Testing for Non-stationarity using Maximum Entropy Resampling: A Misspecification Perspective

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Objective

- Develop an alternative procedure for testing the time-invariance (t-invariance) of the model parameters, by investigating for possible non-stationarity in the primary moments of the process.

Features of our approach

- Focuses on more general forms of heterogeneity (slow moving) rather than sudden shifts.
- Resampling Techniques
- Rolling Window Estimators of the primary moments of the stochastic process

Structural Change: AR(1)

AR(1) model: $y_t = \alpha_0 + \alpha_1 y_{t-1} + u_{t,}\ \{u_t | F_{t-1}\} \sim N(0,\sigma^2)$

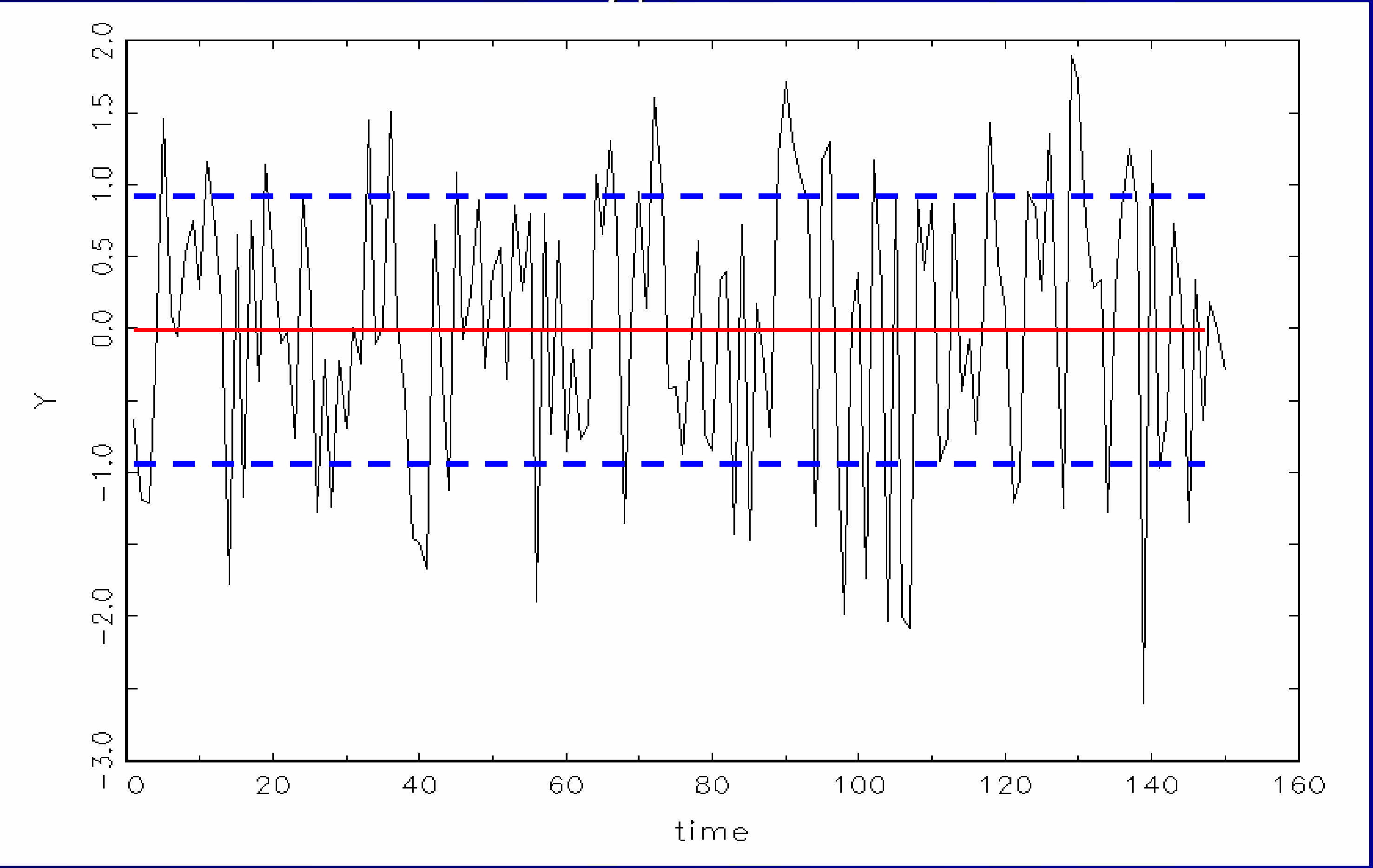
$$\alpha_0 = \alpha_0(t) \quad \text{or} \quad \alpha_1 = \alpha_1(t) \quad \text{or } \sigma^2 = \sigma^2(t)\ ,\ t \epsilon T$$

where $\alpha_0(t)$, $\alpha_1(t)$ and $\sigma^2(t)$ can be any function of time t

- How does t-heterogeneity reveal itself on data plots?

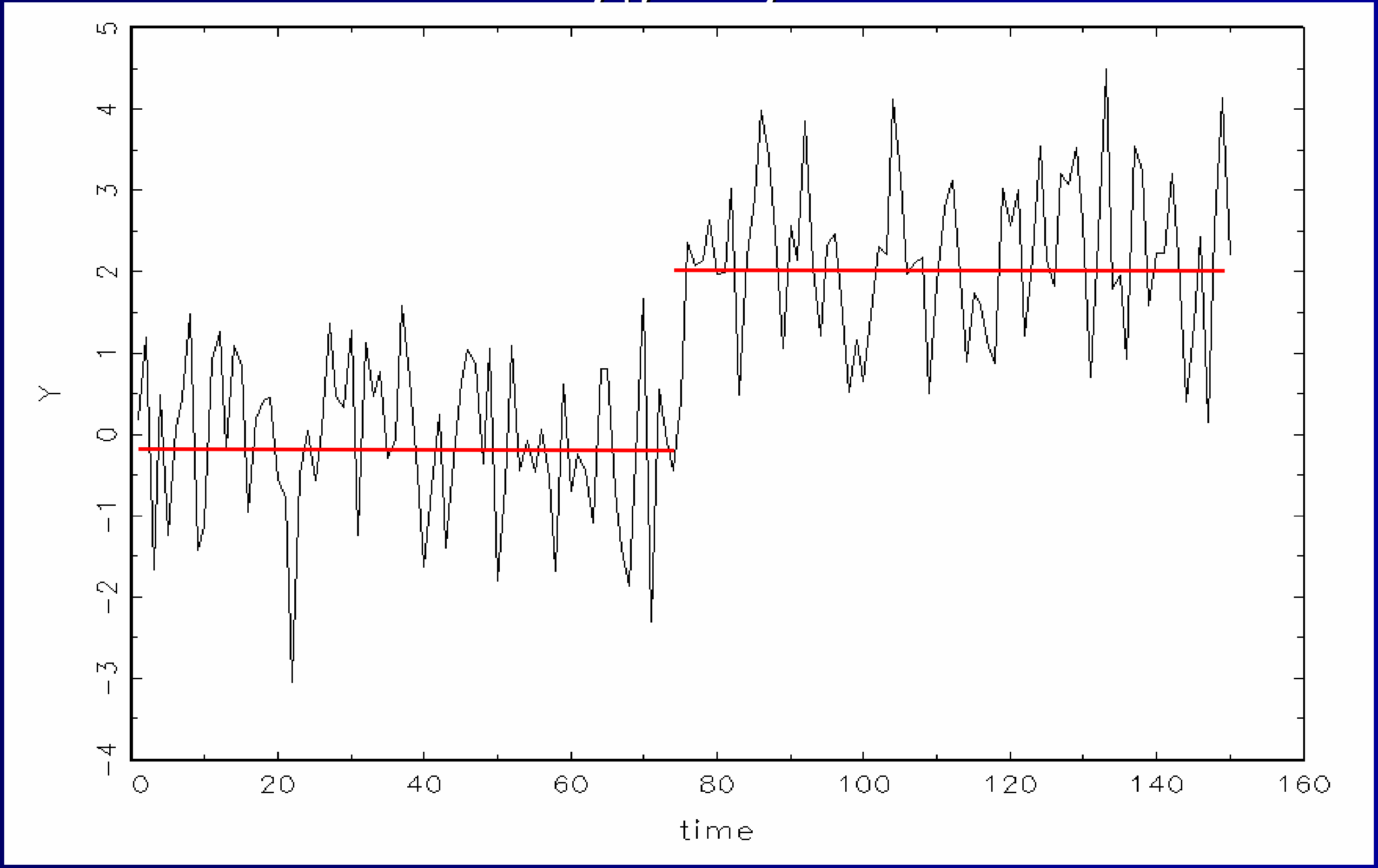
NIID

Simulated NIID series y_t, constant mean and variance



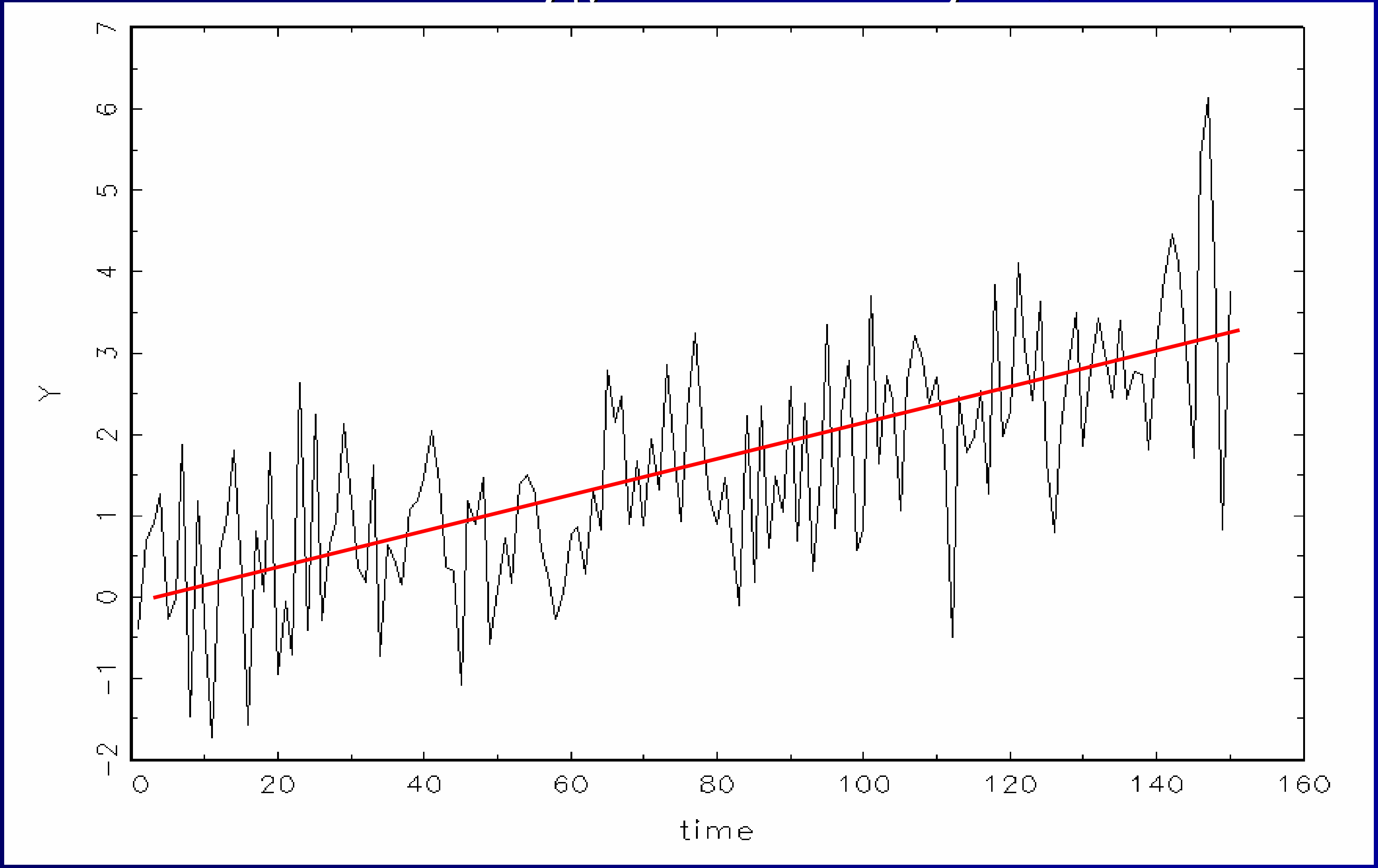
Mean Break

Simulated NI series y_t, single mean break



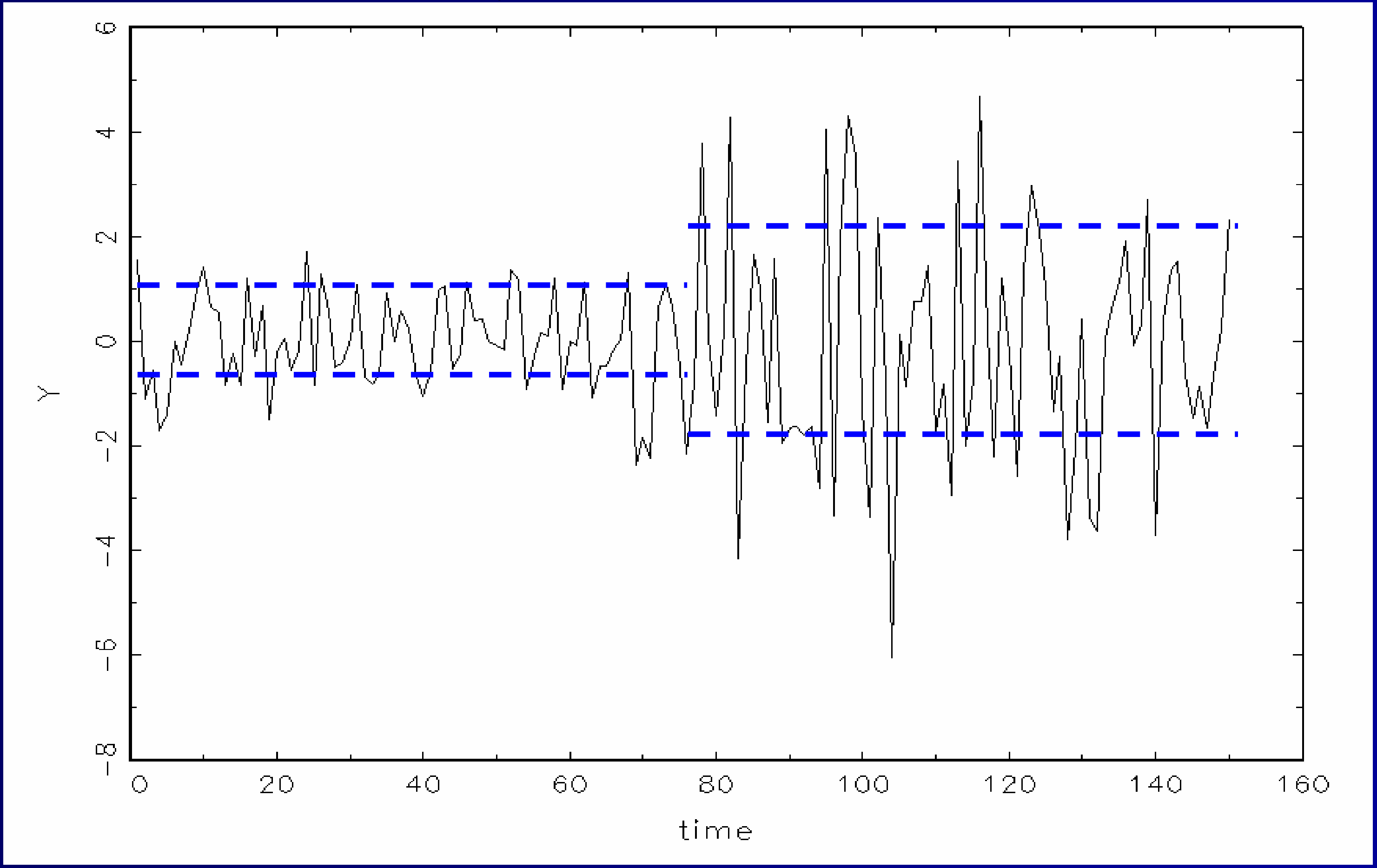
Trending Mean

Simulated NI series y_t, smooth change of the mean



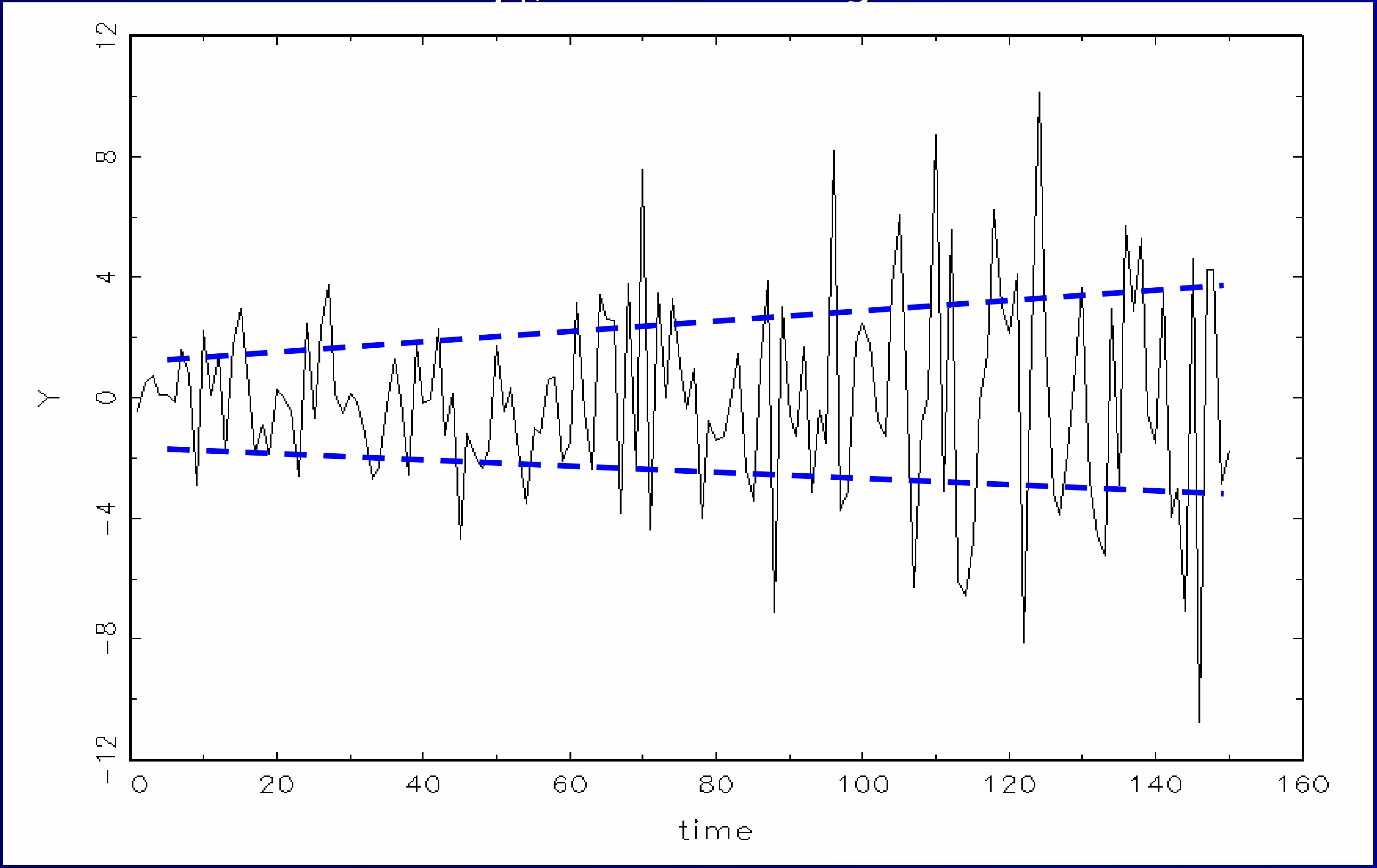
Variance Break

Simulated NI series y_t, single variance break



Trending Variance

Simulated NI series y_t, smooth change of the variance



Consequences of Structural Change

- Models with Time-varying Parameters
- Incorrect Inferences about Economic Relationships
- Inaccurate Forecasts
- Unreliable Policy Evaluation
- Misleading Policy Recommendations

Tests of Structural Change

- Chow (1960)
- Quandt (1960)
- Brown, Durbin and Evans (1975)
- Andrews (1993) and Ploberger (1994)
- Bai and Perron (1998)
- Hansen (2000)
- Perron (2005) –Survey of the literature

Need for Alternative Tests

- Existing tests have low power in detecting smoothly trending t-heterogeneity.

Table 3: Empirical, size-corrected Power of A&P tests under mean trend; α=5%

Trend Function	SupF	ExpF	AveF
$\mu(t) = \mu + 0.02t$	21.82	24.94	23.14
$\mu(t) = \mu + 0.001t + 5(10^{-4}t^2)$	7.61	9.61	9.29
$\mu(t) = \exp(0.01t) + \mu$	18.3	18.98	18.27
$\mu(t) = \left(\frac{5}{1+\exp\left(\frac{-t}{4}\right)}\right) + \mu$	7.45	8.21	6.98

Need for Alternative Tests

- Hansen (JEP 2001)

  "While it may seem unlikely that a structural break could be immediate and might seem more reasonable to allow for structural change to take a period to take effect, we often focus on the simple case of immediate structural break for simplicity and parsimony."

Motivating our Test

AR(1) model

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + u_t , \quad \{u_t | F_{t-1}\} \sim N(0,\sigma^2)$$

Implicit parameterizations:

$$\alpha_0 = (1 - a_1)\mu_y \; , \; \alpha_1 = \frac{\sigma_1}{\sigma_0} \; and \; \sigma^2 = \sigma_0 (1 - \alpha_1^2)$$

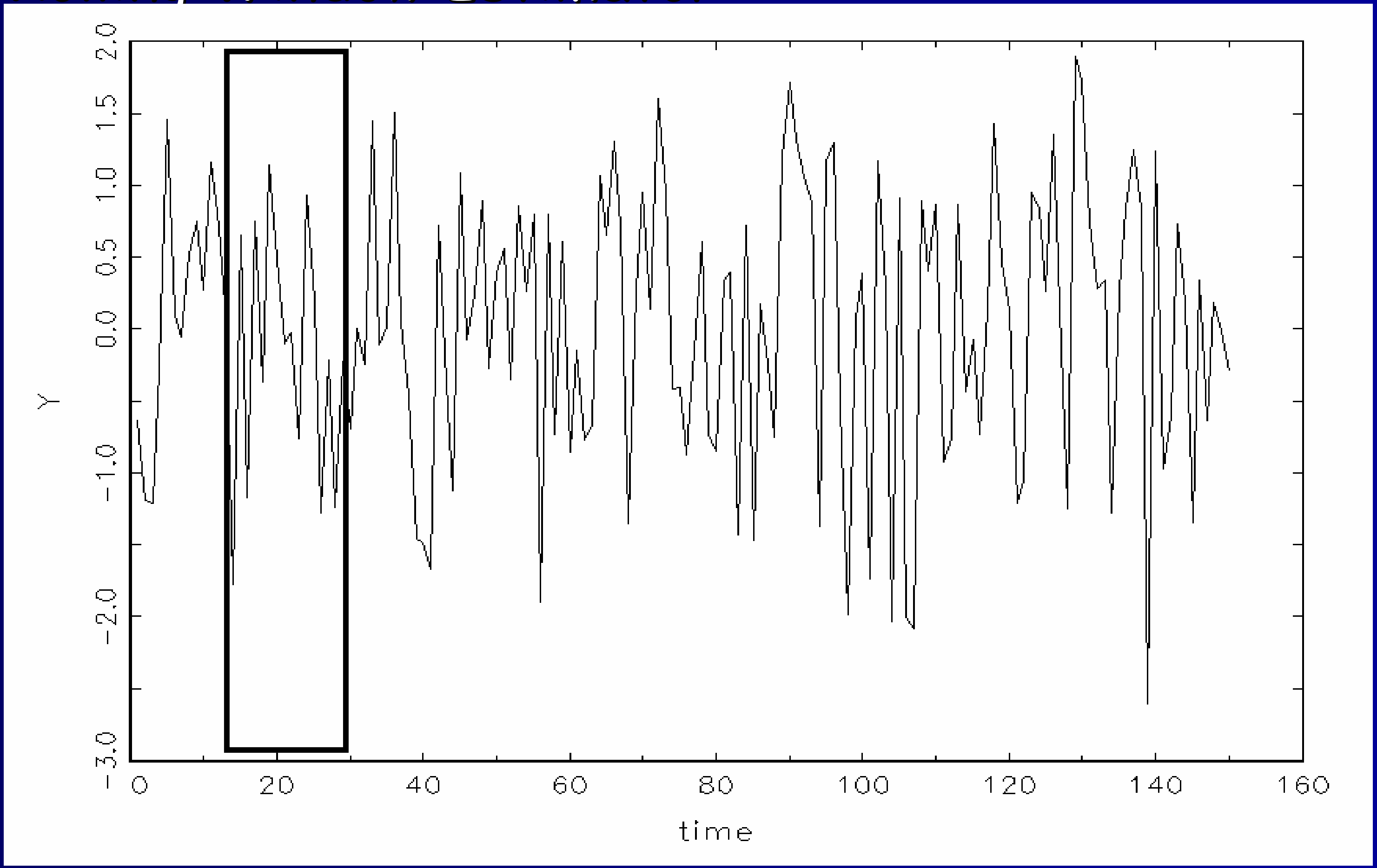
$$where, \; \sigma_1 = Cov(y_t, y_{t-1}) \; and \; \sigma_0 = Var(y_t)$$

Methodology

- Maximum Entropy Bootstrap, Vinod (2004)
  - Reliable resampling algorithm for small sample sizes.
  - It is designed to be robust to deviations from the IID assumption.
- Rolling Window Estimator
- Bernstein Polynomials

MEBR Estimators

Rolling Window Estimator



MEBR Estimators

- Fixed window length ℓ.
  - Small enough to assume local weak stationarity
  - Large enough to be able to estimate the moments
- Create J=100 resamples for the window using ME bootstrap.
- Estimate the sample mean and variance using the $\ell \cdot (J+1)$ observations
- Repeat for each window.

Formulating the Test Statistic

Mean:

Restricted $\hat{\mu}(t_i) = \alpha_0 + \alpha_1 \hat{\mu}(t_i - 1) + u_{r\mu}(t_i)$, vs

Unrestricted $\hat{\mu}(t_i) = \alpha_0' + \alpha_1' \hat{\mu}(t_i - 1) + B_{k,t_i} + u_{u\mu}(t_i)$

$$B_{k,t_i} = \sum_{j=0}^{k} \beta_j \binom{k}{j} t_i^j (1 - t_i)^{k-j}$$

$\mathrm{H_0} : \mu(t_i) = \mu$, for $\mathrm{t_i} = 1 \ldots \mathrm{n} - (\mathrm{l} - 1)$

vs $\mathrm{H_1} : \mu(t_i) \neq \mu$ for any $\mathrm{t_i} = 1 \ldots \mathrm{n} - (\mathrm{l} - 1)$

Formulating the Test Statistic

Variance:

Restricted $\hat{\sigma}^2(t_i) = c_0 + c_1\hat{\sigma}^2(t_i - 1) + u_{r\sigma^2}(t_i)$, vs

Unrestricted $\hat{\sigma}^2(t_i) = c_0^{'} + c_1^{'}\hat{\sigma}^2(t_i - 1) + B_{k,t_i}^{'} + u_{u\sigma^2}(t_i)$

$\mathrm{H}_0 : \sigma(t_i) = \sigma$, for $\mathrm{t_i} = 1 \ldots \mathrm{n} - (\mathrm{l} - 1)$

vs $\mathrm{H}_1 : \sigma(t_i) \neq \sigma$ for any $\mathrm{t_i} = 1 \ldots \mathrm{n} - (\mathrm{l} - 1)$

Simulation Set Up

- 10,000 replications
- Sample size: n = 60, 80 and 100
- $x_t \sim NIID(0,1)$

Simulation Set Up

- Mean Heterogeneity
  - Linear
  - Quadratic
  - Exponential
  - Logistic
  - Single Break at Q_1, Q_2 and Q_3

Simulation Results

Mean t-heterogeneity

Table 6: Trending Mean

Trend Function	$\alpha\%$	$H_0 : \mu$ constant			$H_0 : \sigma^2$ constant		
		n=60	n=80	n=100	n=60	n=80	n=100
Linear trend $\mu(t_i)=\mu+0.02\cdot t_i$	1	11.28	30.85	60.47	2.13	1.69	2.23
	5	27.89	57.99	83.86	6.06	5.13	6.35
	10	40.22	72.03	91.86	10.05	8.98	10.86
Quadratic trend $\mu(t_i)=\mu+10^{-3}\cdot t_i+5\cdot 10^{-4}\cdot t_i^2$	1	18.93	81.41	99.86	1.36	1.75	2.34
	5	48.15	97.35	100	4.67	5.20	6.46
	10	66.52	99.50	100	7.87	8.97	10.55
Exponential trend $\mu(t_i)=\exp(10^{-2}t_i)+\mu$	1	5.06	15.18	40.63	2.17	1.69	2.24
	5	15.05	35.31	67.28	6.30	5.11	6.44
	10	24.15	49.75	80.30	10.47	8.96	10.86
Logistic trend $\mu(t_i)=\left(\frac{5}{1+\exp\left(\frac{-t_i}{4}\right)}\right)+\mu$	1	34.50	31.42	54.05	2.39	2.01	3.03
	5	56.52	50.25	61.89	7.02	6.43	7.42
	10	65.06	59.47	66.98	12.15	10.41	13.53

Simulation Results

Mean t-heterogeneity

Table 8: Single Mean Break							
Mean Break	$\alpha\%$	$H_0 : \mu$ constant			$H_0 : \sigma^2$ constant		
		n=60	n=80	n=100	n=60	n=80	n=100
$\mu(t_i)=\mu+2\sigma\cdot I_{\{t_i \geq \frac{n}{4}\}}$	1	79.90	90.40	96.42	1.30	1.42	1.49
	5	93.91	98.74	99.35	5.10	4.05	4.10
	10	97.00	99.72	99.89	7.80	6.68	7.21
$\mu(t_i)=\mu+2\sigma\cdot I_{\{t_i \geq \frac{n}{2}\}}$	1	16.60	26.89	34.14	0.79	0.83	1.02
	5	50.11	63.61	72.17	3.09	3.46	3.46
	10	71.41	82.65	88.17	5.71	5.73	6.54
$\mu(t_i)=\mu+2\sigma\cdot I_{\{t_i \geq \frac{3n}{4}\}}$	1	6.33	4.99	5.48	1.24	1.12	1.56
	5	23.11	21.62	21.77	4.45	3.79	4.00
	10	39.76	39.50	40.47	7.65	6.48	7.16

Simulation Set Up

- Variance Heterogeneity
  - Linear
  - Quadratic
  - Exponential
  - Single Break at Q_1, Q_2 and Q_3

Simulation Results

Variance t-heterogeneity

Table 7: Trending Variance

Trend Function	$a\%$	$H_0:\mu$ constant			$H_0:\sigma^2$ constant		
		n=60	n=80	n=100	n=60	n=80	n=100
Linear trend $\sigma^2(t_i)=\sigma^2+0.05\cdot t_i$	1	1.79	2.91	4.99	13.57	24.83	36.62
	5	7.29	10.29	14.12	30.02	48.70	62.43
	10	13.19	16.77	22.43	41.87	62.49	75.27
Quadratic trend $\sigma^2(t_i)=\sigma^2+0.03\cdot t_i+0.01\cdot t_i^2$	1	8.53	14.72	20.36	44.68	52.28	60.40
	5	28.37	30.45	38.81	67.35	73.38	79.79
	10	40.14	40.47	49.38	76.92	81.97	87.33
Exponential trend $\sigma^2(t_i)=\sigma^2+\exp(0.02\cdot t_i)$	1	2.53	3.31	7.34	10.41	24.37	46.35
	5	9.29	10.54	19.07	23.81	45.86	69.07
	10	15.42	17.07	28.14	33.14	57.59	78.80

Simulation Results

Variance t-heterogeneity

Table 9: Single Variance Break

Variance Break	$a\%$	$H_0 : \mu$ constant			$H_0 : \sigma^2$ constant		
		n=60	n=80	n=100	n=60	n=80	n=100
$\sigma^2(t_i)=\sigma^2+2\sigma^2 I_{\{t_i\geq\frac{n}{4}\}}$	1	1.78	1.10	2.13	4.81	6.88	15.35
	5	7.49	5.61	7.74	17.83	25.09	43.27
	10	14.31	10.21	14.37	31.31	41.66	61.82
$\sigma^2(t_i)=\sigma^2+2\sigma^2 I_{\{t_i\geq\frac{n}{2}\}}$	1	3.42	2.78	4.10	8.10	9.24	11.78
	5	11.66	9.52	12.24	24.03	28.18	33.50
	10	19.76	15.31	19.54	37.33	43.56	51.85
$\sigma^2(t_i)=\sigma^2+2\sigma^2 I_{\{t_i\geq\frac{3n}{4}\}}$	1	8.66	6.84	9.50	23.30	23.27	23.54
	5	20.49	17.02	21.56	36.82	36.68	37.34
	10	28.85	24.56	29.81	44.22	44.97	46.57

Empirical Illustration

- Data
  - Quarterly Yen/US Dollar 1982Q2-2005Q2
  - Monthly European Total Turnover Index 01/1995-08/2004
  - Annual US Industrial Production 1921-2004
  - Quarterly US Investment 1963Q2- 1982Q4

Empirical Illustration

US-Japan Exchange rate & EU Total Turnover index

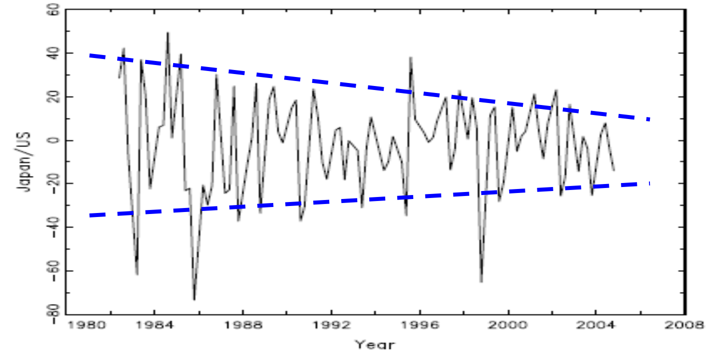


Fig. 1(b): Dem US/Jp Exchange

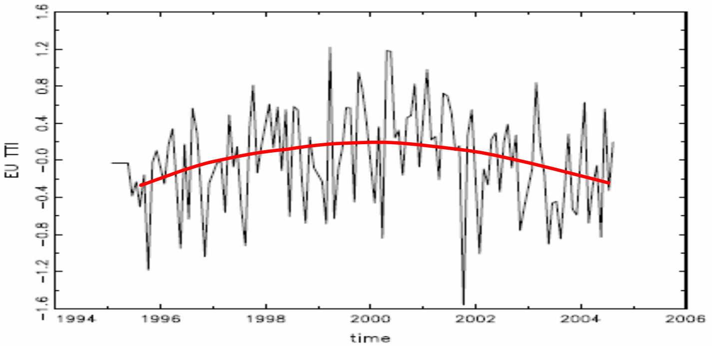


Fig. 2(b): Dem EU TTI

Table 10: Empirical Results

Variable	Sample Size	l	$H_0 : \mu$ constant	$H_0 : \sigma^2$ constant
US/Japan Exchange	91	7	0.403 (0.806)	3.495 (0.011)**
EU Total Turnover	116	10	2.190 (0.075)*	0.499 (0.737)
US Ind Production	84	7	0.114 (0.977)	2.716 (0.036)**
US Investment	80	6	3.698 (0.009)***	1.864 (0.127)

Notes:

1. Entries are test statistics with p-values in parentheses
2. l is the rolling window length
3. (*), (**) , (***) refer to the rejection of the null hypothesis at 10%, 5% and 1% level of significance, respectively.

5. Empirical Illustration cont.

US Industrial Production & US Investment

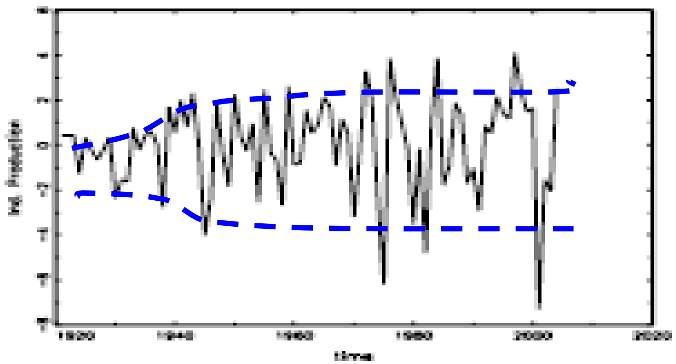


Fig. 3(b): Demem Ind Prod.

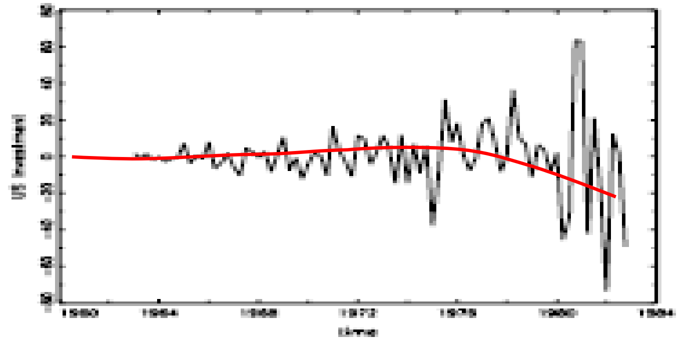


Fig. 4(b): Dem Invest

Table 10: Empirical Results

Variable	Sample Size	l	$H_0 : \mu$ constant	$H_0 : \sigma^2$ constant
US/Japan Exchange	91	7	0.403 (0.806)	3.495 (0.011)**
EU Total Turnover	116	10	2.190 (0.075)*	0.499 (0.737)
US Ind Production	84	7	0.114 (0.977)	2.716 (0.036)**
US Investment	80	6	3.698 (0.009)***	1.864 (0.127)

Notes:

1. Entries are test statistics with p-values in parentheses
2. l is the rolling window length
3. (*), (**) , (***) refer to the rejection of the null hypothesis at 10%, 5% and 1% level of significance, respectively.

Empirical Illustration

Table 11: Exchange Rate Japan/US				
Test Statistic		Andrews p-value	bootstrap p-value	Hetero-Corrected p-value
SupF	6.4256	0.417	0.338	0.572
ExpF	1.1671	0.465	0.451	0.591
AveF	1.9141	0.417	0.403	0.425

Table 12: European Total Turnover Index				
Test Statistic		Andrews p-value	bootstrap p-value	Hetero-Corrected p-value
SupF	9.7149	0.277	0.285	0.086
ExpF	2.4727	0.269	0.335	0.101
AveF	2.8780	0.438	0.489	0.185

Conclusion

- Our test has good power properties for detecting smooth trends even for small samples.
- It is capable of distinguishing between t-heterogeneity arising from changes in the mean or variance of the process.
- It has good power properties for single breaks, suggesting that it can be used in conjunction with the traditional tests to explore a broader variety of possible departures.

Thank You!

7. Remarks

- Identify the optimal polynomial basis for hypothesis testing purposes
  - Ordinary polynomials

$$p_n(x) = \sum_{i=0}^{n} a_i x^i$$

  - Bernstein polynomials

$$BS_{i,n} = \binom{n}{i} x^i (1-x)^{n-i}$$

  - Gram-Schmidt and LS polynomials
    - See def 4 and 5, page 57 , Koutris (2005)

7. Remarks

Table 5.1: Condition Number for polynomials of deterministic trend.[5]

Degree	1	2	3	4	5	6	7	8
Ordinary Pol.	1	19.44	533.5	15952	495218	∞	∞	∞
Bernstein Pol.	3	10.01	35.08	126.56	465.42	1735	6543	24889
Gram Schmidt	2.717	3.344	3.724	3.898	4.188	4.348	4.482	4.599

Table 5.2: Condition Number for polynomials of Uniform IID realization.[1]

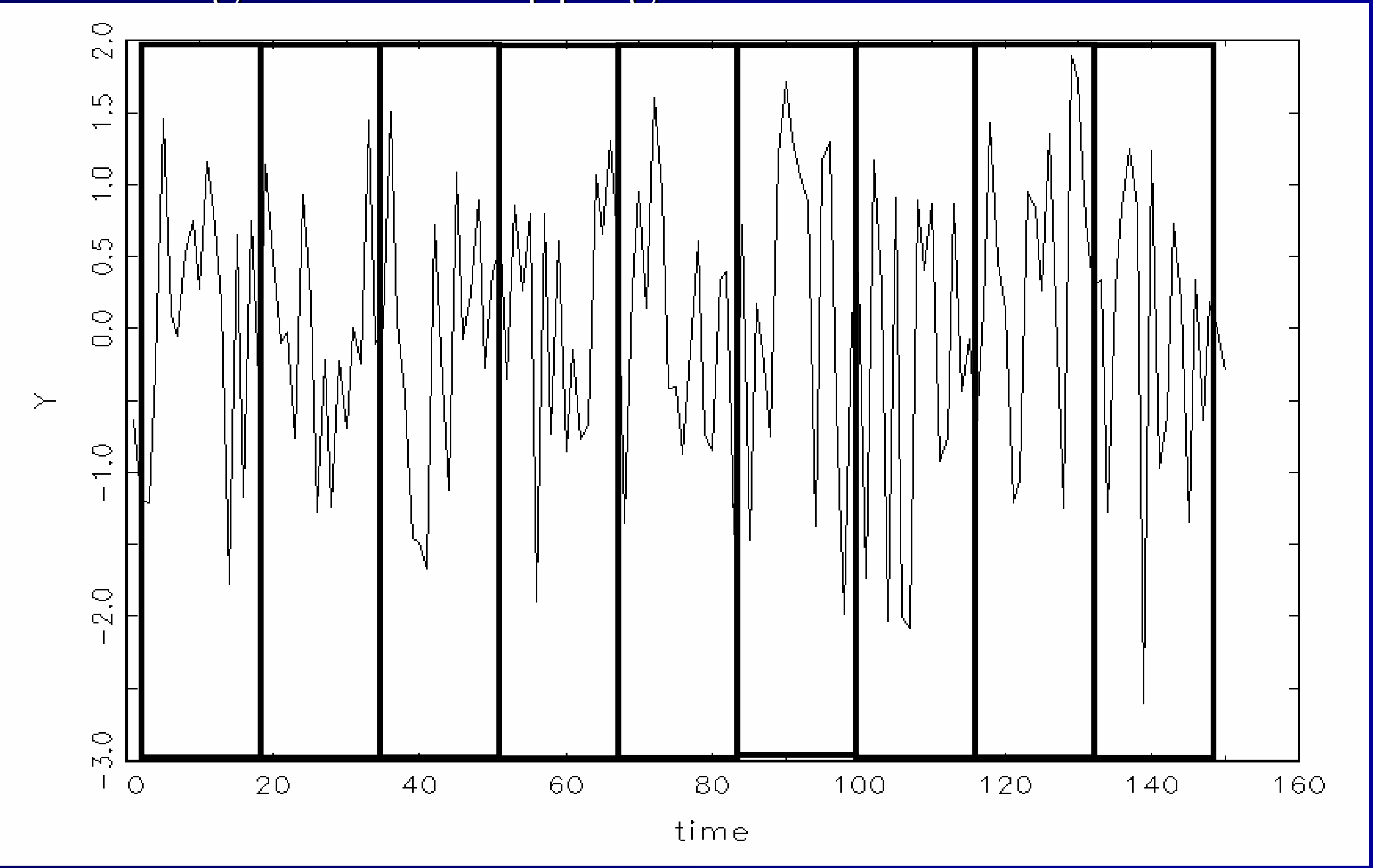
Degree	1	2	3	4	5	6	7	8
Ordinary Pol.	1	22.23	777.4	27137	918977	∞	∞	∞
Bernstein Pol.	3.238	13.22	52.24	199.7	825.6	3927.5	14881	61974
Gram Schmidt	3.119	4.169	5.229	6.204	7.782	11.241	16.322	31.530

Table 5.3: Condition Number for polynomials of Normal IID realization.[1]

Degree	1	2	3	4	5	6	7	8
Ordinary Pol.	1	63.25	511.9	4220.5	50537	672639	∞	∞
Bernstein Pol.	1	16.56	63.41	182.067	920.1	7272.7	21096	73256
Gram Schmidt	4.002	6.612	6.651	37.506	40.124	70.183	113.84	289.27

7. Remarks

- **R**olling **N**on-overlapping **W**indow **E**stimators



7. Remarks

RNWE

Mean:

Restricted $\hat{\mu}(t_i) = \alpha_0 + u_{r\mu}(t_i)$, vs

Unrestricted $\hat{\mu}(t_i) = \alpha_0^{'} + B_{k,t_i} + u_{u\mu}(t_i)$

Variance:

Restricted $\hat{\sigma}^2(t_i) = c_0 + u_{r\sigma^2}(t_i)$, vs

Unrestricted $\hat{\sigma}^2(t_i) = c_0^{'} + B_{k,t_i}^{'} + u_{u\sigma^2}(t_i)$

7. Remarks

Table 3.1: Mean Trend

Trend Function	n	$H_0 : \mu$ constant		$H_0 : \sigma^2$ constant	
		α=5%	α=10%	α=5%	α=10%
Linear trend $\mu(t_i)=\mu+0.02t_i$	150	97.36	98.67	5.02	10.08
	250	99.96	99.99	5.00	10.17
	350	99.98	99.99	5.05	9.98
Quadratic trend $\mu(t_i)=\mu+10^{-2}t_i+5\cdot 10^{-4}t_i^2$	150	98.66	99.38	5.03	10.06
	250	100	100	5.09	10.12
	350	100	100	5.26	10.24
Exponential trend $\mu(t_i) = \exp(0.01t_i)+\mu$	150	91.52	94.97	5.09	10.02
	250	99.99	100	5.55	10.65
	350	100	100	5.45	10.54
Logistic trend $\mu(t_i)=\left(\frac{5}{1+\exp(\frac{-t_i}{4})}\right)+\mu$	150	89.58	95.52	5.24	10.36
	250	98.89	100	5.31	10.57
	350	100	100	5.48	10.68

7. Remarks

Table 3.3: Single Mean Break

Mean Break	n	$H_0 : \mu$ constant		$H_0 : \sigma^2$ constant	
		α=5%	α=10%	α=5%	α=10%
$\mu(t_i)=\mu+2\sigma \cdot I_{\{t_i \geq \frac{n}{4}\}}$	150	72.94	84.26	4.79	9.55
	250	97.94	98.08	4.95	9.93
	350	99.33	99.81	4.90	9.99
$\mu(t_i)=\mu+2\sigma \cdot I_{\{t_i \geq \frac{n}{2}\}}$	150	91.63	96.00	4.93	9.67
	250	99.45	99.87	4.71	9.48
	350	99.96	99.97	4.93	9.82
$\mu(t_i)=\mu+2\sigma \cdot I_{\{t_i \geq \frac{3n}{4}\}}$	150	82.79	90.66	4.86	9.91
	250	96.60	98.69	5.03	10.18
	350	99.83	99.93	5.03	9.89

7. Remarks

Table 3.2: Variance Trend

Trend Function	n	H_0 : μ constant		H_0 : σ^2 constant	
		α=5%	α=10%	α=5%	α=10%
Linear trend $\sigma^2(t_i)=\sigma^2+0.05t_i$	150	5.05	10.17	99.30	99.82
	250	4.91	9.77	100	100
	350	5.01	9.91	100	100
Quadratic trend $\sigma^2(t_i)=\sigma^2+0.03t_i+0.01t_i^2$	150	4.92	10.18	99.98	99.99
	250	4.77	9.63	100	100
	350	5.01	9.93	100	100
Exponential trend $\sigma(t_i)=\exp(0.02t_i)+\sigma^2$	150	5.06	10.10	99.95	99.99
	250	4.94	9.78	100	100
	350	5.04	9.97	100	100

7. Remarks

Table 3.4: Single Variance Break

Variance Break	n	$H_0 : \mu$ constant		$H_0 : \sigma^2$ constant	
		α=5%	α=10%	α=5%	α= 10%
$\sigma^2(t_i) = 1 + 2\sigma^2 I_{\{t_i \geq \frac{n}{4}\}}$	150	5.02	10.15	70.82	88.44
	250	4.96	10.00	96.70	99.51
	350	5.09	10.06	99.88	100
$\sigma^2(t_i) = 1 + 2\sigma^2 I_{\{t_i \geq \frac{n}{2}\}}$	150	5.21	10.14	97.18	99.31
	250	4.99	10.22	99.91	99.96
	350	5.04	10.12	99.99	100
$\sigma^2(t_i) = 1 + 2\sigma^2 I_{\{t_i \geq \frac{3n}{4}\}}$	150	5.15	10.14	97.63	99.35
	250	4.98	10.01	99.80	99.96
	350	5.20	10.14	100	100