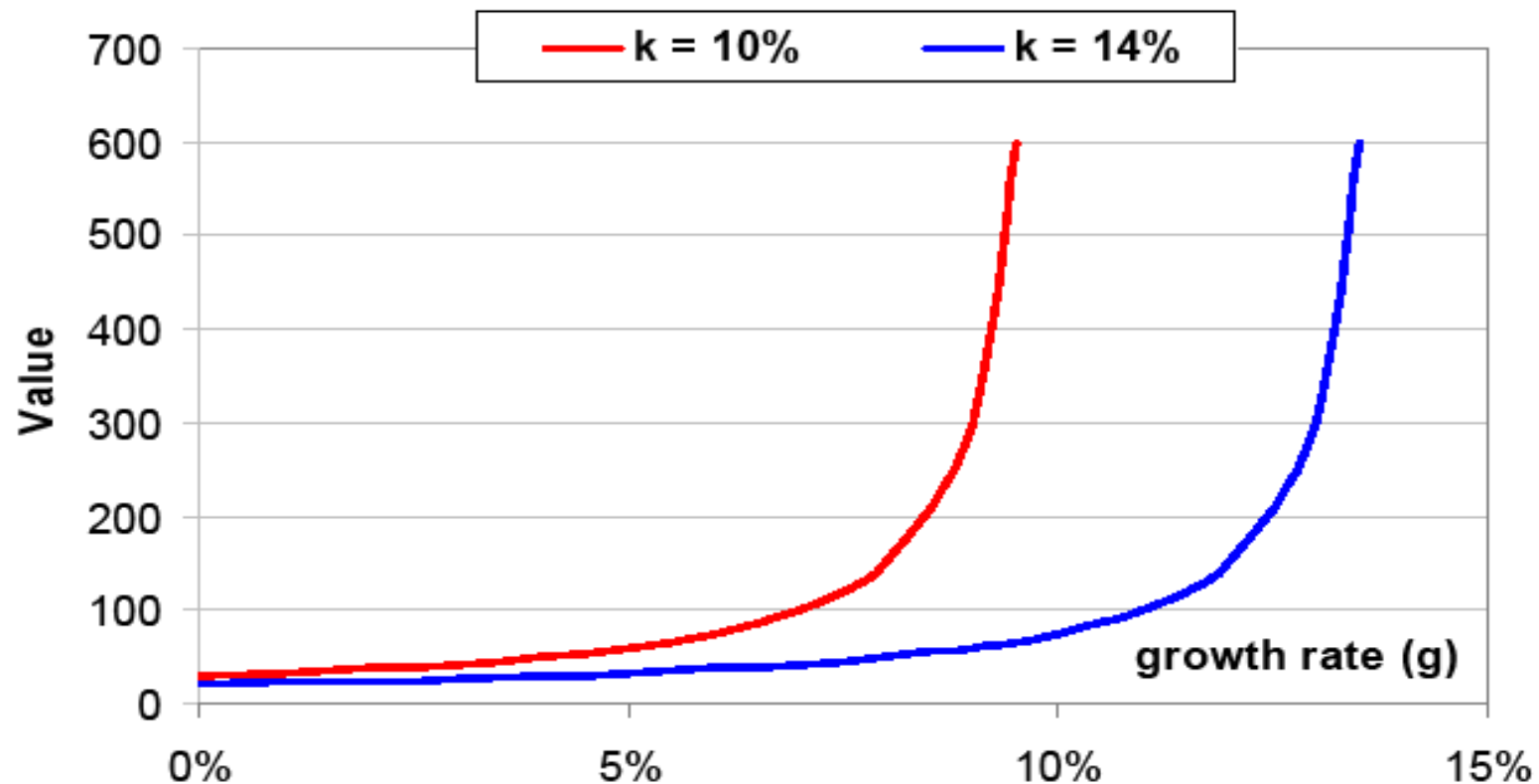
Dividend Discount Model 8

- A simple manipulation of the previous equation shows that the risk-adjusted rate of return (k) equals the dividend yield plus the rate of growth of the dividends (g)

$$\begin{aligned}P_0 &= \frac{D_1}{k-g} \quad (k > g) \\k - g &= \frac{D_1}{P_0} \\k &= \frac{D_1}{P_0} + g \\&= \frac{3}{75} + 0.06 \\&= 10\%\end{aligned}$$

Dividend Discount Model 9

Exhibit 5 – The firm's value is extremely sensitive to the difference between cost of capital and growth rate ($k - g$). This explains why it is so difficult to value a firm



§7 –Excel functions referred to in this Teaching Note

- IRR
- NPV
- PV
- SUM
- SUMPRODUCT

TN 02

Basic calculus and financial mathematics I

This teaching note was prepared by Georgio Questa with help from Dr. Enrico Biffis and Dr. Lorenzo Trapani

1. Functions and limits
2. Continuity, convexity and concavity
3. Derivatives. Taylor series
4. Excel functions used in this note

Appendix (PDF on the website)

General Introduction

- ❑ Calculus is a BIG topic. In this Teaching Note we can only review a few basic concepts that are the most likely to be useful for some modules of our MSc courses
- ❑ Our discussion of these topics must necessarily be conducted relying on intuition and with exclusive reference to univariate calculus to benefit from both its analytical simplicity and ease of visualization
- ❑ There are a few proofs in the appendix, to provide some extra material to the interested reader

§1 – Functions and limits

What is a function

Limits of a function

- ❑ The first use of the word function is credited to Leibniz (1646-1716)
- ❑ Until the mid 1800s the concept of function was that of a relatively simple mathematical formula expressing the relationship between the values of a dependent variable (y) and those on an independent variable (x)
- ❑ In the second half of the 1800s, the concepts of function and limit were generalized and made a lot more rigorous, thereby providing a solid foundation for the further development of calculus

What is a Function 1

- A real-valued mathematical expression, such as the following quadratic equation, has no defined numerical value until you assign a value to (x). Thus, we say that (y) is a function of (x)

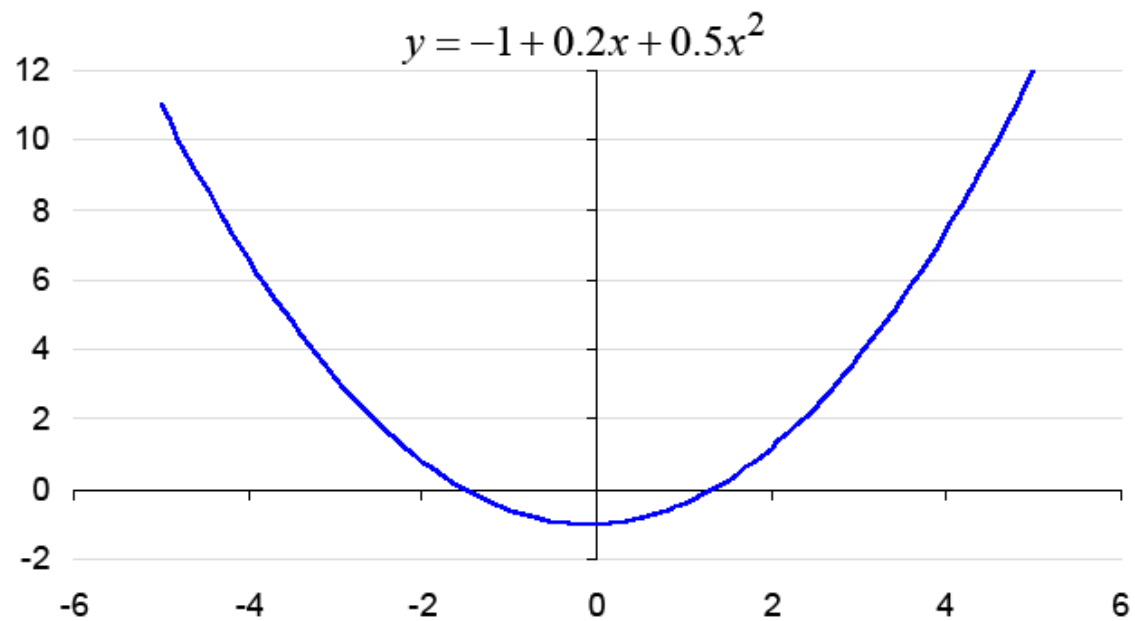
$$y = -1 + 0.2x + 0.5x^2$$

What is a Function 2

- ❑ Functions are also called transformations because they transform the value of (x) into a value (y)
- ❑ If we plot the values of the independent (x) and the dependent (y) variables on the Cartesian plane we obtain the graphic representation of the function

What is a Function 3

Exhibit 1 - Graph of the quadratic function (a parabola) on the Cartesian plane

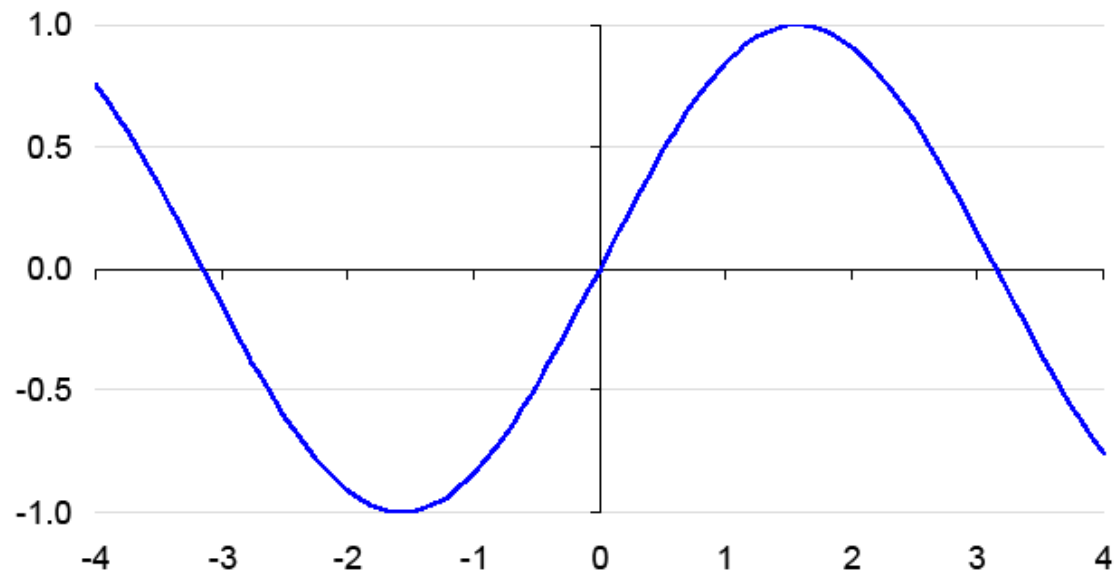


What is a Function 4

- ❑ A function (f) from a set D (domain) to a set R (range) defines a rule which assigns to each element of D a unique element in R
- ❑ A very important element in the definition of a function is the requirement that for every given point on the domain there is one and only one function value.
- ❑ In other words, if you draw a straight line parallel to the y -axis it should cross only once the function's graph (see exhibit 1 & 2)
- ❑ Thus, if there is more than one value of (y) corresponding to one value of (x) we are not dealing with a function (see exhibit 3)

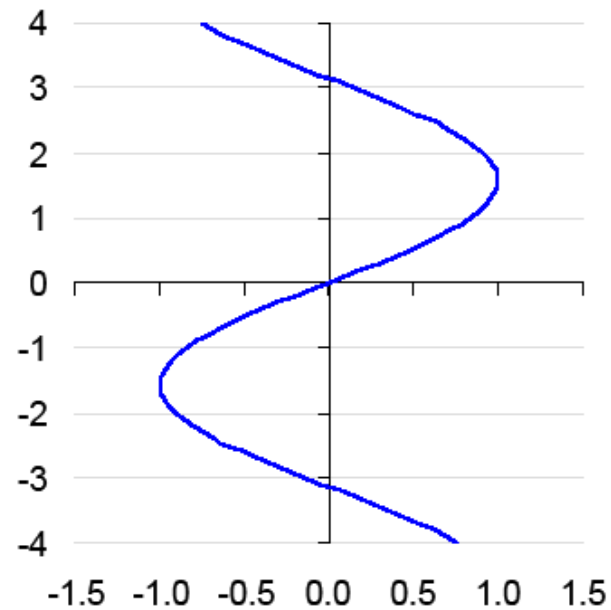
What is a Function 5

Exhibit 2 – This is a sine function, $\text{SIN}(x)$ in Excel, defined on the whole real line. The range is $[-1 \leq y \leq 1]$. For every possible value of (x) there is one, and only one, value of (y)



What is a Function 6

Exhibit 3 – This continuous sinusoidal graph on the $[-1 \leq x \leq 1]$ domain is not a function because there is more than one value of (y) corresponding to any given value of (x)



Limit of a function 1

- ❑ The concept of limit is all-pervasive in calculus and its applications.
- ❑ The rigorous definition of the limit of a function was worked out in the mid 1800s, mainly by the German mathematician Karl Weierstrass
- ❑ The Weierstrass approach is intuitively evident and we shall use it in this TN, albeit with some simplifications

Limit of a function 2

- There are several kinds of limits. Due to time constraints, we shall concentrate only on two limits, chosen for their relevance for our studies. First, the limit of a function that tends to zero for $(x \rightarrow \infty)$, where the symbol \rightarrow stands for “tends to” or “approaches”
- Second, the limit of a function for $(x \rightarrow 0)$. This limit is the cornerstone for the definition and calculation of derivative of a function and will be discussed in §3

Limit of a function 3 (for $x \rightarrow \infty$)

- We now introduce the concept of a limit using what is known as the (δ, ε) approach. Suppose the following statement is true
- Given an arbitrarily small number (ε) , there is a number (δ) such that, for all $x > \delta$, $|f(x) - 0| < \varepsilon$
- Then the limit of a function is zero for $(x \rightarrow \infty)$. We indicate this with the following equation:

$$\lim_{x \rightarrow \infty} f(x) = 0$$

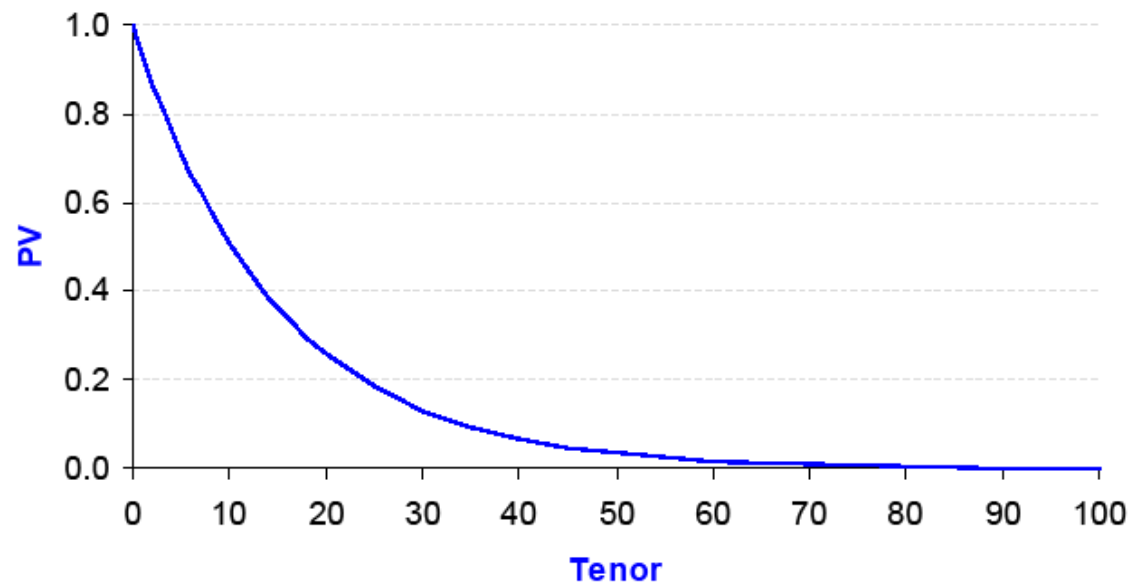
Limit of a function 4 (for $x \rightarrow \infty$)

- Let us now consider the present value (P) of a unit zero coupon bond due in (n) years, given a constant compounded discount rate (y)
- The time-to-maturity is also known as tenor
- The well-known equation for the unit zero-coupon bond price is

$$P = (1 + y)^{-n}$$
$$\lim_{n \rightarrow \infty} (1 + y)^{-n} = 0$$

Limit of a function 5 ($x \rightarrow \infty$)

Exhibit 4 – Price of a unit zero coupon bond for ($n \rightarrow \infty$), at a constant compounded discount rate of 7%



Limit of a function δ ($x \rightarrow \infty$)

- We now want to show numerically that the $(PV \rightarrow 0)$ when $(n \rightarrow \infty)$
- Let us assign a reasonable value to the compounded discount rate, for example $(y = 7\%)$
- We can see that $(\varepsilon \rightarrow 0)$ as $(n \rightarrow \infty)$
- A simple proof, using logs, is included in the Appendix to this TN. The logarithmic function is discussed in TN 3

Tenor	PV
20	0.258419
50	0.033948
100	0.001152
200	0.000001

§2 - Continuity, convexity and concavity

- *Continuous and discontinuous functions*
- *Jumps in (y)*
- *Jumps in both (x) and (y)*
- *Convex and concave functions*

Continuity 1

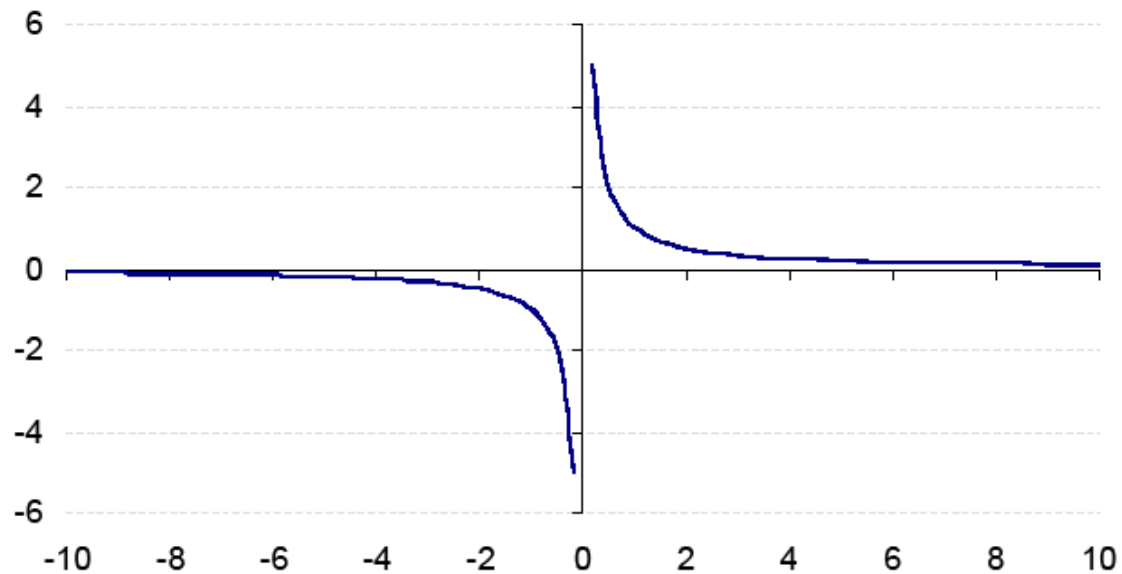
- Relying on intuition, we can say that a uni-variate function $y = f(x)$ is continuous if we can draw its graph on the Cartesian plane without having to lift pen from paper
- The functions that we have seen in exhibits $\{1, \dots, 4\}$ are all continuous
- However there do exist functions that are discontinuous. They may be discontinuous because:
 - Of their algebraic properties (typically, they entail one or more divisions by zero)
 - They are constructed in such a way as to be discontinuous

Continuity 2 (discontinuous function example)

- ❑ A well known example of a discontinuity due to a division by zero is the rectangular hyperbola ($y = 1/x$)
- ❑ Exhibit 5 shows that the rectangular hyperbola tends to infinity when (x) approaches zero from the right ($x \rightarrow 0+$) and to minus infinity when (x) approaches zero from the left ($x \rightarrow 0-$)
- ❑ Clearly, there is a discontinuity when ($x = 0$)

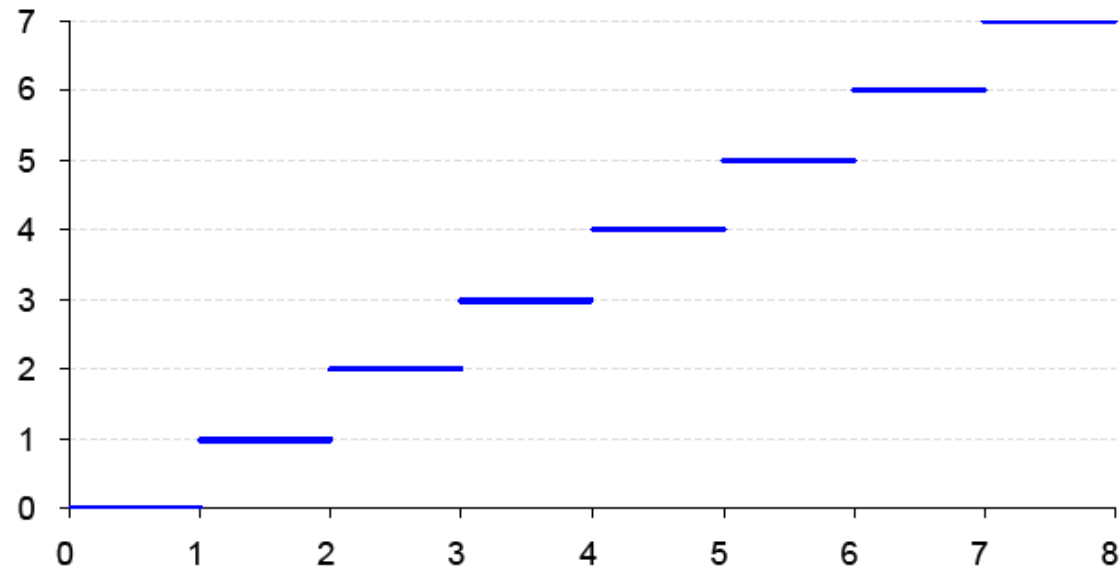
Continuity 3 (discontinuous function example)

Exhibit 5 – This is the graph of a hyperbola, $y = 1/x$. The function is discontinuous at the point $(x = 0)$



Continuity 4 (jumps in y)

Exhibit 6 – This is the plot of the function where $y =$ the integer part of (x) .
The Excel function TRUNC(x) has been used to calculate its value



Continuity 5 (jumps in y)

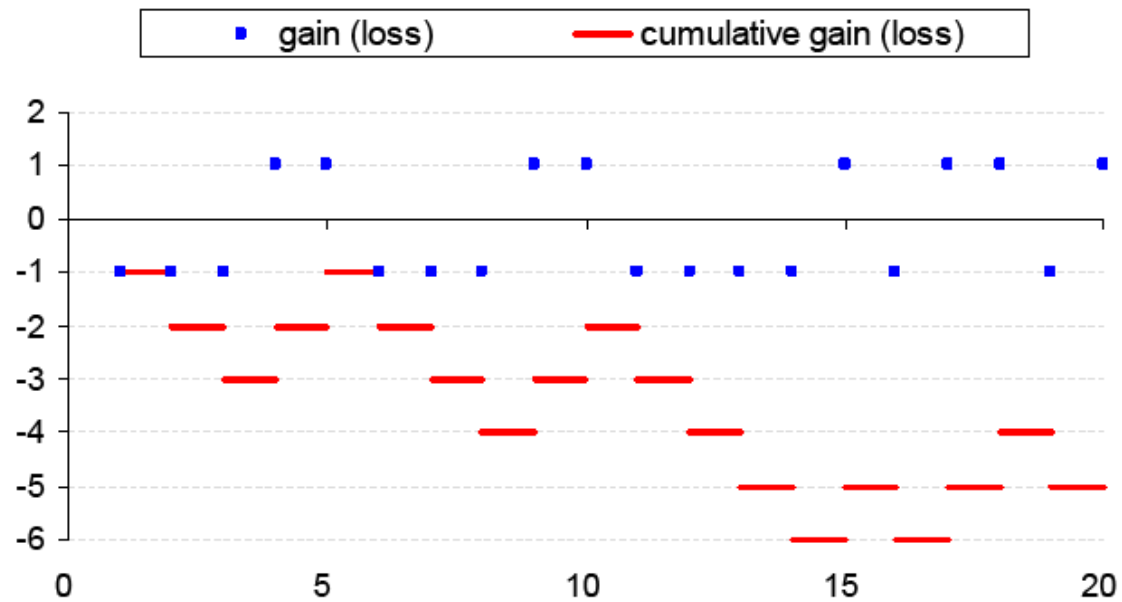
- The function in exhibit 6 is plotted for $(0 < x < 8)$. The value of the function equals the integer part of (x) . The Excel function TRUNC(x) has been used to calculate its value

$$f(x) = \text{TRUNC}(x) = \{1, 2, \dots, 7\}$$

- Therefore, the function is defined on a continuous domain (x) but is discontinuous on (y)
- If we think of (x) as representing time, then the function represents a continuous-time, discontinuous-values process (also known as continuous-time, discrete-states process)

Continuity 6 - Jumps in both (x) and (y)

Exhibit 7 – This is the plot of the outcomes of 20 bets on tossing a fair coin (random path)



Continuity 7 - Jumps in both (x) and (y)

- ❑ One of the two functions depicted in exhibit 7 is discontinuous in (x) and, consequently, in (y)
- ❑ If we think of (x) as time, this function represent a sequence of 20 gains (losses) on betting \$1, at regular intervals, on the outcome of tossing a fair coin (this is also known as a binomial process)
- ❑ The second function (continuous time, discontinuous values) shows the cumulative gain (loss) on the game

Convex and concave functions 1

- A function is convex over a given interval if a secant through any possible couple of points on the curve will always lie above the function itself
- In calculus books convexity is often denoted as upwards concavity
- Convexity has also a different meaning in mathematics (and in many economic and financial applications)
- A set is said to be convex when a straight line through any two points in the set will lie inside the set

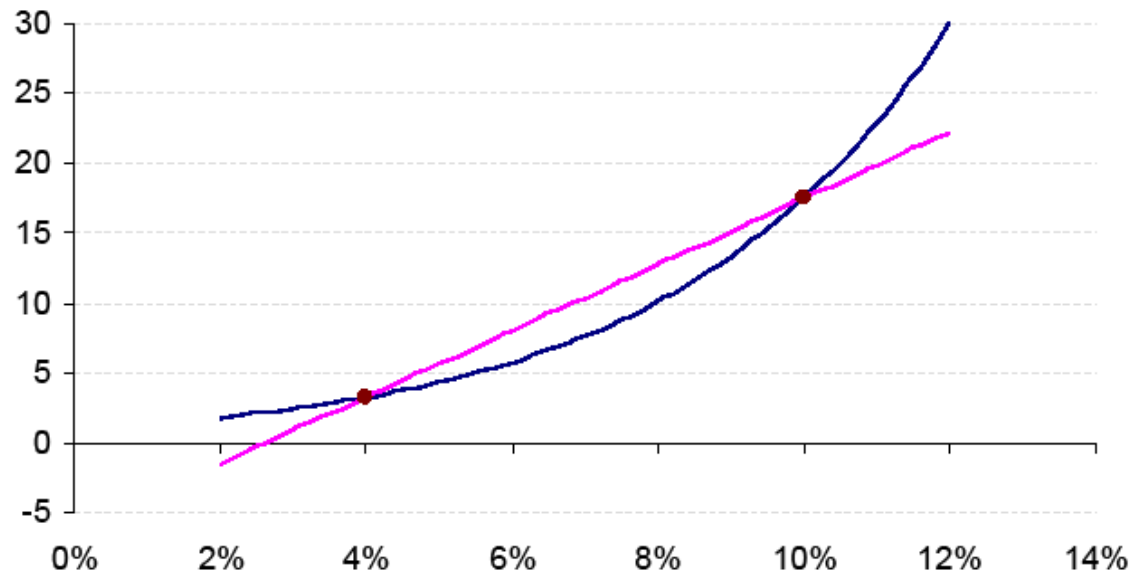


Convex and concave functions 2

- ❑ The following two exhibits (8a & 8b) show that both the future value and the present value (as a function of yield) are convex
- ❑ Convexity is much used in fixed income and will be discussed in some detail in “§4 - Duration and convexity of a zero coupon bond”
- ❑ We should add that convexity is measured by the second derivative of the function (see §3 of this TN)

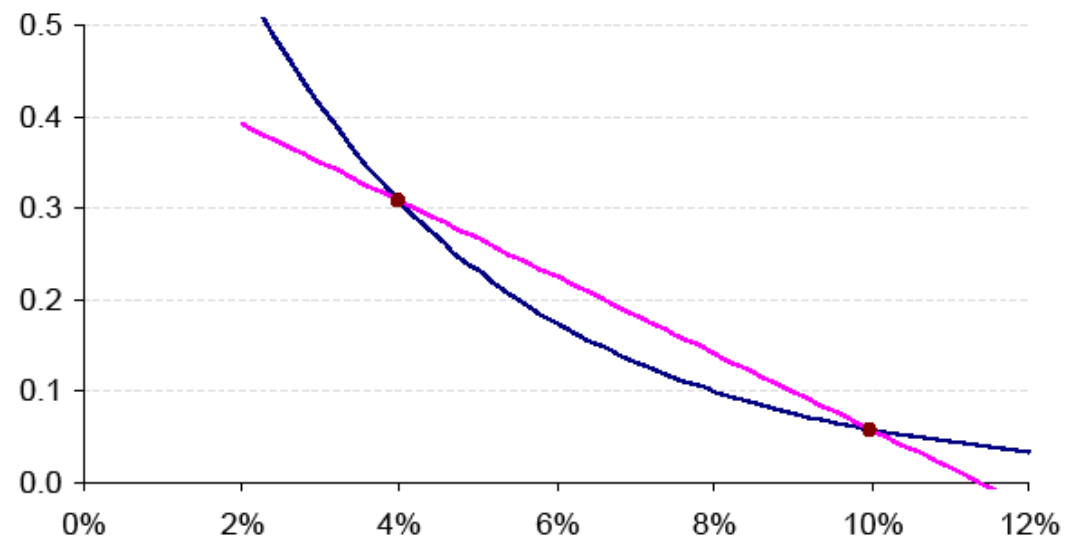
Convex and concave functions 3

Exhibit 8a – This is the graph of the future value of one unit (say 1€) invested at the rate (y) for 30 years. The function is convex



Convex and concave functions 3

Exhibit 8b – This is the graph of the present value, PV, of one unit (say 1€) discounted at the rate (x) for 30 years. We can think of this as the price of a 30-year zero coupon bond. The function is convex

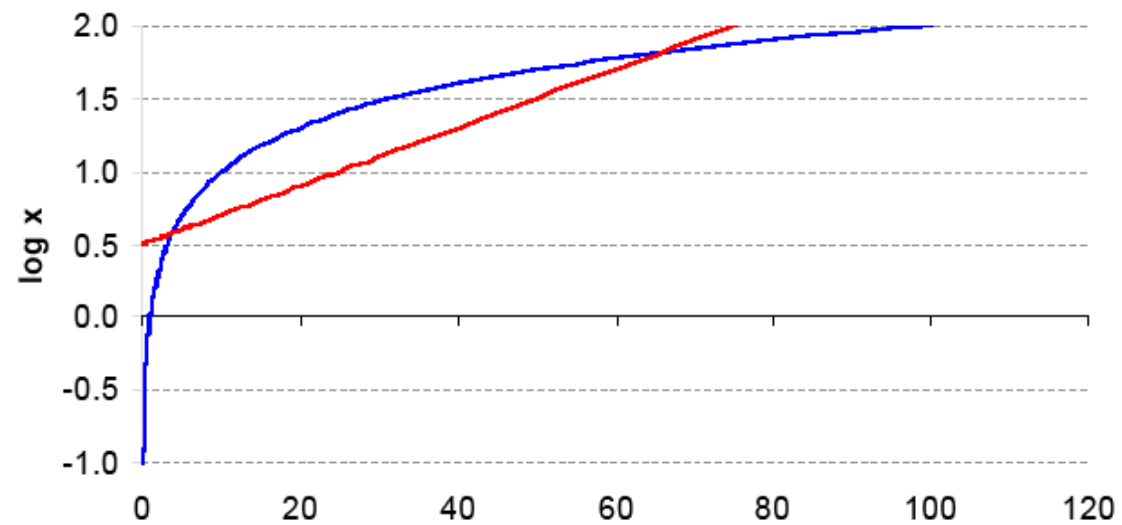


Convex and concave functions 5

- ❑ A function is concave over a given interval if a secant through any possible couple of points on the curve will always lie below the function itself
- ❑ In calculus books this is often denoted as downwards concavity
- ❑ A typical example of a concave function is the logarithmic function (see exhibit 8c)
- ❑ The logarithmic function will be discussed in §3 of TN 3

Convex and concave functions 3

Exhibit 8c – This is the graph of the log function (base 10)



§3 - Derivatives

Derivatives

Derivative and limit

Second derivative

Taylor series

- The problem of calculating the tangent to a curve and the area delimited by a curve were already tackled more than 2000 years ago, with very limited success. The breakthrough came in the late 1600s, due to the work of Fermat, Newton and Leibnitz
- It came as a surprise that the tangent and the area problems are intimately connected. The tangent is calculated with the derivative and the area with the integral (which is also known as the anti-derivative)

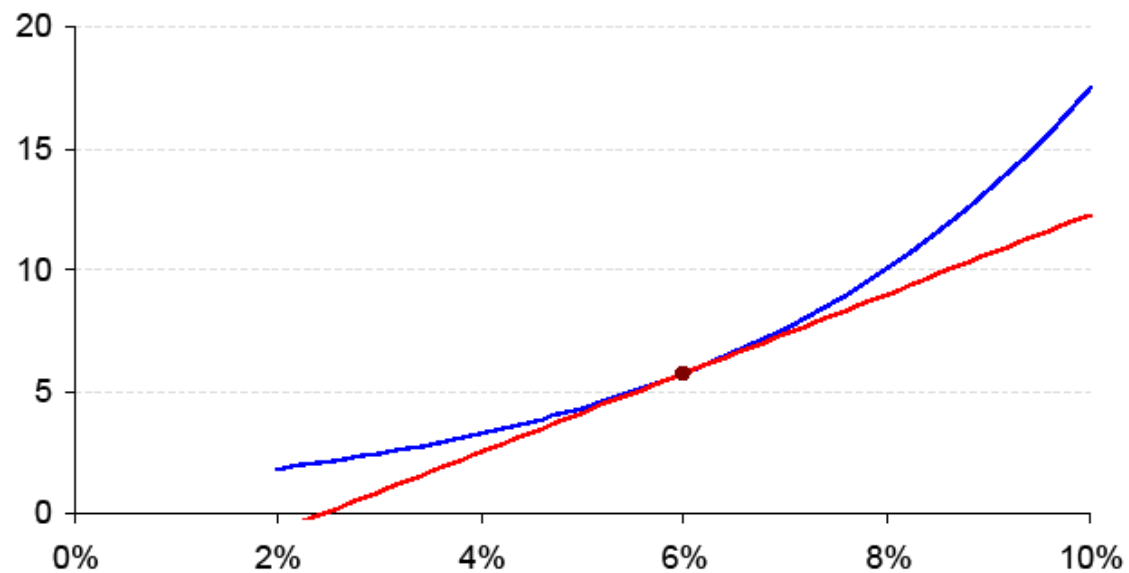
Derivatives 1

- The central idea of differential calculus is the notion of derivative.
- The derivative of a function $f(x)$ in correspondence of a given value of x is a number which measures the slope of the function at that point
- More generally, the derivative of a function is another function that gives the slope of $f(x)$ for each value of x on the domain of the function. Thus, the derivative measures the slope of the tangent to the function
- Given a function $y = f(x)$, its derivative function can be denoted in many different ways:

$$f'(x) \equiv y' \equiv \frac{dy}{dx} \equiv \frac{d}{dx} f(x)$$

Derivatives 2

Exhibit 9 – This graph is the future value of a unit investment, over a 30-year time-horizon, at a constant annual compounded yield. The tangent's slope is visualized for yield = 6%

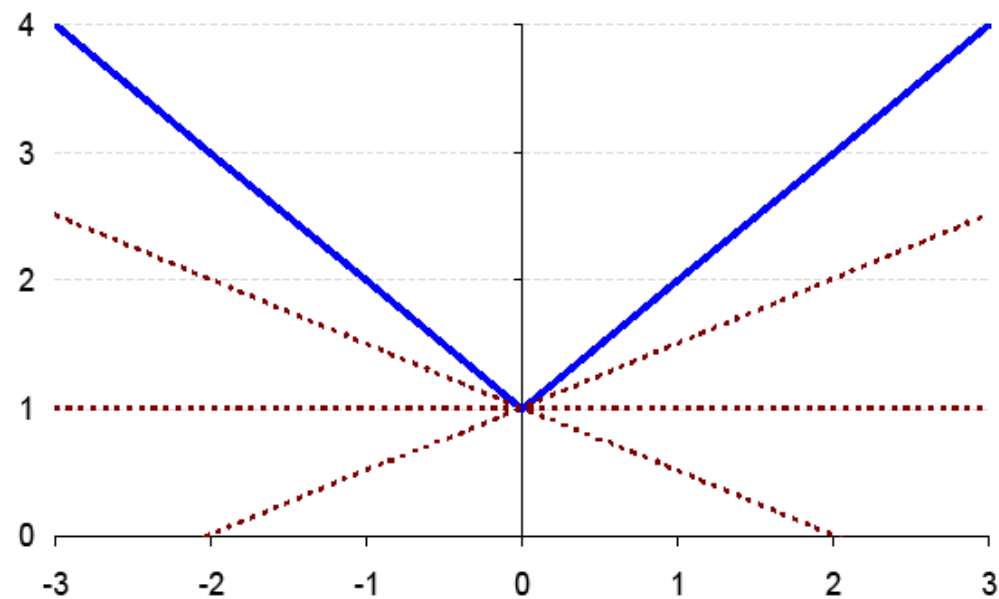


Derivatives 3

- ❑ In exhibit 9, the curve is continuous and does not have corner points.
- ❑ Therefore, it is possible to determine its tangent in any point on the relevant domain
- ❑ Such a function is known as differentiable (or well behaved)
- ❑ A discontinuous function is clearly non differentiable in correspondence of points of discontinuity
- ❑ A function is also non-differentiable in correspondence of corner points
- ❑ Exhibit 10 visualizes that there can be infinite tangents at $f(x = 0)$

Derivatives 4

Exhibit 10 – there are infinitely many tangents at the corner point of the function $y = |x| + 1$ (the $|x|$ symbol indicates the absolute value of x)



Derivative and limit 1

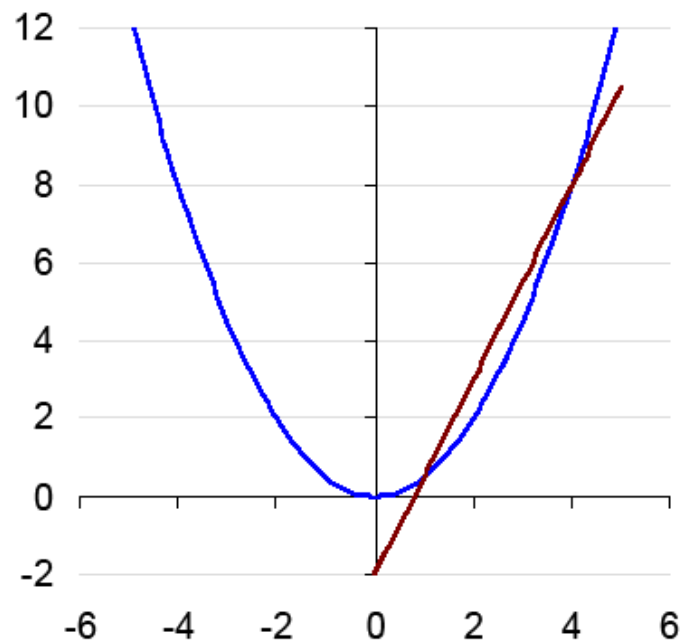
- Let us consider a well-behaved differentiable function, such as a simple quadratic
- The slope of a secant (s) through two points of the function is given by the following equation, where $f(x)$ and $f(x + h)$ are the values at which the secant will intersect the function

$$S = \frac{f(x+h) - f(x)}{h}$$

Derivatives and limit 2

Exhibit 11 – Secant (for $x = 1$ & $x = 4$) to the function

$$y = bx^2 = 0.5x^2$$



Derivative and limit 3

- Exhibit 11 is based on the following numerical values (the secant intersects the quadratic at points: $x = 1$ & $x = 4$)

$$y = b x^2 = 0.5x^2, x = 1, h = 3$$
$$S = \frac{f(x+h) - f(x)}{h}$$
$$S = \frac{0.5(1+3)^2 - 0.5(1)^2}{3} = 2.5$$

Derivative and limit 4

- This slope (s) as we have discussed so far is clearly the average slope of the function
- Think of a plane at take off, that climbs 8,000 meters in 250 Km (the 250 Km are measured relative to the ground). The average slope of the climb is 3.2%
- Suppose now we want to determine the climbing angle over a smaller time interval corresponding to, for example, a 10 meters climb from 7,000 to 7,010 meters, which takes 0.9 seconds.
- To do this, we would simply divide the 10 meters by the ground distance covered while climbing, say 200 meters
- The slope 0.05 so obtained would not represent the instantaneous climb angle, but it would certainly be very close to the instantaneous slope

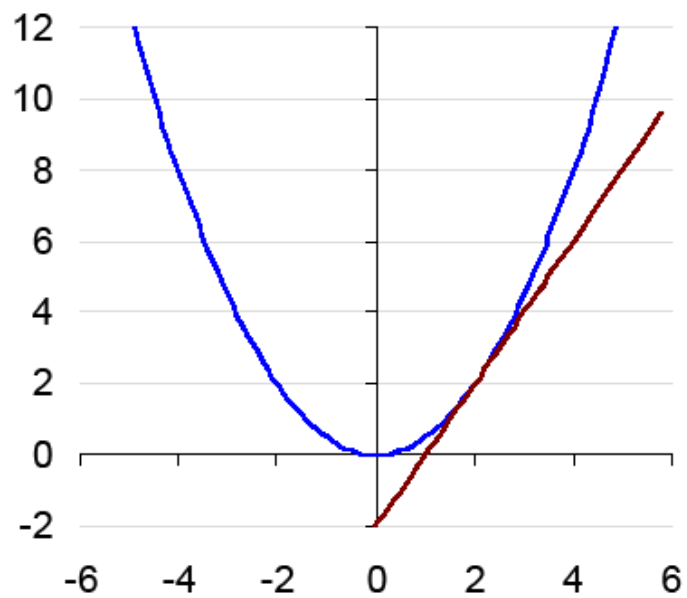
Derivative and limit 5

- The derivative measures the instantaneous slope of the function at single points
- Therefore, it seems natural to calculate it with the same equation that we used for the secant, letting ($h \rightarrow 0$). With reference to the quadratic equation $y=bx^2=0.5x^2$ we get:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{b(x+h)^2 - b(x)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\overbrace{b(x)^2} + 2b(x)(h) + \overbrace{b(h)^2} - b(x)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2b(x) + b(h)}{1} \\ &= 2bx\end{aligned}$$

Derivative and limit 6

Exhibit 11 – Tangent (for $x = 2$) to the function
 $y = 0.5x^2 \Rightarrow y' = x$



Second derivative 1

- The second derivative of a function is simply the derivative of the first derivative
- Just as in the case of the first derivative the second derivative can also be denoted in several possible ways. For example given a function $y=f(x)$:

$$y' \equiv f'(x) \equiv \frac{dy}{dx} \equiv \frac{d}{dx} f(x)$$
$$y'' \equiv f''(x) \equiv \frac{d}{dx} \left(\frac{dy}{dx} \right) \equiv \frac{d^2 y}{dx^2} \equiv \frac{d^2}{dx^2} (f(x))$$

Second derivative 2

- In our quadratic function example, we obtain the first derivative as a linear function $y=x$.
- Thus, the second derivative will be a constant (the slope of a straight line is clearly a constant)

$$y' = 2bx = 2(0.5)x = x$$
$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(x) = 1$$

Taylor series 1

- It can be proved that, over a relatively contained interval, a well behaved function can be approximated using its derivatives
- We could legitimately ask why we should use this approximation when we can calculate the function
- There are two answers:
 - Using only the first derivative we get a linear approximation, which can be much simpler to compute and very useful as an approximation (for example the concept of a bond's duration is based on this idea)
 - The Taylor series approach is the cornerstone for proving some properties of a function

Taylor series 2

- The factorial of a natural number n (i.e. n belongs to the set $\{1,2,3, \dots\}$) is simply the product $1 \times 2 \times \dots \times n$
- Factorials can be calculated with the Excel function FACT(N)
- In the Taylor series expression given below, the symbol "!" stands for factorial

$$\begin{aligned} f(x) &\cong f(a) \\ &+ f'(a)(x - a) \\ &+ \frac{f''(a)}{2!} (x - a)^2 \\ &+ \frac{f'''(a)}{3!} (x - a)^3 \\ &+ \dots \end{aligned}$$

§4 - Excel functions referred in this note

- ❑ FACT(N): N must be a natural number. $\text{FACT}(N) = 1 \times 2 \times \dots \times N$
- ❑ LOG10(x): base 10 logarithm. The domain of the function is $(0 \leq \infty)$
- ❑ SIN(x): The sine function of (x), where (x) is measured in degrees
- ❑ TRUNC(x): Truncates a number (any real number), eliminating decimals

TN 03

Basic calculus and financial mathematics II

This teaching note was prepared by Georgio Questa with help from Dr. Enrico Biffis and Dr. Lorenzo Trapani

1. Some rules to calculate derivatives
2. Duration and convexity of a zero coupon bond (Optional)
3. The exponential and the log functions

Excel functions used in this note

Appendix (PDF on the website)

§3.1 - Some rules to calculate derivatives

- *Adding a constant to a function*
- *Multiplying a function by a constant*
- *Derivative of a linear function*
- *Derivative of a sum of functions*
- *The power rule*
- *The chain rule*
- *Example: the derivative of the price of a bond*
- *As we may well imagine, there is a large set of rules to find derivative functions. In this § we shall examine only a few of them, chosen on the basis of their immediate relevance for finance applications*
- *The derivative of the natural exponential and logarithmic functions will be discussed in §3.3*
- *The proofs of the power and chain rules can be found in the appendix (PDFs on the website)*

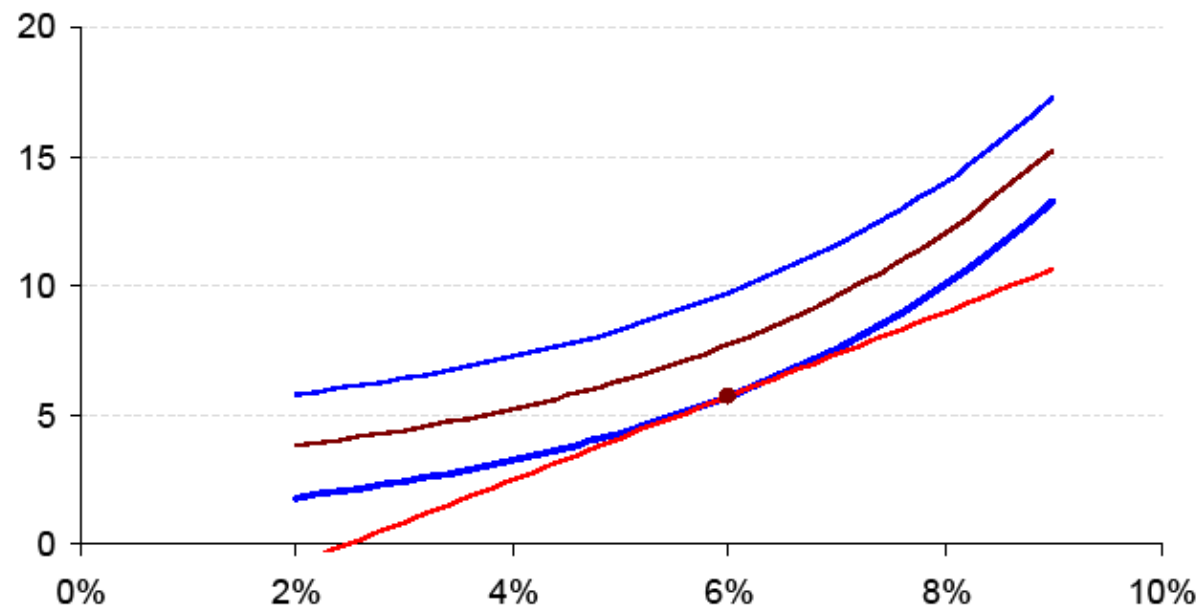
Adding a constant to a function 1

- Adding a constant to a function does not change its derivative
- The added constant will simply cause a parallel shift in the function, without changing its slope
- Exhibit 12 provides a visualization of the parallel shift

$$\begin{aligned}g(x) = a + f(x) \Rightarrow g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\{a + f(x+h)\} - \{a + f(x)\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x)\end{aligned}$$

Adding a constant to a function 2

Exhibit 1 – Parallel shift due to adding a constant to a function. The derivatives of the 3 functions are equal



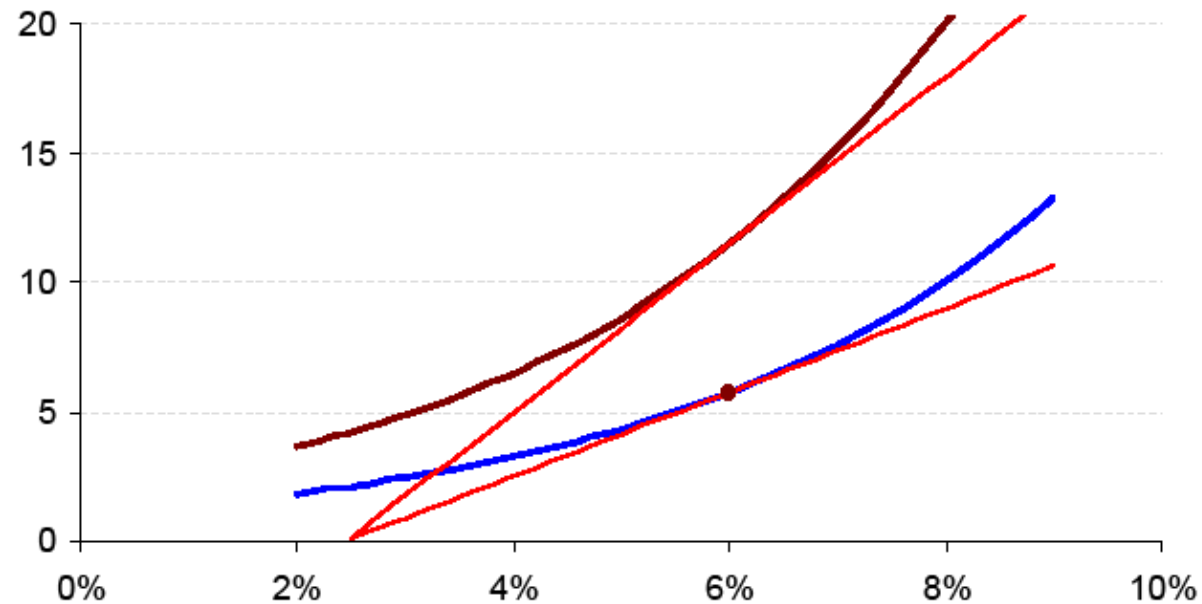
Multiplying a function by a constant 1

- Let a function $g(x)$ equal a function $f(x)$ multiplied by a constant (λ)
- The derivative of $g(x)$ is equal to the derivative of $f(x)$ multiplied by the constant (λ)

$$\begin{aligned}g(x) = \lambda f(x) \Rightarrow g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\{\lambda f(x+h)\} - \{\lambda f(x)\}}{h} \\ &= \lambda \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lambda f'(x)\end{aligned}$$

Multiplying a function by a constant 2

Exhibit 2 – The lower curve in this graph visualizes the future value of \$1 invested at a constant compounded rate for 30 years. The upper curve is the future value of \$2 ($\lambda = 2$)



Derivative of a linear function

- The derivative of a linear function is a constant. The result is intuitively evident because a straight line has a constant slope
- However, we can easily prove this result. This also provides an example of the sum of a constant and of the product of a constant rules

$$\begin{aligned}g(x) = a + bx \Rightarrow g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\{a + b(x+h)\} - \{a + bx\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{bh}{h} \\ &= b\end{aligned}$$

Derivative of a sum of functions 1

- The derivative of a sum of functions equals the sum of their derivatives
- This result is intuitively evident when we consider the following equation

$$\begin{aligned} s(x) = f(x) + g(x) &\Rightarrow s'(x) = \lim_{h \rightarrow 0} \frac{s(x+h) - s(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\{f(x+h) + g(x+h)\} - \{f(x) + g(x)\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

The power rule

- The power rule allows us to differentiate a power of (x) even if the power is not an integer but any real number
- Here is the relevant equation:

$$y = x^a \Rightarrow y' = ax^{a-1}$$

The chain rule

- The chain rule is one of the most widely used rules to find the derivative of a function
- It applies whenever we have a composite function (function of a function)
- The following equations show a composite function and its
- derivative, obtained with the chain rule

$$y = f[g(x)], \quad u = g(x)$$
$$y' = \frac{d}{du} f(u) \frac{d}{dx} u = \frac{dy}{du} \frac{du}{dx}$$

Example: the derivative of the price of a bond 1

- In TN 1 we have seen that the price of a fixed rate bond is given by the following equation, where (y) stands for the yield to maturity
- To keep things simple we assume that the coupon payments (C) are constant and paid at the end of each year until maturity (n)
- We also assume that the principal of the bond is (F) is paid together with the last coupon

$$P = C [(1+y)^{-1} + (1+y)^{-2} + \dots + (1+y)^{-n}] + F(1+y)^{-n}$$

Example: the derivative of the price of a bond 2

- A simple inspection shows that, to find the derivative of the bond's price relative to (y) , we could use the following theorems:
 - chain rule
 - sum of a constant rule
 - the sum of functions rule
 - multiplication by a constant rule

Example: the derivative of the price of a bond 3

- Let us start with the discount factors inside the square brackets and multiplying the principal (F)

$$\begin{aligned}\frac{d}{dy}(1+y)^{-j} &= -j(1+y)^{-j-1} \frac{d}{dy}(1+y) \\ &= -j(1+y)^{-j-1} \\ \therefore \frac{d}{dy}P &= C[-(1+y)^{-2} - 2(1+y)^{-3} + \dots \\ &\quad -n(1+y)^{-n-1}] - nF(1+y)^{-n-1}\end{aligned}$$

§3.2 - Duration and convexity of a zero coupon bond Optional

- *Duration metrics for zeros*
- *Dollar duration approximation for a zero coupon bond*
- *Dollar value of one basis point (DV01)*
- *Duration (modified duration)*
- *Convexity approximation*

Duration metrics for zeros 1

- *Duration metrics are based on linear approximations of the price yield function (which is usually convex)*
- *Linear expressions are easy to use because they depend only on one number and allow quick and rough comparisons between securities.*
- *The following two statements are equivalent; however, the second is a lot more compact and easier to grasp quickly, even in the buzz of a dealing room:*
 - *The price of Bond A will decrease by 0.90% if its YTM increases by 10 basis points ($0.9\% = 9 \times 0.1\%$), while the price of bond B will decrease by 0.30% ($0.3\% = 3 \times 0.1\%$). Therefore, Bond A is 3 times more interest-rate sensitive than bond B.*
 - *The duration of bond A is 9, versus 3 for bond B.*

Duration metrics for zeros 2

- We can use a linear risk measure because the price/yield functions for a coupon bond and for a zero are rather flat (low convexity) and a linear approximation is acceptable over a limited range yield movements
- This means that a 1st order (Taylor series) approximation is acceptable
- The tangent (slope) of the price/yield curve is also known as dollar duration
- Dollar duration for a zero coupon bond (and for straight coupon bonds) is a negative number but is nearly always quoted as a positive one

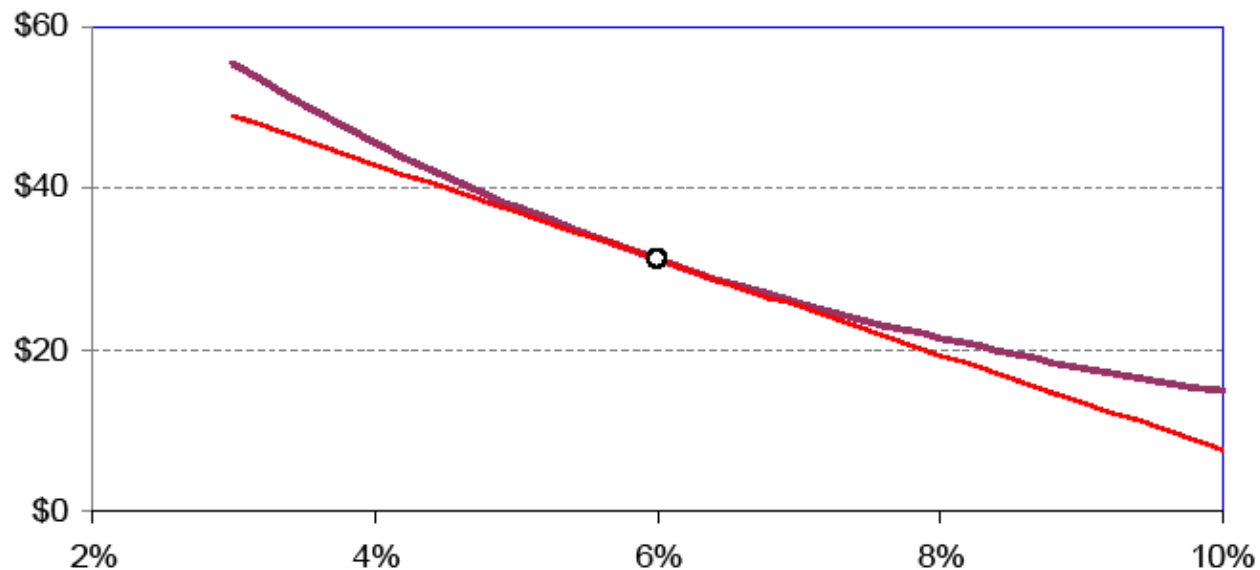
Dollar duration approximation for a zero 1

- P = price of the zero
- F = face value of the zero (we shall use 100)
- Y = spot yield at which we are calculating duration
- T = tenor of the zero
- $D_{\$}$ = Dollar duration
- The 1st equation is simply the price of a zero; the 2nd equation is the derivative of the price as a function of (Y): the 3rd equation is the first order approximation

$$P = F(1 + Y)^{-T}$$
$$D_{\$} = \frac{dP}{dY} = -T(F)(1 + Y)^{-T-1} = -T \frac{P}{1+Y}$$
$$\Delta P \approx D_{\$} \Delta Y$$

Dollar duration approximation for a zero 2

Exhibit 3 – Dollar duration approximation for a zero coupon bond,
face value = \$100, tenor T = 20 years, measured at Y = 6%



Dollar duration approximation for a zero 3

- Consider a 20-year zero coupon bond (face value = \$100), priced to yield 6% per annum (spot yield, Y)
- Let us compute, using the dollar duration approximation, by how much the price would decline if the yield jumped to 6.10% ($\Delta Y = 0.001$)
- Dollar duration is expressed in monetary units over yield

$$P = F(1 + Y)^{-T} = 100(1.06)^{-20} = 31.180473$$
$$D_{\$} = -T \frac{P}{1+Y} = -20 \frac{31.180473}{1.06} = -588.3108$$
$$\Delta P \approx D_{\$} \Delta Y = -588.3108(0.001) = -0.5883108$$

Dollar duration approximation for a zero 4

- The \$0.588311 drop in price computed with the linear dollar duration approximation is rather accurate.
- Due to the price-yield function convexity, duration overestimates the loss due to an increase in (Y) , and underestimates the gain when (Y) decreases. This is sometimes referred to as gain from convexity
- Gains from convexity have long been pursued by portfolio managers and by traders

Dollar value of one basis point 1

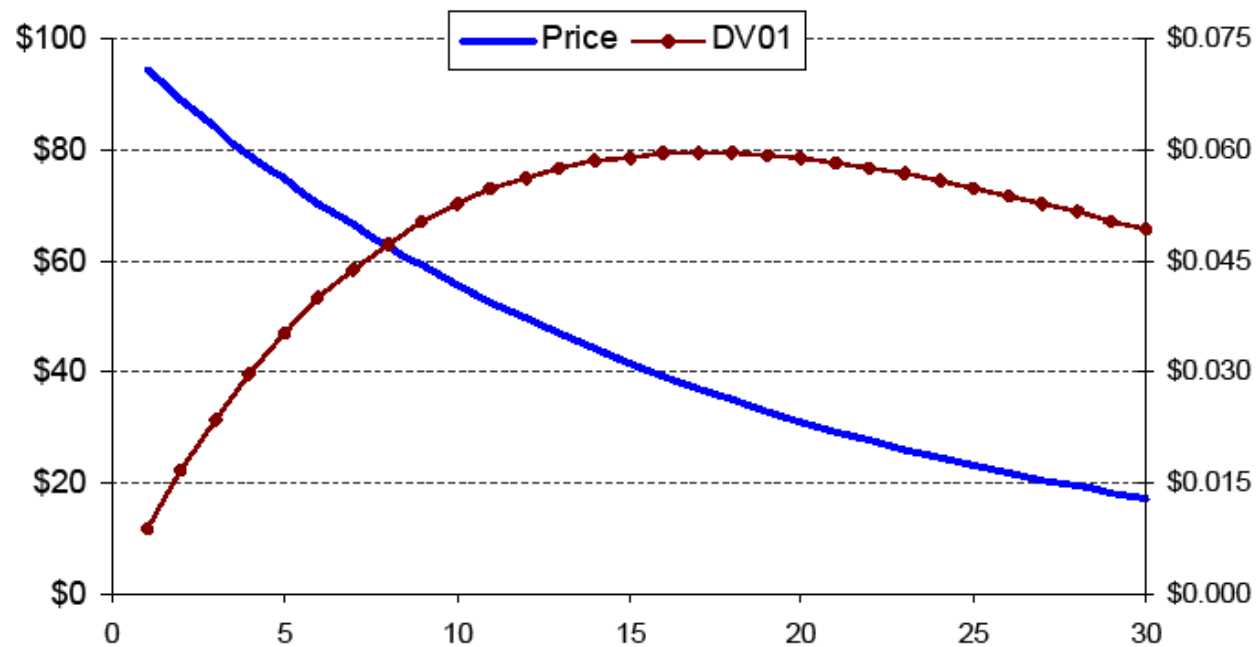
- The dollar value of one basis point (DV01), also known as the price value of one basis point (PV01), is a commonly used duration metric
- DV01 is simply the absolute price change of a bond due to a 0.01% (one basis point) increase in yield
- The price change is expressed in monetary units, thereby the name *dollar value*
- DV01 is a negative number but is often quoted as an absolute value. The equation is: $DV01 = D\$ \times 0.0001$

Dollar value of one basis point 2

- The DV01 of a long-dated zero coupon bond first increases as a function of term to maturity, and then declines, as shown in exhibit 4. This is due to two counteracting factors at play:
 - The price-elasticity increases with maturity; therefore the DV01 increases rapidly as the term increases.
 - Given a face value (\$100 in exhibit 2), the price of the zero coupon bond declines with the increase in tenor, and this produces a price-compression of dollar duration and DV01

Dollar value of one basis point 3

Exhibit 4 – Price and DV01 as a function of tenor for a for a \$ 100 face value zero coupon bond yielding 6%



Duration (modified duration)

- To avoid the price compression problem, we use modified duration which is simply dollar duration divided by the bond price (calculated at the yield at which we compute duration)
- Modified duration (often denoted as duration) is therefore the **proportional** price change due to an absolute change in yield

$$D = \frac{D_{\$}}{P} = \frac{dP}{dY} \frac{1}{P} = -T \frac{P}{1+Y} \frac{1}{P} = -\frac{T}{1+Y}$$

Example

- Consider a 12 year zero coupon bond (face value = \$100), priced to yield 6% per annum. Use modified duration to compute by how much the price would decline if the yield jumped to 6.20% ($\Delta Y = 20$ bp)
- $P = \$100(1.06)^{-12} = \49.696936
- $D = -12/1.06 = -11.320755$
- $\Delta P = P \times D \times \Delta Y$
- $= \$49.696936 \times (-11.3208) \times 0.002$
- $= -\$1.125214$

Linking the duration metrics

- The duration metrics mentioned so far are linked by the following equations

$$DV01 = 0.0001D\$$$

$$D = \frac{D\$}{P} = \frac{DV01}{0.0001P}$$

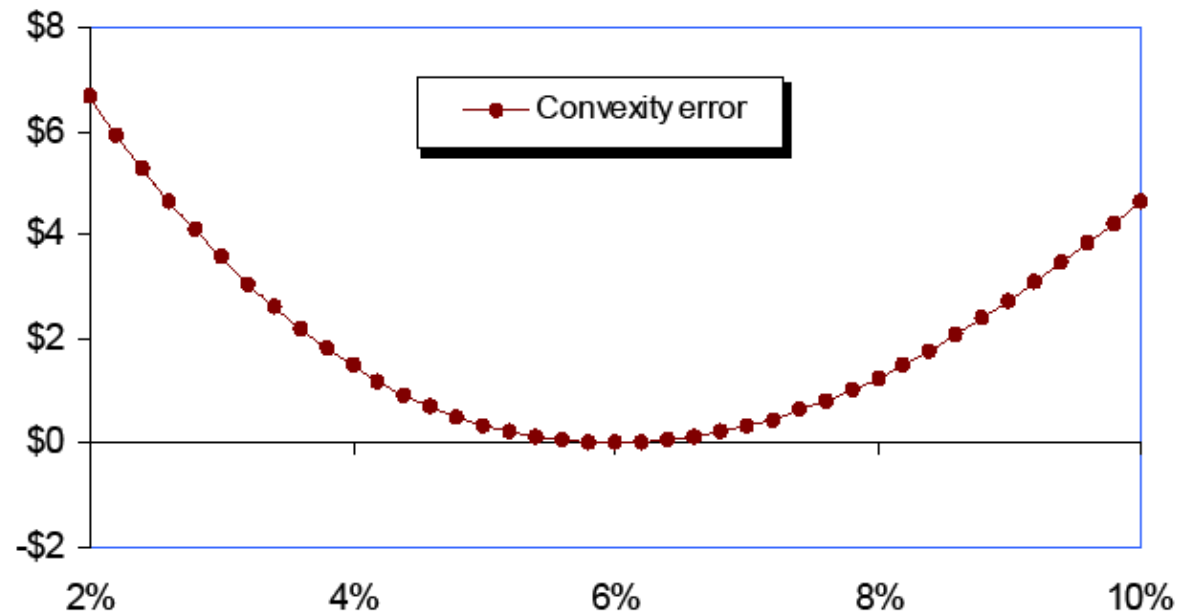
The convexity of a zero coupon bond 1

- The convexity of a function quantifies its curvature, and is measured by its second derivative
- In the case of a zero coupon bond, this implies that convexity is a quadratic function of tenor, as shown in the following equation (for a proof see appendix)

$$\begin{aligned}\frac{dP}{dY} &= -T(1 + Y)^{-(T+1)} \\ \frac{d^2P}{dY^2} &= (T + 1)T(1 + Y)^{-(T+2)} \\ &= (T^2 + T)(1 + Y)^{-(T+2)}\end{aligned}$$

The convexity of a zero coupon bond 2

Exhibit 5 – This slide visualizes the magnitude of the convexity error using the duration approximation to the price of a zero coupon bond (face value = \$100, tenor = 12 years, duration computed at $Y = 6.00\%$)



Convexity approximation 1

- A smooth function can be approximated using both the 1st and the 2nd derivatives (second order approximation). This property can be used to write an equation to approximate the price-yield function in terms of dollar duration and convexity (measured by the 2nd derivative)
- Note: in options-speak, Delta, and Delta-Gamma denote respectively the first and second order approximations

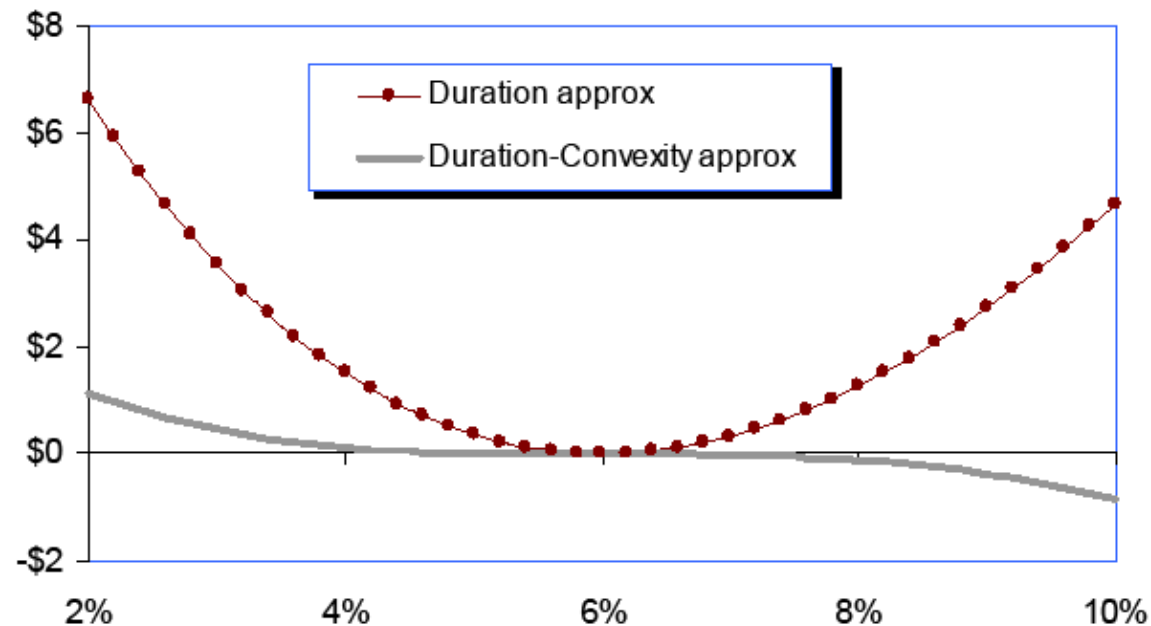
Convexity approximation 2

- The 2nd order approximation (duration and convexity) is very accurate, but (contrary to duration) is not much used in business life
- Duration is a rough measure but has the advantage of being expressed by one number. This is the equation, where (C) indicates convexity
- In the following equation (P') indicates the first derivative and (P'') the second

$$\begin{aligned}\Delta P &= P'(Y)\Delta Y + \frac{1}{2}P''(Y)(\Delta Y)^2 \\ &= D_{\$}\Delta Y + \frac{1}{2}C_{\$}(\Delta Y)^2\end{aligned}$$

Convexity approximation 3

Exhibit 6 – Duration and convexity approximation to the price of a zero (F = \$100, T = 12 years, yield at which duration is computed = 6.00%). To make the figure easily readable, the approximations are measured by their errors relative to the true value of the zero.



§3.3 - The exponential and the log functions

- The exponential function
- The natural exponential function
- The log and the natural log functions
- The exponential and the logarithmic functions are a cornerstone of calculus and are widely used in economic and financial applications

The exponential function 1

- Let us start with the exponential function, which is the standard tool when modeling proportional growth (both positive or negative)
- If we calculate the values of the function for $x = \{0, 1, 2, \dots, n\}$, we obtain a geometric sequence (TN 1, §1)
- In fact, we can prove that proportional growth can be modeled with, and only with, the exponential function
- The equation for the general exponential function, with arbitrarily chosen base (a) is:

$$y(x) = a^x \quad (a \geq 0)$$

The exponential function 2

- Some standard properties of the exponential function $y(x)=a^x$ ($a \geq 0$) can be summarized as follows:

$$a^0 = 1$$

$$a^1 = a$$

$$a^{-x} = \frac{1}{a^x}$$

$$a^{x+h} = a^x a^h$$

$$a^{x-h} = \frac{a^x}{a^h}$$

$$(a^x)^h = a^{xh}$$

The natural exponential function 1

- The natural exponential function has base ($e = 2.718281\dots$), and, for better readability, can be written as $\text{EXP}(x)$. Excel has a spreadsheet function, $\text{EXP}(x)$, to perform the calculation
- The value of (e) can be calculated with either of the following two equations:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$
$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

The natural exponential function 2

- We shall see (when discussing logarithms) that any generic exponential function can be easily transformed into a natural exponential function
- The symbol (e) was chosen by the great Swiss mathematician Leonard Euler, who is credited with much of the development of a consistent theory of exponential and logarithmic functions
- The widespread use of the natural exponential function is due to the remarkably useful mathematical property that its derivative equals the function itself

$$\frac{d}{dx} e^x = e^x$$

The log and the natural log functions 1

- ❑ The logarithmic function is the inverse of the exponential function
- ❑ In other words, given a real number ($x > 0$), its logarithm is the exponent of a base ($a > 0$) that will satisfy the following equations
- ❑ In practice we use only two logarithms function, one with base-10 and the other with base-e (called the natural logarithmic function)
- ❑ The base-10 logarithm is usually abbreviated as log, while the base-e is abbreviated with ln
- ❑ Excel has two spreadsheet functions LOG10(x) and LN(x) to calculate base-10 and natural logarithms

The log and the natural log functions 2

- Let us now explore, with four self-explanatory equations what we mean precisely when we say that the logarithmic and exponential functions are the inverse of each other

$$y = 10^x$$

$$\log(y) = x$$

$$y = e^x$$

$$\ln(y) = x$$

The log and the natural log functions 3

- The logarithmic function, irrespective of its base, have the following properties, which are the mirror-image of those of the exponential function
- In the following equations we use natural logarithms

$$\ln(e) = 1$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$\ln(a^b) = b \ln(a)$$

The log and the natural log functions 4

- Using the properties of the logarithmic function we can easily transform an arbitrarily-based exponential function into a natural exponential function
- As an example, let use the future value (v), at a fixed compounded rate ($y = 0.07$), of a unit investment over a time-horizon ($t = 10$)

$$v = (1 + y)^t = 1.967151$$

$$R = \ln(1 + y) = 0.067658648$$

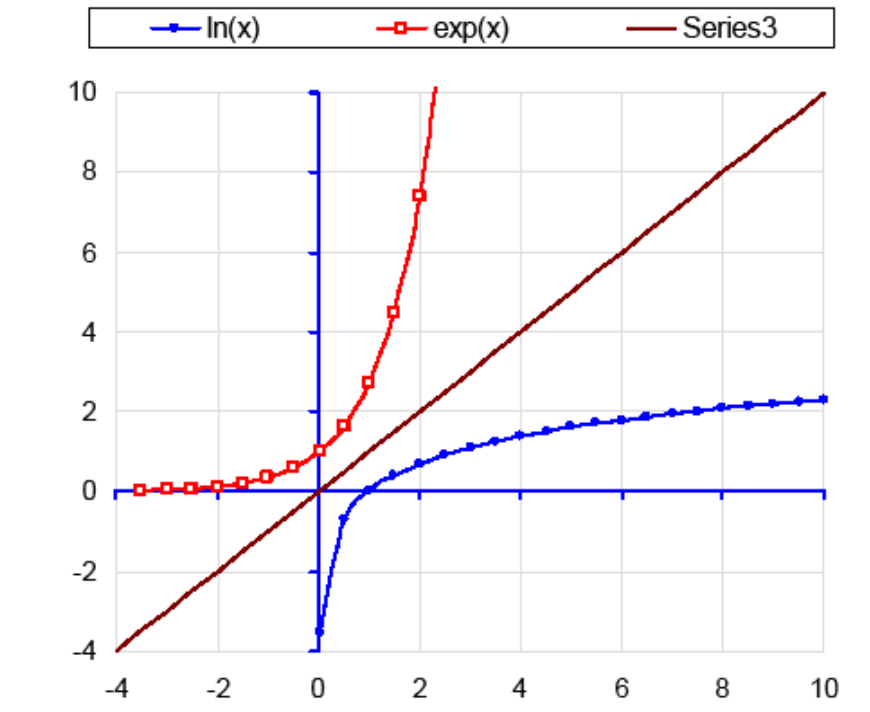
$$v = \exp(Rt) = 1.967151$$

The log and the natural log functions 5

- We have seen how logarithms can be used to express yields in ln-rates (R). This subject will be discussed in much detail in TN 0
- Let us discuss one more application
- Consider a time series that increases in a roughly exponential way
- The problem with this series is that it cannot be used in a linear regression model (it is not linear)
- Clearly, taking the log of the series solves the linearity problem

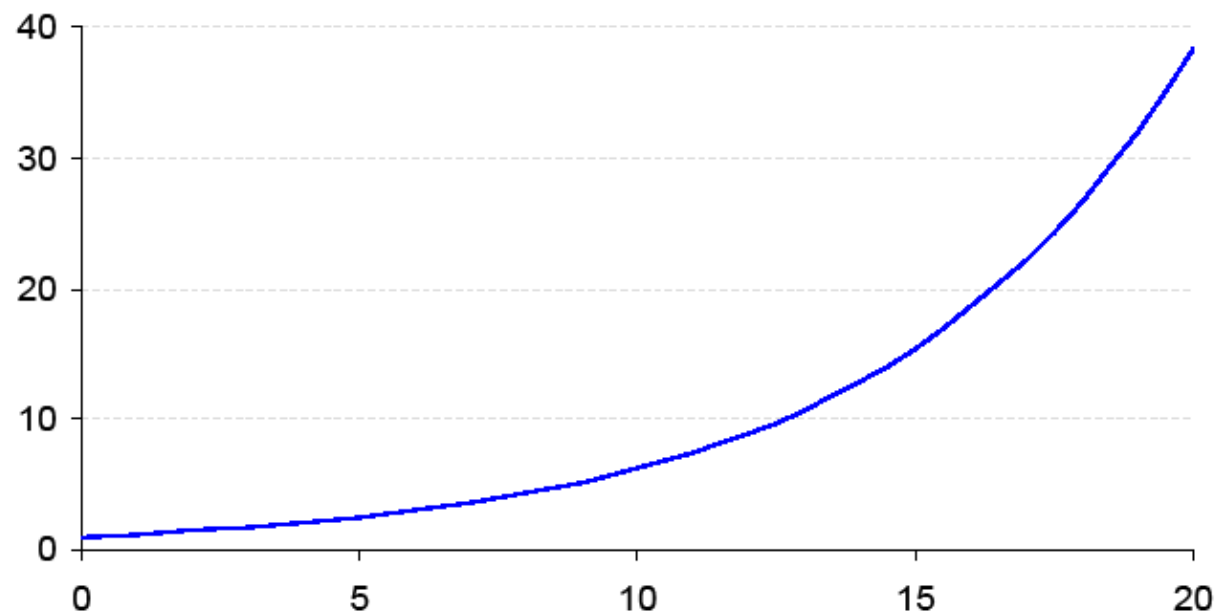
Graphical Illustration 1

Exhibit 7 – Graph of the $\exp(x)$ and $\ln(x)$ functions



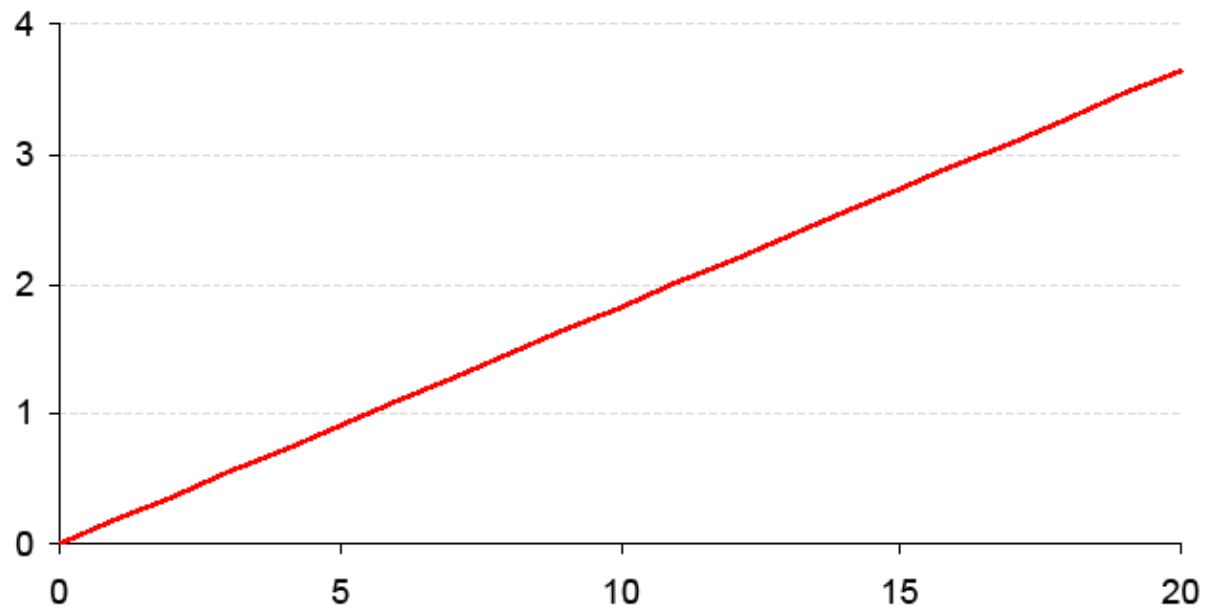
Graphical Illustration 2

Exhibit 8 - The original exponential series, growing at 20% per year



Graphical Illustration 3

Exhibit 9 – The graph of the natural logarithm of the original series



Excel functions used in this note

- LN(x): natural logarithm, with base (e)
- LOG10(x): base 10 logarithm
- SIN(x): The sine function of (x), where (x) is measured in degrees

TN 04

Yield Metrics

This teaching note was prepared by Georgio Questa with help from Dr. Enrico Biffis and Dr. Lorenzo Trapani

1. Spot yields
2. Discount factors & value relatives
3. Bank discount rate & simple interest
4. Classic compounded yield and U.S. bond yield quotation
5. Log yield (continuously compounded yield)
6. Key terms
7. References

§1 – Spot yields

Spot yields 1

- ❑ Spot yields (often referred to as zero-coupon yields) are defined with reference to fixed-income financial instruments reimbursed at maturity, without any intermediate payment of coupons and/or principal. For example:
- ❑ Discount securities (treasury bills, commercial paper, zero coupon bonds, etc.)
- ❑ Other short-term fixed-income instruments, such as: bank deposits (term deposits), coupon bonds with only the last coupon left, etc.

Spot yields 2

- ❑ However, most long-term fixed income securities and loans typically pay coupons before maturity
- ❑ Coupon payments also apply to widely used derivatives such as interest-rate and currency swaps
- ❑ Therefore, we could reasonably think that spot yields are relevant only for short-term instruments, and for investment & trading strategies based on zero-coupon bonds

Spot yields 3

- ❑ In reality, the spot yield approach is a lot more powerful than traditional yield-to-maturity analysis, and is now widely used to price and structure all fixed income securities
- ❑ Derivatives and a number of structured products are priced based on spot and forward yields
- ❑ In turn, forward yields are calculated based on spot yields
- ❑ The spot yield approach helps in gaining a much better understanding of risk measures, such as duration and convexity
- ❑ The spot yield approach is now being fully adopted by high level finance textbooks, such as Berk and De Marzo (2006)

§2 - Discount factors & value relatives

Discount Factors

- Take a security that promises to pay unconditionally the amount $V(t)$ at some future time, and is priced at $P(0)$ today. We define its discount factor $d(t)$ as $d(t)=P(0)/V(t)$.
- Example. If the price of a t-bill with face value of €100 is quoted at €97.50, its discount factor is 0.975
- Using a ratio between future and present value is a first step towards comparing the returns on different assets, irrespective of the size of the investment
- Note that the discount factor does not have a currency dimension. It is a pure number.

Value relatives

- A discount factor is the present value of a unit payment due at a future time (t)
- The future value at time- t of a unit investment at time-0 is known as value relative. We shall indicate it with $v(t)$
- Note that, like the discount factor, the value relative does not have a currency dimension. It is a pure number

$$v(t) = \frac{V(t)}{P(0)} = \frac{1}{d(t)}$$

Rates of Return I

- ❑ Clearly, we need to measure returns taking into account the length of the investment horizon
- ❑ A 6% return over one year is very different from a 6% quarterly return
- ❑ In other words, we need some measure of the intensity of return per unit of time, something analogous to the concept of speed in physics
- ❑ The intensity of return problem is solved by computing rates of return per unit of time. Market convention dictates that time is measured in years, for both short-term and long-term investments
- ❑ While speaking of time dimension, let us recall that volatility is also measured with reference to a 1-year time-horizon

Rates of Return II

- ❑ Introducing the time element entails that we can use one of several different yield metrics. These metrics are not fungible; different applications require different metrics
- ❑ Bank discount rate, used to quote treasury bills and commercial paper
- ❑ Simple yield. This is a linear growth measure, used in money market transactions and in a large number of derivatives (FRAs, futures, swaps, etc.)
- ❑ Classic compounded yield (this is not much used in derivative and risk management applications)
- ❑ Log yield, also known as continuously compounded yield. Log yields are used in option pricing and risk management

§3 – Bank discount rate and simple interest

Bank discount rate 1

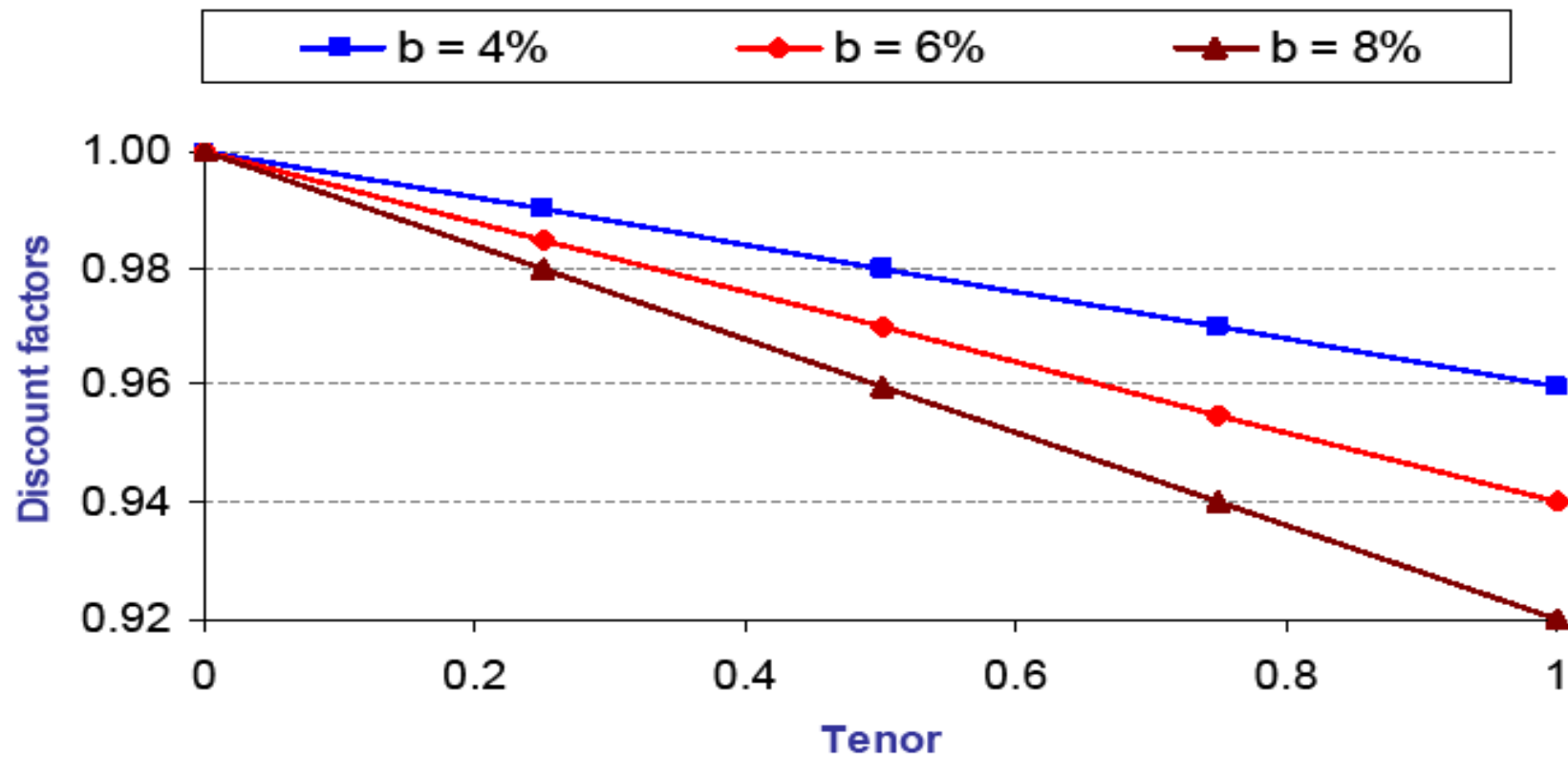
- ❑ This is widely used as a quotation convention for short-term securities (original maturities not exceeding one year) such as t-bills and commercial paper
- ❑ Despite its wide use it is somewhat of an historical relic. Hopefully one day it will be abandoned because it is a non-transparent and redundant yield metric

Bank discount rate 2

- ❑ The bank discount rate is a linear discount metric with reference to both the time-to-maturity (tenor) and the discount rate. See exhibits 1 & 2
- ❑ Bank discount rates are **never** used as a growth rate
- ❑ Indicating with (b) the bank discount rate, the discount factor, over a time-horizon (t) is defined as $d(t) = 1 - b \cdot t$

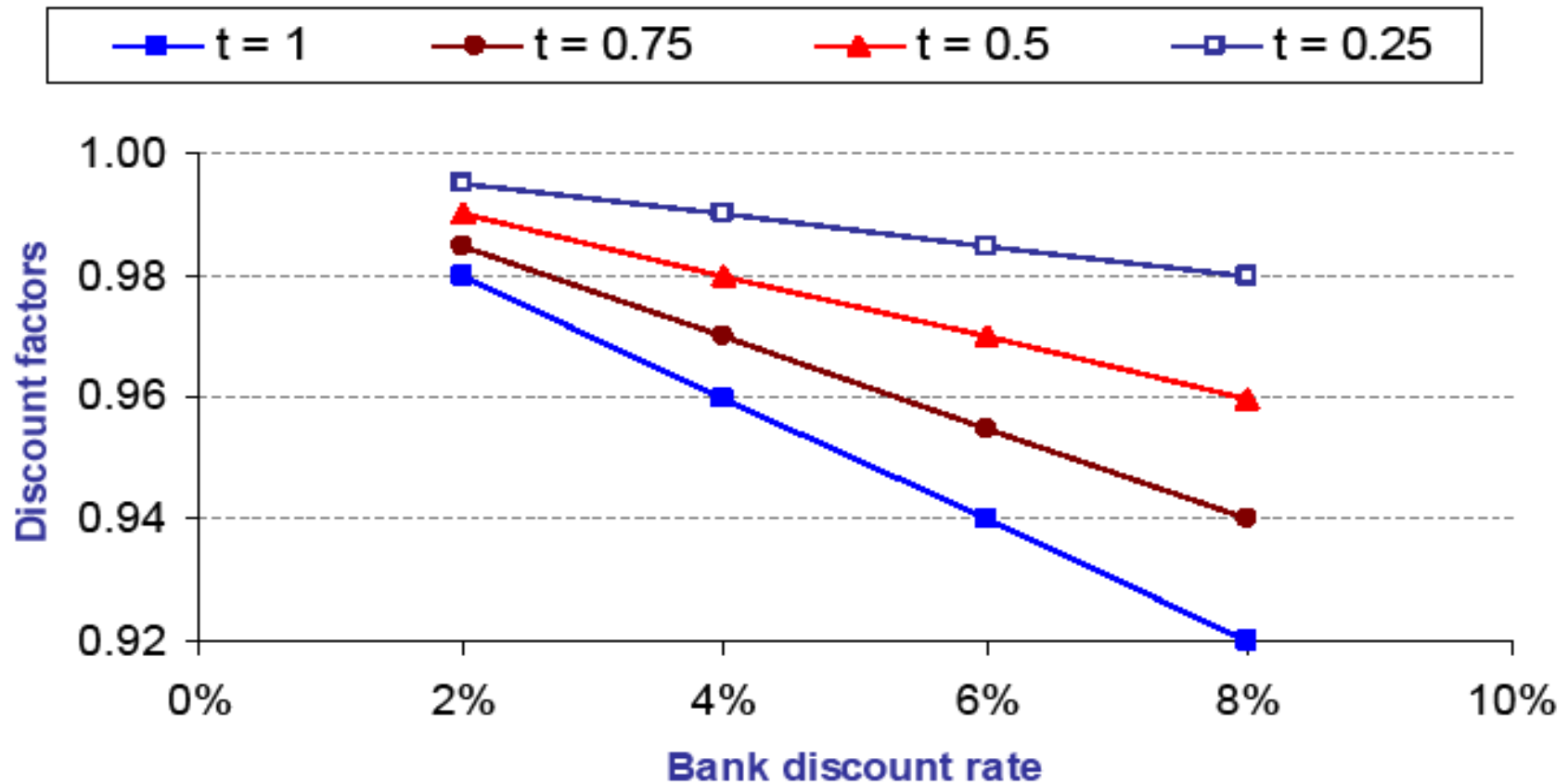
Bank discount rate 3

Exhibit 1 – discount factors as a function of tenor using bank discount rates



Bank discount rate 4

Exhibit 2 – discount factors as a function of bank discount rates



Bank discount rate 5 (numerical example)

- Bank discount rates are quoted with reference to the “days/360” day count
- This means that (t) equals “effective days / 360” for a 1-year bill
- Consider a 6-month t-bill, face value €1,000, quoted at 4.00%
- If we assume that the 6-month tenor we are considering has 183 days, the bill’s price will be:
$$€979.6667 = €1,000 (1 - 0.04 \times 183/360)$$

Simple Interest 1

- The simple rate, which we shall denote with r is a linear growth metric $v(t) = 1 + r \cdot t$
- It is widely used for money market transactions, such as bank deposits
- For example, the widely used LIBOR rates (London Interbank Offered Rate) are quoted as simple rates
- LIBOR rates are quoted for a number of currencies (\$, €, £, ¥, etc.). Values quoted as simple rates (spot yields) are for maturities up to one year

Simple Interest 2 (numerical example)

- Simple rates are quoted with reference to the “days/360” day count for a number of currencies (including the U.S. dollar)
- Example. We borrow \$10,000,000 at 6.00% for one quarter (91 days).
- The amount of interest due in three months time will be:
$$\$151,666.67 = \$10,000,000 \times 0.06 \times 91 \div 360$$

Simple rate & discount factors 1

- The simple rate can be used to compute discount factors and we can calculate discount factors using the simple rate
- While $v(t)$ is a linear function of time, $d(t)$ clearly is not.

$$d(t) = \frac{1}{v(t)} = \frac{1}{1+(r)(t)}$$

Why use simple rates 1

- We need to use an apparently crude linear metric such as simple yield because it has two very important (and interrelated) properties that turn out to be very useful both when aggregating securities in a portfolio (this is also known as cross-section aggregation) and when dealing with expected returns
- First, the simple rate of return on a portfolio equals the weighted average of the returns of the individual assets, where the weights are the initial invested values. This is shown in exhibit 4

Why use simple rates 2

Exhibit 4 – Computing simple return for a portfolio, term = 1 year. The simple rate of return computed on the portfolio (the shaded number) equals the rate of return obtained as the weighted average of the returns on the individual securities (in bold). To keep things as general as possible, two of the assets are assumed to have negative weights (borrowing or selling short)

Security	V(0)	Weight	V(t)	r	Wgted -r
1	90	90.00%	105	16.67%	15.00%
2	75	75.00%	79	5.33%	4.00%
3	-60	-60.00%	-61	1.67%	-1.00%
4	-45	-45.00%	-40	-11.11%	5.00%
5	40	40.00%	31	-22.50%	-9.00%
Portfolio	100	100.00%	114	14.00%	14.00%

Why use simple rates 3

- Second, when considering the probability distribution of an investment's outcomes, the expected yield of the possible outcomes $E[r]$ equals the yield on the expected outcome.
- Thus, the standard deviation of simple rates of return equals the standard deviation of the outcomes per unit invested. This is shown in exhibit 5

Why use simple rates 4

Exhibit 5 – Probability distribution of simple returns. the probability distribution is a simple binomial with symmetric probabilities implementation (at each step the up and down probabilities are equal)

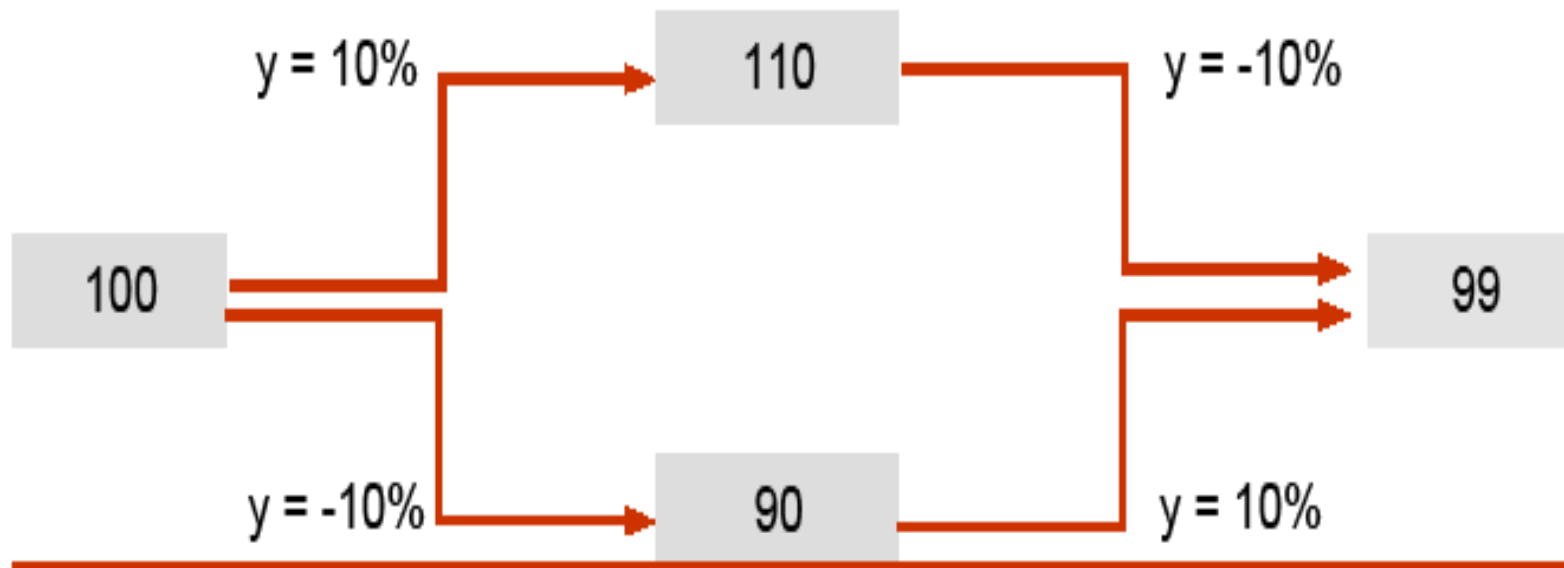
Prob	v(t)	r
1.56%	0.80	-20.00%
9.38%	0.90	-10.00%
23.44%	1.00	0.00%
31.25%	1.10	10.00%
23.44%	1.20	20.00%
9.38%	1.30	30.00%
1.56%	1.40	40.00%
E[] =	1.10	10.00%
std =	12.25%	12.25%

Simple rate time-series aggregation inconsistency

- Unfortunately, simple yields are not adequate for describing time sequences of returns. This is known as the time aggregation problem

Illustration

Exhibit 6 – Simple yield time-aggregation inconsistency



Measuring currency returns using simple rates

- ❑ The inadequate time-aggregation property of simple rates pops up when we want to measure currency returns
- ❑ The problem, sometimes referred to as Siegel's paradox, is that you get two different absolute values of currency returns, depending on whether you use direct or indirect quotations
- ❑ This is shown in exhibit 7

Illustration

Exhibit 7 – Currency returns with simple yields. At time-0, the \$/€ quotation of 1.2 is identical to a €/ \$ quotation of 0.83333. If the \$/€ rate increases by 10% (to 1.32) the €/ \$ rate decreases by 9.09%

	Time-0		Time-t	r
\$ / €	1.20	→	1.32	10.00%
€ / \$	0.833333	→	0.757576	-9.0909%

§4 – Classic compounded yield & U.S. bond yield quotation

Proportional growth 1

- In most cases (especially for medium and long-term fixed income securities) we need a proportional yield metric. This means that the invested amount is assumed to grow over time, at a given yield rate, in proportion to the amount outstanding
- This requires the use of an exponential function (it can be mathematically proved that only exponential functions will do)
- In classic compounded yield we have the following equation, where (Y) is the annual compounded yield rate

$$v(t) = \frac{1}{d(t)} = (1 + Y)^t$$

$$\therefore Y = [v(t)]^{\frac{1}{t}} - 1$$

Proportional growth 2 (Numerical Example)

- Compute the discount factor and the spot yield for a 4 year zero coupon bond, face value of \$100, priced at \$80

$$d(4) = \frac{80}{100} = 0.80$$

$$v(4) = \frac{100}{80} = 1.25$$

$$Y(4) = (1.25)^{\frac{1}{4}} - 1 = 5.7371$$

Proportional growth 3 (Graphical Illustration)

Exhibit 9 – Proportional (exponential) growth of the value of a zero coupon bond, face value 100, reimbursed at par, and yielding 6.10%



The U.S. bond yield quotation

- In the U.S. bond market, yields are quoted as if they were capitalized twice a year. The historical origin of this peculiar convention reflects the fact that in the U.S. (like in the UK) bonds traditionally pay interest twice a year
- Therefore, when the bond trades at par, its yield to maturity equals the coupon rate.
- In other words, the U.S. bond yield convention means that the quoted yield is the semiannual yield multiplied by two. Indicating with the subscript (us) this metric, we have:

$$v(t) = \left(1 + \frac{Y_{US}(t)}{2}\right)^{2t}$$
$$\therefore Y_{US}(t) = 2(\sqrt{1 + Y} - 1)$$

§5 – Log yield
(continuously compounded yield)

Log yield 1

- With classic compounded yield, the yield rate appears in the base of the exponential function. With log yield, that we shall indicate with (R), the yield rate is in the exponent of the natural exponential function
- When using log yields, day count is $act/365$ (elapsed days/365)

$$v(t) = \frac{1}{d(t)} = \exp(Rt) = e^{Rt}$$

$$\therefore R = \frac{\ln(v(t))}{t} = \frac{-\ln(d(t))}{t}$$

Log yield 2 (Numerical Example)

- Example of log yield calculation. A one-year t-bill is priced at 95.00. Its log yield is computed as follows

$$d(t) = \frac{95}{100} = 0.95$$

$$\therefore R = \frac{-\ln(d(t))}{t} = \frac{-\ln(0.95)}{1} = 5.1293\%$$

Continuously compounded yield 1

- Log yield is often denoted as 'continuously compounded yield' because the value relative $v(t)$ that you obtain using the log rate (R) is identical to that obtained using (R) simple rate compounded an infinite number of times over the interval $[0, t]$
- The following result is proved in the appendix:

$$e^R = \lim_{n \rightarrow \infty} \left[1 + \frac{R}{n} \right]^n$$

Continuously compounded yield 2

(numerical example)

Exhibit 10 – Numerical example of the continuous compounding property

n	$\left(1 + \frac{R}{n}\right)^n$
1	1.050000
2	1.050625
4	1.050945
12	1.051162
365	1.051267
exp R	1.051271

A common misconception 1

- ❑ You are likely to come across the statement that continuous compounding should be used only when interest is capitalized with a very high frequency
- ❑ For example, let us quote from a well-known fixed income textbook/professional manual:
- ❑ *“Eurobonds pay annual coupons, U.S. treasuries’ coupons are semiannual, and GNMMAs make monthly payments. As the coupons become more frequent, it becomes more accurate to assume exponential continuous compounding.”* (GNMMAs are bonds issued by the Government National Mortgage Association, an agency of the U.S. government.)

A common misconception 1

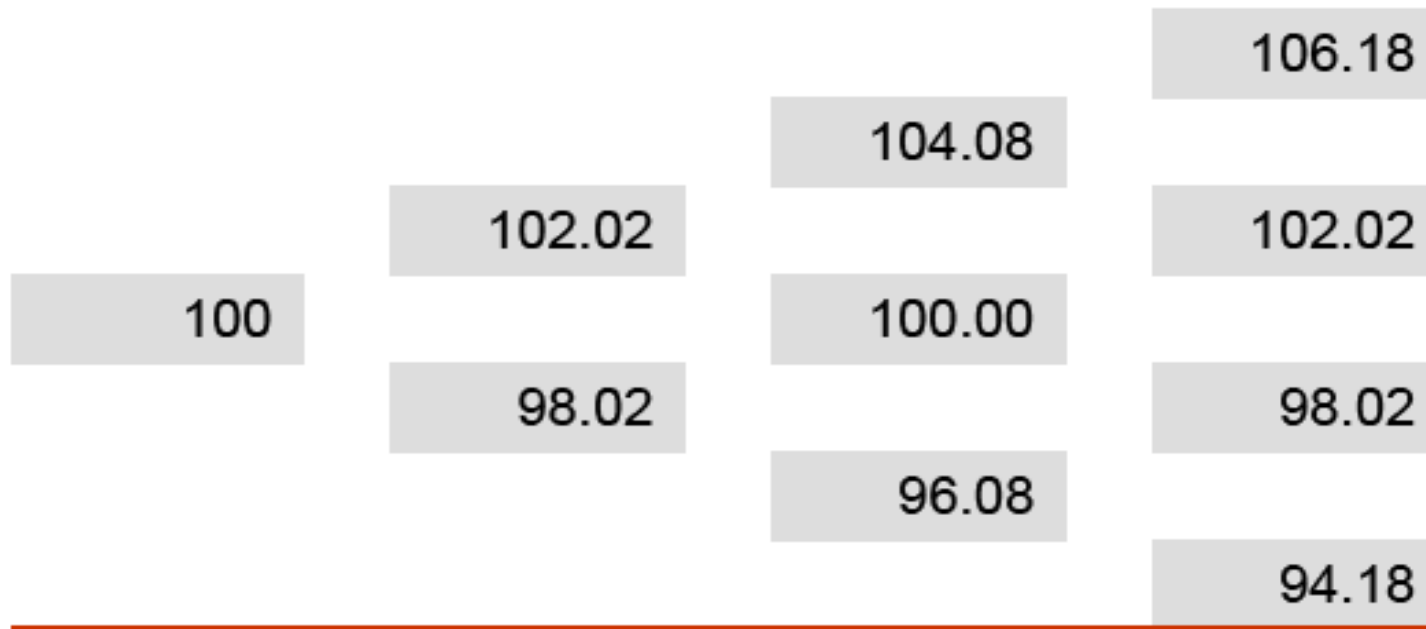
- ❑ This does not make sense
- ❑ Given a value relative $v(t)$, the log rate (R) is simply an efficient way to measure the rate of growth (positive or negative), and has nothing to do with the coupon payment frequency
- ❑ In fact, we use log yields for all sort of securities, including zero coupon bonds (which have no coupon payment at all)

Log yields and time-series aggregation 1

- Log yields (and classic compounded yields) have a coherent time aggregation
- This is shown in the following exhibits



Log yields and time-series aggregation 2

Exhibit 11 – A binomial diffusion using log rates will never produce negative prices (as in the binomial option pricing model and the Black Scholes equation)



Log yields and currency returns

Exhibit 12 – Log rates produce coherent rates of change for currency returns

	Time-0		Time-t	R
\$ / €	1.20		1.32	9.5310%
€ / \$	0.833333		0.757576	-9.5310%

Log rates and cross-section aggregation 1

- ❑ The flip side of the time series aggregation consistency is that log rates cannot be used for cross section aggregation, as shown in exhibit 13
- ❑ This is due to the convexity of log rates

Log rates and cross-section aggregation 2

Exhibit 13 – This exhibit is based on the same asset returns of exhibit 2. The log rate computed on the portfolio values (shaded) is materially different from the weighted average of the assets' log rates

Security	V(0)	Weight	V(t)	R	Wgted -R
1	90	90.00%	105	15.42%	13.87%
2	75	75.00%	79	5.20%	3.90%
3	-60	-60.00%	-61	1.65%	-0.99%
4	-45	-45.00%	-40	-11.78%	5.30%
5	40	40.00%	31	-25.49%	-10.20%
Portfolio	100	100.00%	114	13.10%	11.88%

§6 – Key terms

- Bank discount rate
- Binomial diffusion
- Commercial paper
- Continuously compounded yield
- Coupon payments
- Cross-section aggregation
- Discount factor
- Forward yields
- FRAs
- Futures
- LIBOR rates
- Proportional growth
- Simple yield
- Spot yields
- Swaps
- Tenor
- Time-series aggregation
- Time-to-maturity
- Treasury bills
- Value relative
- Yield-to-maturity (YTM)
- Zero coupon bonds

§7 - References

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