On bootstrapping the likelihood ratio test of stationarity

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Abstract

The distribution of the LR (likelihood ratio) test of the null hypothesis of stationarity is unknown. An LBI/LM (locally best invariant and one-sided lagrange multiplier) test however has a well-known limiting distribution and critical values. This paper proposes a bootstrap version of the LR test of stationarity and compares it with various form, parametric, nonparametric and bootstrapped, of the LBI/LM test for the same hypothesis. The bootstrap samples are obtained from the Kalman filter innovations imposing the null hypothesis of stationarity. Monte Carlo simulations show that, for the random walk plus noise or local level model, the bootstrap LR test achieves the correct size and it is rather more powerful than the LBI/LM for medium-sized deviations from the null hypothesis. For more elaborate models the bootstrap LR test appears even more attractive. In the case of a random walk trend plus AR(1) cycle, the performance of the bootstrap LR test is significantly superior, also in the neighborhood of the null hypothesis, than all other options. One distinctive advantage of the bootstrap LR test is that it allows to avoid the issue of bandwidth choice and the reliance on asymptotic critical values when the observation sample is small. Empirical illustrations on the use of the test are provided. All computations have been carried out using the Ox matrix programming language and the Ssfpack set of routines for state space models.

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1 Introduction (sketched)

Testing the null hypothesis of stationarity has received considerable attention in recent years. For a Gaussian random walk plus white noise error model, Nyblom and Makelainen (1983) have obtained a locally most powerful test, in the sense that it maximizes the slope of the power function at the null hypothesis. The test, that is also one-sided lagrange multiplier, will be denoted as LBI/LM. A non-parametric modification, known as KPSS test, has been subsequently proposed by Kwiatkwoski et al. (1992) in order to allow for serial correlation in the error process.

This paper intends to explore the behaviour of the likelihood ratio (LR) test of stationarity. Its distribution is unknown but resampling simulation methods can be implemented to obtain an approximation. The interest in the LR test stems for at least two reasons: (i) the Neyman-Pearson lemma suggests basing the tests on the likelihood principle; (ii) the classical results on asymptotic equivalence between the likelihood ratio, the lagrange multiplier and the Wald tests breaks down for testing hypotheses on the boundary of the parameter space.

Bootstrap resampling methods for parameter estimation and testing have become increasingly popular. Horowitz (1997, 2001), Berkowitz and Kilian (2000), Li and Maddala (1996) are excellent surveys with emphasis on econometrics.

Stoffer and Wall (1991, 2003) consider parameter estimation in state space models by bootstrapping from the Kalman filter innovations. Although they do not with deal with constructing bootstrap tests, their approach seems well suited for the stationarity tests, where the underlying models have an unobserved component representation that can be easily put in state space form. Indeed Franco et al. (1999) modifies the Stoffer and Wall procedure to obtain bootstrap tests in a state space framework that closely mirrors the approach taken here. Bootstrap tests for unit roots are instead investigated in Nankervis and Savin (1996), Ferretti and Romo (1996) and Burridge and Taylor (2002), among others.

As Andrews (2000) points out, while the standard bootstrap technique may not be appropriate for estimation of parameters that lie on the boundary of the parameter space, bootstrap tests are however expected to be consistent and achieve the correct size (as the number of bootstrap replications increases) also for this case.

In this paper we propose a bootstrap version of the LR test of stationarity and we compare it with various forms, parametric, nonparametric and bootstrapped, of the LBI/LM test for the same hypothesis. The bootstrap samples are obtained from the Kalman filter innovations imposing the null hypothesis of stationarity. The main differences with respect to Franco et al. (1999) proposal are that we do not recenter the Kalman filter residuals and the treatment of initial conditions. Monte Carlo simulations show that, for the random walk plus noise or local level model, the bootstrap LR test achieves the correct size and it is rather more powerful than the LBI/LM for medium-sized deviations from the null hypothesis. For more elaborate models the bootstrap LR test appears even more attractive. In the case of a random walk trend plus AR(1) cycle, the performance of the bootstrap LR test is significantly superior, also in the neighborhood of the null hypothesis, than all other options. Finally, one distinctive advantage of the bootstrap LR test is that it allows to avoid the issue of bandwidth choice and the reliance on asymptotic critical values when the observation sample is small.

In summary the paper proceeds as follows. Section 2 reviews the LBI/LM test of stationarity and the KPSS modification for weakly dependent error processes. In section 3 we consider the simple random walk plus noise or local level model and detail the bootstrap procedure for the LR test. Section 4 generalizes the previous set up to a model made up of a random walk trend plus a general state space process without unit roots at the zero frequency. The performance of the bootstrap LR tests of Section 3 and 4 is evaluated by extensive Monte Carlo simulations. Section 5 provides illustrative examples on the use of the tests and Section 6 concludes.

2 Stationarity tests

The basic set up of stationarity tests is the structural time series model, $t = 1, \ldots, T$,

$$y_t = d_t + \mu_t + \varepsilon_t, \quad \varepsilon_t \sim NIID(0, \sigma_\varepsilon^2), \tag{1}$$

$$\mu_{t+1} = \mu_t + \eta_t, \quad \eta_t \sim NIID(0, \sigma_\eta^2). \tag{2}$$

where $d_t = \mathbf{x}_t'\beta$ is a deterministic component, with $\mathbf{x}_t$ being a $(p \times 1)$, $p < T$, fixed sequence, whose first element is fixed at unity throughout (so that (1) always contains an intercept term), with associated parameter vector β. The initial value μ_1 is assumed to be zero with no loss of generality. The model disturbances ε_t and η_t are i.i.d gaussian and mutually independent; $q^2 = \sigma_\eta^2/\sigma_\varepsilon^2$ is the so-called signal-to-noise ratio. The case of d_t being a constant term corresponds to the local level model of Harvey (1989).

A stationarity test is a test of the null hypothesis $H_0 : q = 0$ against the alternative $H_1 : q > 0$. King and Hillier (1985) show that the LBI/LM test of stationarity for the model (1)-(2) rejects when

$$LBI = T^{-2}\hat{\sigma}_{\varepsilon}^{-2}\sum_{t=1}^{T}\left(\sum_{j=1}^{t} e_j\right)^2 > \ell \tag{3}$$

where e_t, $t = 1, \ldots, T$, are the Ordinary Least Squares (OLS) residuals obtained from the regression of y_t on $\mathbf{x}_t$, $\hat{\sigma}_{\varepsilon}^2 = T^{-1}\sum_{t=1}^{T} e_t^2$, and ℓ is a suitably chosen constant. The test maximizes the slope of the power function at the origin and it is invariant to affine transformations of the data. For the local level model, i.e when $\mathbf{x}_t = 1$ for all t, the LBI/LM test was derived in Nyblom and Makelainen (1983); in that case $e_t = y_t - T^{-1}\sum_{j=1}^{T} y_j$ are the demeaned observations.

The null limiting distribution of the LBI/LM statistic (3) is non-standard and depends on d_t. Although Gaussianity has been assumed in (1)-(2), the same limiting results also hold under considerably weaker, martingale difference, conditions on $\{\varepsilon_t, \eta_t\}$; see Stock (1994,pp.2745,2794-2799) for details. Assuming that the vector $\mathbf{x}_t$ satisfies the following conditions (see Phillips and Xiao, 1998): there exists a scaling matrix $\boldsymbol{\delta}_T$ and a bounded piecewise continuous function $\mathbf{x}(r)$ such that (a) $\boldsymbol{\delta}_T \mathbf{x}_{\lfloor Tr \rfloor} \to \mathbf{x}(r)$ as $T \to \infty$ uniformly in $r \in [0,1]$, and (b) $\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr$ is positive definite, then

$$LBI \Rightarrow \int_0^1 \mathbb{B}_{\mathbf{x}}(r)^2 dr \tag{4}$$

where $\Rightarrow$ denotes weak convergence and

$$\mathbb{B}_{\mathbf{x}}(r) = \mathbb{W}(r) - \int_0^1 \mathbf{x}(r)'d\mathbb{W}(r)\left(\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr\right)^{-1}\int_0^r \mathbf{x}(s)ds, \quad r \in [0,1], \tag{5}$$

with $\mathbb{W}(r)$ being a standard Brownian motion process.

For the local level model, $\mathbf{x}_t = 1$, the limiting distribution holds with $\mathbb{B}_{\mathbf{x}}(r) \equiv \mathbb{B}_1(r) = \mathbb{W}(r) - r\mathbb{W}(1)$, $r \in [0,1]$, a standard Brownian bridge process, and the right member of (4) is a first level Cramér-von Mises distribution, denoted CvM_1. Where $\mathbf{x}_t = (1,t)'$, as in Nyblom (1986), $\mathbb{B}_{\mathbf{x}}(r) \equiv \mathbb{B}_1(r) - 6r(1-r)\int_0^1 \mathbb{B}_1(s)ds$, a standard second level Brownian bridge process. Where $\mathbf{x}_t = (1, t, ..., t^{p-1})'$, $1 \leq p < \infty$, $\mathbb{B}_{\mathbf{x}}(r)$ takes the form of a standard p-th level Brownian bridge processes (see MacNeill, 1978), and the right member of (4) is a pth level generalised Cramér-von Mises distribution, denoted CvM_p. The upper 5% fractiles of CvM_1 and CvM_2 are 0.461 and 0.149, respectively. A further generalisation that allows for structural breaks in the deterministic trend is given in Busetti and Harvey (2001).

Under the (fixed) alternative hypothesis $H_1 : q > 0$, the LBI/LM statistic (3) is $O_p(T)$, regardless of the specific form of $\mathbf{x}_t$. This result is demonstrated in, *inter alia*, Leybourne and McCabe (1994) and KPSS. The limiting distribution under the *local* alternative hypothesis $H_c : q = c/T$ is given in Tanaka (1996, p.368) and Taylor (2003). For the two cases $\mathbf{x}_t = 1$ and $\mathbf{x}_t = (1, t)'$ the limiting power function is graphed in Tanaka (1996, p.389).

Kwiatkowski et al. (1992) generalise (1)-(2) to the case where the error term ε_t is a weakly dependent process that satisfies the α-mixing conditions of Phillips and Perron (1988,p.336), with long run variance $\sigma^2_{LR} = \lim_{T\to\infty} T^{-1} E\left(\sum_{t=1}^{T} \epsilon_t\right)^2$. In such cases, they suggest replacing the OLS variance estimator $\hat{\sigma}^2$ in (3) by the Newey-West non-parametric estimator of the long run variance

$$\hat{\sigma}^2_{LR}(m) = T^{-1} \sum_{t=1}^{T} e_t^2 + 2T^{-1} \sum_{i=1}^{m} w(i, m) \sum_{t=i+1}^{T} e_t e_{t-i}, \tag{6}$$

where $w(i, m) = 1 - i/(m+1)$, $i = 1, \ldots, m$, m the lag-truncation parameter. The rate conditions $m \to \infty$ and $m = o(T^{1/2})$ as $T \to \infty$ are sufficient to ensure that $\hat{\sigma}^2_{LR} \to^p \sigma^2_{LR}$ under both the null hypothesis and the local alternative that the long run variance of η_t is $\sigma^2_{LR} c^2/T^2$; see Stock (1994,p.2797-99). The resulting non-parametrically modified test statistic, denoted as

$$KPSS(m) = \frac{\sum_{t=1}^{T} \left(\sum_{j=1}^{t} e_j\right)^2}{T^2 \hat{\sigma}^2_{LR}(m)}, \tag{7}$$

is asymptotically pivotal and mantains the same Cramèr-von Mises limiting distribution (4). This is denoted as KPSS test. It turns out that this non-parametric test is still consistent but at the slower rate of $O_p(T/m)$.

An alternative way to account for serial correlation in the error term ε_t is to fit a parametric time series model. This is the approach taken by Harvey and Streibel (1997) and Leybourne and Mc-Cabe (1994), where it is showed that the parametric test is $O_p(T)$ under the alternative hypothesis, exactly as for the LBI/LM test (3) with white noise errors. While Leybourne and Mc-Cabe (1994) estimate an autoregressive model for the error term ε_t, Harvey and Streibel (1997) propose to fit a more general state space model to the data and construct the statistic (3) by replacing the OLS residuals e_t by the Kalman filter innovations. The innovations are computed by estimating the model parameters under the alternative hypothesis and then running the Kalman filter with the zero variance restriction of stationarity; the filter is initialized using the smoothed estimate of the initial condition. It is demonstrated that the limiting null distribution still mantains the Cramer-von Mises representation (4).

3 The Bootstrap LR test for the local level model

The local level model, or random walk plus noise, corresponds to equations (1)-(2) with $d_t = \beta_0$, a constant term interpretable as the initial value of the random walk component. This model is thoroughly analysed in Harvey (1989) and Durbin and Koopman (2001, chapter 2), inter alia.

The time domain likelihood function is obtained by the prediction error decomposition performed by the Kalman filter. Let $m_{t+1} = E\left(\mu_{t+1} \middle| Y_t\right)$, $P_t = Var\left(\mu_{t+1} \middle| Y_t\right)$, where $Y_t = \{y_1, y_2, ..., y_{t-1}, y_t\}$ is the set of all observations up to time t. Assuming a diffuse initial condition for the state variable, the Kalman filter is given by the following equations,

$$
\begin{aligned}
v_t &= y_t - m_t, & (8)\\
F_t &= P_t + \sigma_\varepsilon^2, & (9)\\
K_t &= P_t / F_t, & (10)\\
m_{t+1} &= m_t + K_t v_t, & (11)\\
P_{t+1} &= P_t(1 - K_t) + \sigma_\eta^2, & (12)
\end{aligned}
$$

with $m_1 = 0$ and $P_1 = \kappa \rightarrow \infty$ (note that $m_2 \rightarrow y_1$, $P_2 \rightarrow \sigma_\varepsilon^2(1 + q^2)$ and thus the diffuse Kalman filter is equivalent to treating y_1 as fixed, i.e. with the proper prior that $\mu_1 \sim N(y_1, \sigma_\varepsilon^2)$); v_t are the Kalman filter innovations, that have zero mean and variance equal to F_t; K_t is the so-called Kalman gain.

The (diffuse) log-likelihood for the local level model is

$$
\log \ell\left(Y_T; \sigma_\varepsilon^2, \sigma_\eta^2\right) = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \sum_{t=2}^{T} \left(\log F_t + \frac{v_t^2}{F_t} \right), \qquad (13)
$$

cf. Harvey (1989), where v_t, F_t are obtained from (8)-(12). The likelihood can be easily maximized by some numerical optimization algorithm, perhaps concentrating out the variance of the noise σ_ε^2. In this paper we use the Ssfpack 2.3 set of routines for state-space models of Koopman et al. (1999); in particular, (13) is computed using the *Ssflik* function and maximization is carried out by the BFGS algorithm implemented in Ox.

The restricted likelihood, $\log \ell \left(Y_T;\ \sigma_\varepsilon^2, 0\right)$, is obtained after setting $\sigma_\eta^2 = 0$ in the Kalman filter recursions. As above, it can be computed with the Ssfpack function *Ssflik* on restricting to zero the variance of the random walk component.

The likelihood ratio test statistic for $H_0 : q = 0$ against $H_1 : q > 0$ is therefore given by

$$LR = -2\left(\log \ell\left(Y_T;\ \widehat{\sigma}_\varepsilon^2, 0\right) - \log \ell\left(Y_T;\ \widetilde{\sigma}_\varepsilon^2, \widetilde{\sigma}_\eta^2\right)\right), \tag{14}$$

where the $\widehat{}$ denote maximization under the null and $\widetilde{}$ maximization under the alternative.

The distribution of the LR statistic is unknown. A bootstrap version of the LR test, however, can be obtained by the following procedure.

(1) Estimate the local level model by maximum likelihood, e.g. using the Ssfpack set of routines, and let $\widetilde{\mu}_1$ be the smoothed estimate of the initial condition of the non-stationary state variable (cf. Koopman and Durbin, 2001, for the smoothing recursions). Run the Kalman filter recursions (8)-(12) starting from the estimated initial condition, i.e. setting $m_1 = \widetilde{\mu}_1$, $P_1 = 0$, and store the Kalman filter innovations v_t and their variances F_t.

(2) Generate B bootstrap samples by random sampling, with replacement, from the $T-1$ standardized Kalman filter innovations $\left\{v_2/\sqrt{F_2}, v_3/\sqrt{F_3}, ..., v_T/\sqrt{F_T}\right\}$. Denote the bootstrap standardized innovations by

$$\overline{v}_t^b, \qquad b = 1, ..., B, \quad t = 2, ..., T.$$

(3) From the bootstrap innovations construct the bootstrap data using the Kalman filter recursions under the null hypothesis of stationarity, i.e. setting $\sigma_\eta^2 = 0$, $(t = 2, ..., T,\ \ b = 1, ..., B)$

$$y_t^b = m_t^b + F_t^{-\frac{1}{2}} \overline{v}_t^b, \tag{15}$$

$$m_{t+1}^b = m_t^b + K_t F_t^{\frac{1}{2}} \overline{v}_t^b, \tag{16}$$

$$y_1^b = \widetilde{\mu}_1. \tag{17}$$

(4) For each bootstrap replication $b = 1, ..., B$ estimate the local level model using the bootstrap dataset $Y_T^b = \left\{y_1^b, y_2^b, ..., y_T^b\right\}$ and compute the bootstrap LR statistic

$$LR^b = -2\left(\log \ell\left(Y_T^b;\ \widehat{\sigma}_\varepsilon^2, 0\right) - \log \ell\left(Y_T^b;\ \widetilde{\sigma}_\varepsilon^2, \widetilde{\sigma}_\eta^2\right)\right), \tag{18}$$

where $\ell\left(Y_T^b;\ .,.\right)$ is computed as in (13) but using the bootstrap dataset Y_T^b.

(5) Let κ_α^* be the upper α-fractile from the distribution of the bootstrap LR statistic (18). Then the test rejects the null hypothesis $H_0 : q = 0$ at the α significance level if

$$LR > \kappa_\alpha^*.$$

Our bootstrap procedure differs from Stoffer and Wall (1991) in that in our case the bootstrap samples are generated under the null hypothesis of stationarity. This guarantees correct size and good power of the test. This is also analogous to what happens for unit root tests, where the boostrap samples need to be generated imposing the unit root. Our test is very close to that of Franco et al. (1999), the main differences being in the non recentering of the Kalman filter innovations and in the treatment of initial conditions.

The performance of the bootstrap LR test is evaluated by a set of Monte Carlo simulation experiments. We have generated $N = 10,000$ replications of the local level data generating process for a range of values of the signal-to-noise ratio $q^2 = c^2/T^2$, with $c = 0, 2.5, 5, 10, 25$; $c = 0$ corresponds to the null hypothesis, while $c > 0$ allows to evaluate the power under the (local) alternative. The variance of the noise is set to $\sigma_\varepsilon^2 = 1$. We presents results for $T = 25, 50, 100$. The number of bootstrap replications is set to $B = 1000$.

Table 1 contains the percentage rejection frequencies of the standard LBI/LM test, the bootstrap LR test and also of the boostrapped version of the LBI/LM test, the latter two in the columns headed LR* and LBI* respectively. The LBI* test has been obtained by the boostrap distribution of the LBI/LM statistic (3) computed using the boostrap dataset $Y_T^b = \left\{y_1^b, y_2^b, ..., y_T^b\right\}, b = 1, ..., B$.

Consider first the results for $T = 25$. Notwithstanding we are using the asymptotic critical values (0.347 for $\alpha = 10\%$ and 0.461 for $\alpha = 5\%$) the size of the LBI/LM test is very close to the nominal one while the power closely corresponds to the limiting power function as computed by Tanaka (1996, p.???). Overall the performance of the boostrap LR test seems superior: though slightly oversized, it displays significant power gains for $c \geq 10$, while power is comparable with that of the LBI test for lower c. Note that, as the LBI test maximizes the slope of the power function in the neighborhood of the null hypothesis, no gain was expected for low values of c. It is also interesting to see that the boostrap distribution of the LBI* test closely replicates the true limiting distribution, which can be seen as an indirect confirmation of the correctness of the bootstrap procedure (1)-(5).

The results for the bigger sample sizes $T = 50$ and 100 are very similar to those for $T = 25$; this is clearly seen as we are presenting simulated rejection frequencies under the local alternative hypothesis. Note however that if we keep the size of the alternative hypothesis $\sigma_\eta^2 = c^2/T^2$ fixed the power increases with the sample size and so the tests are consistent; e.g. for $\sigma_\eta^2 = 0.01$ the power of the LR* test is 20.43, 38.42, 72.37 for $T = 25, 50, 100$ respectively.

In summary, for the local level model the bootstrap LR test does as well as the LBI test, which is the best theoretical option, near the null hypothesis but it presents significant power gains for medium sized deviations from the null.

4 The Bootstrap LR test for a general state space model

In this section we consider the following generalization of the local level model (1)-(2),

$$\begin{aligned} y_t &= d_t + \mu_t + \psi_t, & (19)\\ \mu_{t+1} &= \mu_t + \eta_t, \quad \eta_t \sim NIID(0, \sigma_\eta^2), & (20)\\ \psi_t &= \underline{Z}_t \underline{\alpha}_t + \varepsilon_t, \quad \varepsilon_t \sim NIID(0, \sigma_\varepsilon^2), & (21)\\ \underline{\alpha}_{t+1} &= \underline{\Phi \alpha}_t + \underline{u}_t, \quad \underline{u}_t \sim NIID(0, \underline{Q}), & (22) \end{aligned}$$

where d_t is a deterministic component, μ_t is a random walk trend, ψ_t is a random component which follows the time-invariant state space model (21)-(22), where $\underline{\alpha}_t$ is a $p \times 1$ state variable and Z_t is deterministic $1 \times p$ vector. The disturbances η_t, ε_t, u_t are mutually independent. The initial conditions are specified as follows: $\underline{\alpha}_1 \sim N(0, \underline{P}_1)$, $\mu_1 \sim N(0, \kappa)$, $\kappa \to \infty$. We assume that the eigenvalues of $\underline{\Phi}$ are inside the unit circle to ensure that there are no unit roots at zero frequency in ψ_t. The representation of ψ_t is quite general, e.g. it includes any stationary and invertible ARMA process.

Clearly y_t has itself a state space representation

$$\begin{aligned} y_t &= d_t + Z_t \alpha_t + \varepsilon_t, \quad \varepsilon_t \sim NIID(0, \sigma_\varepsilon^2), & (23)\\ \alpha_{t+1} &= \Phi \alpha_t + u_t, \quad u_t \sim NIID(0, Q), & (24) \end{aligned}$$

where $Z_t = (1, \underline{Z}_t')'$, $\alpha_t = (\mu_t, \underline{\alpha}_t')'$, $u_t = (\eta_t, u_t')'$, $\Phi = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\Phi} \end{pmatrix}$, $Q = \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \underline{Q} \end{pmatrix}$, and with the initial conditions modified accordingly. Letting $a_{t+1} = E(\alpha_{t+1} | Y_t)$, $P_t = Var(\alpha_{t+1} | Y_t)$, where $Y_t = \{y_1, y_2, ..., y_{t-1}, y_t\}$ is the set of all observations up to time t, the Kalman filter is given by the following equations,

$$
\begin{aligned}
v_t &= y_t - d_t - Z_t \alpha_t, & (25)\\
F_t &= Z_t P_t Z_t' + \sigma_\varepsilon^2, & (26)\\
K_t &= \Phi P_t Z_t' F_t^{-1}, & (27)\\
a_{t+1} &= \Phi a_t + K_t v_t, & (28)\\
P_{t+1} &= \Phi P_t (\Phi - K_t Z_t) + Q, & (29)
\end{aligned}
$$

see e.g. Durbin and Koopman (2001). As regards the initial conditions, we set the first element of a_1 equal to zero, the top-left element of P_1 equal to $\kappa \to \infty$, the last p elements of a_1 equal to their unconditional mean of zero and the lower-right $p \times p$ submatrix of P_1 equal to the unconditional variance of $\underline{\alpha}_t$, $vec\,(var\,(\underline{\alpha}_t)) = (I - \underline{\Phi} \otimes \underline{\Phi})^{-1}\, vec\left(\underline{Q}\right)$.

Assume first that the deterministic component d_t is known and let $\theta = \left(\underline{\theta}', \sigma_\eta^2\right)'$ be the unknown parameters in (23)-(24). Then the log-likelihood of the model is

$$
\log \ell \left(Y_T; \underline{\theta}, \sigma_\eta^2\right) = -\frac{T}{2} \log\left(2\pi\right) - \frac{1}{2} \sum_{t=2}^{T} \left(\log F_t + \frac{v_t^2}{F_t}\right), \qquad (30)
$$

i.e. it has the same form as (13) but with v_t, F_t obtained from the recursions (25)-(29). The restricted likelihood, $\log \ell\left(Y_T; \underline{\theta}, 0\right)$, is obtained after setting $\sigma_\eta^2 = 0$ in the Kalman filter recursions above. The likelihood ratio test statistic for testing $H_0 : \sigma_\eta^2 = 0$ against $H_1 : \sigma_\eta^2 > 0$ is then given by

$$
LR = -2\left(\log \ell\left(Y_T; \widehat{\underline{\theta}}, 0\right) - \log \ell\left(Y_T; \widetilde{\underline{\theta}}, \widetilde{\sigma}_\eta^2\right)\right), \qquad (31)
$$

where the $\widehat{\ }$ denote maximization under the null and $\widetilde{\ }$ maximization under the alternative.

If $d_t = x_t'\beta$, where β are uknown parameters (e.g. the coefficients of a trend function), one would evaluate the likelihood function (30) with v_t in (25) replaced by

$$
v_t = y_t - x_t'\beta_{GLS} - Z_t \alpha_t,
$$

where β_{GLS} is the Generalised Least Square estimator of β. This can be obtained by an Ordinary Least Squares regression of the transformed observations $y_{t,GLS}$ on $x_{t,GLS}$, where the GLS transform is given by the same Kalman filter (25)-(29); see Harvey (1989, chapter 3). Alternatively one could augment α_t, Z_t by the deterministic components that correspond to, respectively, β and x_t; see Harvey (1989).

The bootstrap version of the LR test can then be obtained by the following procedure, which is the direct generalization of that in section 3.

(1) Estimate the model (23)-(24) by maximum likelihood, e.g. by the Ssfpack set of routines, and let $\widetilde{\mu}_1$ be the smoothed estimate of the initial condition in first element of the state vector (cf. Koopman and Durbin, 2001, for the smoothing recursions). Run the Kalman filter recursions (25)-(29) starting from the estimated initial condition, i.e. setting the first element of a_1 equal to $\widetilde{\mu}_1$ and the top-left element of P_1 equal to 0; store the Kalman filter innovations v_t and their variances F_t.

(2) Generate B bootstrap samples by random sampling, with replacement, from the $T-1$ standardized Kalman filter innovations $\left\{v_2/\sqrt{F_2}, v_3/\sqrt{F_3}, ..., v_T/\sqrt{F_T}\right\}$. Denote the bootstrap standardized innovations by

$$\overline{v}_t^b, \qquad b = 1, ..., B, \quad t = 2, ..., T.$$

(3) From the bootstrap innovations construct the bootstrap data using the Kalman filter recursions under the null hypothesis of stationarity, i.e. setting $\sigma_\eta^2 = 0$, $(t = 2, ..., T, \;\; b = 1, ..., B)$

$$\begin{aligned} y_t^b &= d_t + Z_t a_t^b + F_t^{-\frac{1}{2}} \overline{v}_t^b, && (32) \\ a_{t+1}^b &= \Phi a_t^b + K_t F_t^{\frac{1}{2}} \overline{v}_t^b, && (33) \\ y_1^b &= \widetilde{\mu}_1. && (34) \end{aligned}$$

(4) For each bootstrap replication $b = 1, ..., B$ estimate the local level model using the bootstrap dataset $Y_T^b = \left\{y_1^b, y_2^b, ..., y_T^b\right\}$ and compute the bootstrap LR statistic

$$LR^b = -2\left(\log \ell\left(Y_T^b; \widehat{\underline{\theta}}, 0\right) - \log \ell\left(Y_T^b; \widetilde{\underline{\theta}}, \widetilde{\sigma}_\eta^2\right)\right), \qquad (35)$$

where $\ell\left(Y_T^b; ., .\right)$ is computed as in (30) but using the bootstrap dataset Y_T^b.

(5) Let κ_α^* be the upper α-fractile from the distribution of the bootstrap LR statistic (35). Then the test rejects the null hypothesis $H_0 : \sigma_\eta^2 = 0$ at the α significance level if

$$LR > \kappa_\alpha^*.$$

Notice that the state space model for ψ_t can also be extended to include components with unit roots at any non-zero frequency, e.g. it can account for seasonal unit roots. Only few modifications to the procedure described above need to be done. In particular, there will be more than one element in the state vector with a diffuse prior and the summation term in the diffuse log-likelihood (30) would start at some $t > 2$ (depending on the type of non-stationary components included); see Harvey (1989) for detailed explanations on how to handle the initial conditions in this case. A further generalization to the above state space model for ψ_t would be to allow the system matrices $\underline{\Phi}$, $\underline{Q}$ to be time-varying; obvious modifications to the Kalman filter recursions would then be required, see e.g. Durbin and Koopman (2001).

The performance of the bootstrap LR test in evaluated in the context of a simple trend-cycle model, where the trend μ_t is a random walk and the cycle ψ_t is an AR(1) process, i.e.

$$\begin{aligned} y_t &= \beta_0 + \mu_t + \psi_t, & (36) \\ \mu_{t+1} &= \mu_t + \eta_t, \quad \eta_t \sim NIID(0, \sigma_\eta^2), & (37) \\ \psi_{t+1} &= \rho\psi_t + u_t, \quad u_t \sim NIID(0, \sigma_u^2). & (38) \end{aligned}$$

Let $q^2 = \sigma_\eta^2/\sigma_u^2$ be the ratio of the variances of the trend and the cycle disturbances. We have generated $N = 5000$ Monte Carlo replications of the data generating process (36)-(38) under the local alternative hypothesis $q^2 = c^2/T^2$, for $c = 0$, 2.5, 5, 10, 25, 50, with σ_u^2 set equal to 1. The bootstrap LR test has been computed as described above with $B = 500$ replications. For the bootstrap samples the initial values of the parameters in the likelihood maximization are set equal to the ML estimates $\widetilde{\sigma}_u$, $\widetilde{\rho}$ but with $\sigma_\eta^2 = 0$; although, in principles, one would like that the numerical optimization algorithm achieves the same maximum irrespective of the initial conditions, in this model the optimization results for the boostrap samples are somehow sensitive to the initial conditions (while this was not the case for the local level model of the previous section). This choice for the initial parameters permits to have good size properties for the bootstrap LR test.

Tables 2-5 present percentage rejection frequencies for the bootstrap KPSS test, the parametric LBI/LM test of Harvey and Streibel (1997), the bootstrap LR test and a bootstrap version of the LBI/LM test, the latter obtained from the boostrap distribution of the statistic (3), computed without correction for serial correlation (i.e. with $e_t = y_t - \overline{y}$). Results are presented for sample sizes of $T = 25$, 50, 100.

In all the tables below the $KPSS(m)$ statistics (7) are computed across four values of the lag truncation parameter $m = 0$, $m(4)$, $m(8)$ and $m(12)$, where, following Kwiatowski et al. (1992), $m(x)$ is given by the formula

$$m(x) = integer\left(x\,(T/100)^{1/4}\right); \tag{39}$$

note that $KPSS(0)$ corrsponds to the LBI/LM statistic (3). The choice of lag truncation parameter reflects a trade-off between size and power; i.e., in general higher m corresponds to better size properties but lower power for the test. Simulation experiments in Kwiatowski et al. (1992) provide some evidence for using $m(4)$ for a moderate sample size and $m(12)$ for larger samples.

Consider first Table 2, which contains the results for $\rho = 0.5$ when the nominal significance level is $\alpha = 10\%$. While the KPSS test is strongly oversized for $m(x) \leq m(4)$, the parametric LBI test appears rather undersized in the small samples of $T = 25$ and $T = 50$ observations (the limiting null distribution, a Cramer-von Mises with one degree of freedom, does not appear to provide an adequate approximation in this context). Overall, the size properties of the bootstrap tests are very good: the LR test suffers of some undersizing for $T = 25$, while the size of the bootstrap LBI statistic is very close to the nominal 10% for all sample sizes. More interesting are the power properties of the tests. Consider first the small sample of $T = 25$ observations. Near the null hypothesis, e.g. when $c \leq 5$, the parametric LBI test, being undersized, also suffers from low power, while the $KPSS(m(12))$ and the boostrap LBI test behave very similarly. On the other hand, despite being undersized, huge gains in terms of power would come from using the boostrap LR test: e.g. for $c = 2.5$ (or $\sigma_\eta = 0.1$) the power of LR* is 29% against just around 12% for the $KPSS(m(x))$, $x = 8, 12$, and 7% of the parametric LBI test. For medium dinstance from the null hypothesis the LR* test becomes relatively less attractive with respect to the parametric LBI and the KPSS, but it is still more powerful. It is interesting to notice that for this sample size of $T = 25$ observations the power function of LR* appears to reach a maximum at $c = 25$ (or $\sigma_\eta = 1$).

Similar considerations apply for the bigger sample sizes of $T = 50$, 100 observations. The boostrap LR* test has a clear power advantage over the other options for small deviations from the null hypothesis. It is also interesting to notice that, in contrast to the local level model, the simulated small sample power of LR* does not provide a good approximation to the limiting power function: for $c > 10$ the power figures are very different for the three cases of $T = 25, 50, 100$. Despite the fact that for a given sample size the power of LR* is not monotonically increasing with σ_η (it starts very steeply and then reaches a maximum at a certain value of the signal-to-noise ratio q^2), the test certainly appears to be consistent. Consistency requires that, for a given non-zero signal-to-noise ratio, the probability of rejecting the null hypothesis increases as the sample size grows. In the case of Table 2 when, for example, $\sigma_\eta = 0.1$ the simulated probability of rejecting the null hypothesis of stationarity is 29%, 41%, 54% when $T = 25$, 50, 100 respectively; when $\sigma_\eta = 0.5$ the simulated power is 76%, 89% for $T = 50$, 100 respectively.

Table 3 contains the corresponding results of Table 2 but for tests run at the nominal significance level of $\alpha = 5\%$. The most important differences concern the size of the KPSS test which now is much more sensitive to the number of observations: for $T = 25$ the appropriate bandwidth parameter would be $m(4)$ while for $T = 50$, 100 one would need to choose $m(8)$, $m(12)$

respectively. In this case the choice of an appropriate value of the bandwidth parameter has important consequences since the power of the KPSS test turns out to be very low for higher values of m. The parametric LBI test is still undersized, while the size bootstrap LR^* test seems more reliable than in the previous case. The power properties of the tests remain qualitatively as described for Table 2.

Table 4 presents the results of the tests run at the nominal significant level of $\alpha = 10\%$ when the autoregressive parameter is $\rho = 0.8$. The high serial correlation in the cyclical component makes the KPSS test rather unreliable: its size is well above the nominal 10% and thus there is a substantial risk of spuriously rejecting the null hypothesis. On the other hand, the boostrap LR* test seems to work well and it is overall preferable to the both the parametric and bootstrap LBI tests.

Finally, Table 5 contains the corresponding results when $\rho = -0.5$. The size of the LR^* test appears very close to the nominal 10% also for a sample of $T = 25$ observation; on the other hand the KPSS test suffers from undersizing (as opposed to the oversizing that typically occurs when $\rho > 0$). As concerns power, the relative ranking of LR^*, parametric LBI and KPSS is mantained as in the previous experiments. For $T = 100$ the performance of the parametric LBI test is very close to that of LR^*.

5 Empirical illustrations

Figure 1 shows the logarithm of the consumption expenditure in tobacco products in UK, for the period 1980Q1-1996Q1. The trend is clearly downward sloping and there is a marked seasonal component. A stationarity test applied to the first differences can be used to assess whether the slope of the trend is fixed or (non-stationary) stochastic. A random walk trend plus noise plus seasonal component provides a satisfactory fit to the first differences of this series, with an estimate of the signal-to-noise ratio equal to 0.18. The LBI statistic applied to first differenced (and taking into account of the seasonal component) takes the value of 0.428, which would not reject the null hypothesis of stationarity at the 5% significance level. On the contrary, the likelihood rato statistic is equal to 3.475 which corresponds to a strong rejection of the null hypothesis, the 5% an 1% bootstrap critical values being 1.303 and 2.952 respectively.

Figure 2 shows the US inflation rate for the period 1970Q1-1990Q4, measured as the first difference of the logarithm of the Consumer's Price Index. For this series Busetti and Taylor (2003) found evidence for a switch from a I(1) to I(0) at the time of the early nineties. Here we are concerned in testing whether the US inflation rate is an I(1) process for the period of the seventies up to the nineties. The KPSS statistics take values of 0.357, 0.241, 0.213 when computed with lag truncation parameters $m(4)$, $m(8)$, $m(12)$ respectively. As the asymptotic 10% and 5% critical values are 0.347 and 0.461, the KPSS test does not provide much evidence for a unit root. A simple random walk trend plus AR(1) model can be seen as a first approximation to the time series behavior of US inflation: the signal-to-noise ratio is estimated equal to 0.76, while the AR coefficient is 0.22; the Box-Ljung test on the residuals has a p-value of 0.03. Based on this model, the likelihood ratio statistic takes value of 1.54 and thus provides evidence for a unit root as the bootstrap 10%, 5%, 1% critical values have been computed as 0.641, 1.445, 3.995 respectively. In this case even more evidence for non-stationarity is provided by the parametric Harvey-Streibel test, the statistic being equal to 0.905, higher than the asymptotic 1% critical values of 0.748.

6 Conclusions

The paper has proposed a bootstrap version of the likelihood ratio test of stationarity for a general state space model with a random walk trend component. The boostrap samples are generated using the Kalman filter innovations and imposing the null hypothesis of stationarity. Monte Carlo simulation experiments demonstrate that the bootstrap LR test works well and seems overall preferable to both the KPSS test and to parametrically modified versions of the LBI statistic. For the simple local level model there are power gains from using the bootstrap LR test for medium-sized deviations from the null hypothesis. These gains appear more significant for more elaborate models, where the finite sample power of the bootstrap LR test near the null hypothesis appears much higher than the power of any other test considered here. The implementation of the bootstrap test is quite simple once a state space model with a random walk trend has been fitted to the data. In addition, our bootstrap procedure can also be straightforwardly extended to the cases of testing against an integrated random walk (or smooth trend) and testing against nonstationary seasonal components.

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		T=25			T=50			T=100		
		LBI/LM	LBI/BOOT	LR/BOOT	LBI/LM	LBI/BOOT	LR/BOOT	LBI/LM	LBI/BOOT	LR/BOOT
	c=0	10.46	10.75	10.99	9.92	10.37	10.53	10.91	10.99	10.70
	c=2.5	20.03	20.61	20.47	19.60	20.14	20.09	20.32	20.61	20.47
α=10%	c=5	37.79	38.58	39.23	38.21	38.42	39.89	38.60	38.88	40.31
	c=10	61.21	61.65	68.44	65.68	65.78	71.42	66.91	67.24	72.37
	c=25	83.40	83.80	92.35	89.47	89.54	95.66	91.99	91.98	96.72
	c=0	5.07	5.62	5.73	4.98	5.28	5.39	5.46	5.73	5.53
	c=2.5	12.03	13.15	12.51	12.01	12.69	12.47	12.62	13.08	13.08
α=5%	c=5	27.40	28.87	29.32	28.93	29.43	30.49	29.62	30.17	31.40
	c=10	51.57	52.80	60.14	56.44	57.20	63.95	58.71	59.00	65.20
	c=25	75.61	76.93	89.74	83.72	84.25	94.12	87.93	88.17	95.59

Table 1: Rejection frequencies (x100) for the LBI/LM, the bootstrap LBI and the bootstrap LR test of stationarity. The data generating process is the local level model; N=10,000, B=1000.

			KPSS(0)	KPSS(m(4))	KPSS(m(8))	KPSS(m(12))	LBI/PARAM	LBI/BOOT	LR/BOOT
	c=0	(σ_η=0)	41.54	16.46	11.10	10.56	3.82	9.50	6.20
	c=2.5	(σ_η=0.1)	46.12	20.74	14.76	13.88	7.50	12.38	32.04
T=25	c=5	(σ_η=0.2)	54.84	27.98	21.04	19.24	12.82	17.30	39.52
	c=10	(σ_η=0.4)	69.14	42.56	33.16	28.78	23.04	26.76	52.18
	c=25	(σ_η=1.0)	85.06	58.80	46.50	40.14	40.24	31.06	60.26
	c=50	(σ_η=2.0)	89.56	62.90	49.82	41.96	54.12	29.22	55.06
	c=0	(σ_η=0)	45.38	18.44	13.18	11.94	5.20	10.32	9.16
	c=2.5	(σ_η=0.05)	49.14	20.80	14.16	13.10	8.64	10.96	26.68
T=50	c=5	(σ_η=0.1)	57.72	30.08	23.14	20.58	16.52	17.62	37.90
	c=10	(σ_η=0.2)	72.56	47.06	37.40	33.20	31.12	31.76	55.84
	c=25	(σ_η=0.5)	91.06	69.00	56.60	50.96	47.70	48.88	74.80
	c=50	(σ_η=1.0)	96.46	76.32	62.68	55.88	52.48	49.36	73.36
	c=0	(σ_η=0)	48.74	16.72	13.08	11.92	6.02	10.02	9.72
	c=2.5	(σ_η=0.025)	51.14	19.34	15.68	14.56	11.86	11.78	20.98
T=100	c=5	(σ_η=0.05)	60.44	29.04	24.14	21.80	19.56	19.84	31.04
	c=10	(σ_η=0.1)	74.70	45.82	39.14	35.90	36.28	34.72	50.20
	c=25	(σ_η=0.25)	92.82	72.82	63.26	57.68	66.12	60.18	80.84
	c=50	(σ_η=0.5)	98.42	83.56	71.72	64.84	70.20	63.92	89.44

Table 2: Rejection frequencies (x100) for the KPSS, the parametric LBI, the bootstrap LBI and the bootstrap LR tests of stationarity. The data generating process is the random walk plus AR(1) model with ρ=0.5; N=5000, B=500. The significance level is α=10%.

			KPSS(0)	KPSS(m(4))	KPSS(m(8))	KPSS(m(12))	LBI/PARAM	LBI/BOOT	LR/BOOT
	c=0	(σ_η=0)	29.10	4.66	0.26	0.00	0.96	3.26	2.86
	c=2.5	(σ_η=0.1)	33.74	7.74	0.44	0.06	5.14	5.08	28.62
T=25	c=5	(σ_η=0.2)	42.50	13.24	1.22	0.22	9.40	7.74	36.60
	c=10	(σ_η=0.4)	58.60	26.26	3.90	0.24	16.84	12.50	50.10
	c=25	(σ_η=1.0)	76.94	42.86	9.26	0.28	30.74	15.02	57.68
	c=50	(σ_η=2.0)	83.16	47.62	11.44	0.50	43.22	12.68	50.04
	c=0	(σ_η=0)	32.52	9.80	4.40	2.24	2.06	4.66	4.52
	c=2.5	(σ_η=0.05)	35.60	10.92	4.70	2.22	6.74	5.04	21.00
T=50	c=5	(σ_η=0.1)	46.22	19.14	9.24	4.46	12.56	9.62	32.08
	c=10	(σ_η=0.2)	63.16	35.02	20.80	11.84	25.60	19.68	50.66
	c=25	(σ_η=0.5)	85.56	58.68	41.38	26.36	39.84	34.18	73.44
	c=50	(σ_η=1.0)	93.48	67.24	48.08	32.60	43.04	33.88	72.50
	c=0	(σ_η=0)	35.32	9.00	5.80	4.50	3.84	4.96	5.00
	c=2.5	(σ_η=0.025)	38.26	11.14	7.50	5.62	9.34	6.24	12.18
T=100	c=5	(σ_η=0.05)	48.14	19.32	13.74	10.28	16.02	11.78	21.40
	c=10	(σ_η=0.1)	65.14	34.70	27.00	21.50	32.20	24.58	40.54
	c=25	(σ_η=0.25)	88.88	63.24	52.64	44.62	60.46	49.70	76.68
	c=50	(σ_η=0.5)	96.58	74.80	62.08	52.70	62.46	52.84	88.18

Table 3: Rejection frequencies (x100) for the KPSS, the parametric LBI, the bootstrap LBI and the bootstrap LR tests of stationarity. The data generating process is the random walk plus AR(1) model with ρ=0.5; N=5000, B=500. The significance level is α=5%.

			KPSS(0)	KPSS(m(4))	KPSS(m(8))	KPSS(m(12))	LBI/PARAM	LBI/BOOT	LR/BOOT
	c=0	(σ_η=0)	72.24	31.52	18.26	14.12	5.98	15.46	4.36
	c=2.5	(σ_η=0.1)	73.96	34.42	20.16	16.02	8.00	17.38	26.76
T=25	c=5	(σ_η=0.2)	75.62	36.68	22.84	17.78	10.86	17.90	28.90
	c=10	(σ_η=0.4)	78.18	43.16	28.84	23.16	17.96	20.86	33.78
	c=25	(σ_η=1.0)	86.80	56.44	43.44	36.36	38.96	24.44	42.90
	c=50	(σ_η=2.0)	89.64	62.14	48.84	40.92	54.12	24.60	46.22
	c=0	(σ_η=0)	83.90	38.86	23.10	18.26	5.18	15.04	5.70
	c=2.5	(σ_η=0.05)	83.62	38.72	22.76	17.42	3.58	13.08	24.96
T=50	c=5	(σ_η=0.1)	84.60	42.04	25.90	20.26	6.10	15.76	29.10
	c=10	(σ_η=0.2)	87.40	47.72	32.60	26.32	10.90	20.08	36.46
	c=25	(σ_η=0.5)	93.48	64.12	49.92	42.84	27.00	33.12	52.00
	c=50	(σ_η=1.0)	96.58	74.14	58.84	52.14	42.94	39.28	57.64
	c=0	(σ_η=0)	88.16	37.56	23.12	17.94	5.48	12.14	8.18
	c=2.5	(σ_η=0.025)	89.06	35.98	23.52	18.82	3.86	11.92	23.50
T=100	c=5	(σ_η=0.05)	89.26	39.76	26.52	21.10	7.42	13.98	28.96
	c=10	(σ_η=0.1)	91.58	47.30	32.74	26.76	12.34	17.96	36.40
	c=25	(σ_η=0.25)	95.78	64.90	51.36	44.46	28.08	33.24	55.84
	c=50	(σ_η=0.5)	98.36	78.04	65.02	57.92	42.10	46.82	67.66

Table 4: Rejection frequencies (x100) for the KPSS, the parametric LBI, the bootstrap LBI and the bootstrap LR tests of stationarity. The data generating process is the random walk plus AR(1) model with ρ=0.8; N=5000, B=500. The significance level is α=10%.

			KPSS(0)	KPSS(m(4))	KPSS(m(8))	KPSS(m(12))	LBI/PARAM	LBI/BOOT	LR/BOOT
	c=0	(σ_η=0)	0.62	3.94	5.02	6.40	5.06	5.64	10.98
	c=2.5	(σ_η=0.1)	6.00	16.20	15.20	15.04	26.74	20.06	44.24
T=25	c=5	(σ_η=0.2)	24.90	35.84	29.88	26.18	49.14	41.90	67.16
	c=10	(σ_η=0.4)	52.84	52.32	42.80	37.32	67.84	59.84	84.00
	c=25	(σ_η=1.0)	80.28	62.52	50.42	43.14	67.48	58.24	90.96
	c=50	(σ_η=2.0)	87.84	64.08	50.48	43.16	63.22	42.56	81.16
	c=0	(σ_η=0)	0.56	4.98	6.28	6.86	7.78	8.20	10.80
	c=2.5	(σ_η=0.05)	5.28	17.70	17.82	17.64	28.22	24.16	35.20
T=50	c=5	(σ_η=0.1)	25.48	43.36	40.00	36.90	55.66	51.08	63.94
	c=10	(σ_η=0.2)	55.80	62.14	54.36	49.46	80.34	71.92	86.02
	c=25	(σ_η=0.5)	86.02	75.74	62.62	56.24	89.66	82.02	98.04
	c=50	(σ_η=1.0)	95.50	78.74	64.90	57.92	76.86	71.00	98.78
	c=0	(σ_η=0)	0.24	5.26	6.58	7.28	12.26	9.00	10.08
	c=2.5	(σ_η=0.025)	5.06	20.64	22.30	22.10	31.60	28.00	31.46
T=100	c=5	(σ_η=0.05)	24.06	44.32	43.30	41.44	58.42	53.46	59.24
	c=10	(σ_η=0.1)	55.34	67.56	62.66	57.90	84.18	77.40	85.46
	c=25	(σ_η=0.25)	88.60	84.06	73.38	66.62	97.70	94.10	99.00
	c=50	(σ_η=0.5)	97.40	88.04	75.60	68.32	96.06	93.16	99.90

Table 5: Rejection frequencies (x100) for the KPSS, the parametric LBI, the bootstrap LBI and the bootstrap LR tests of stationarity. The data generating process is the random walk plus AR(1) model with ρ=-0.5; N=5000, B=500. The significance level is α=10%.

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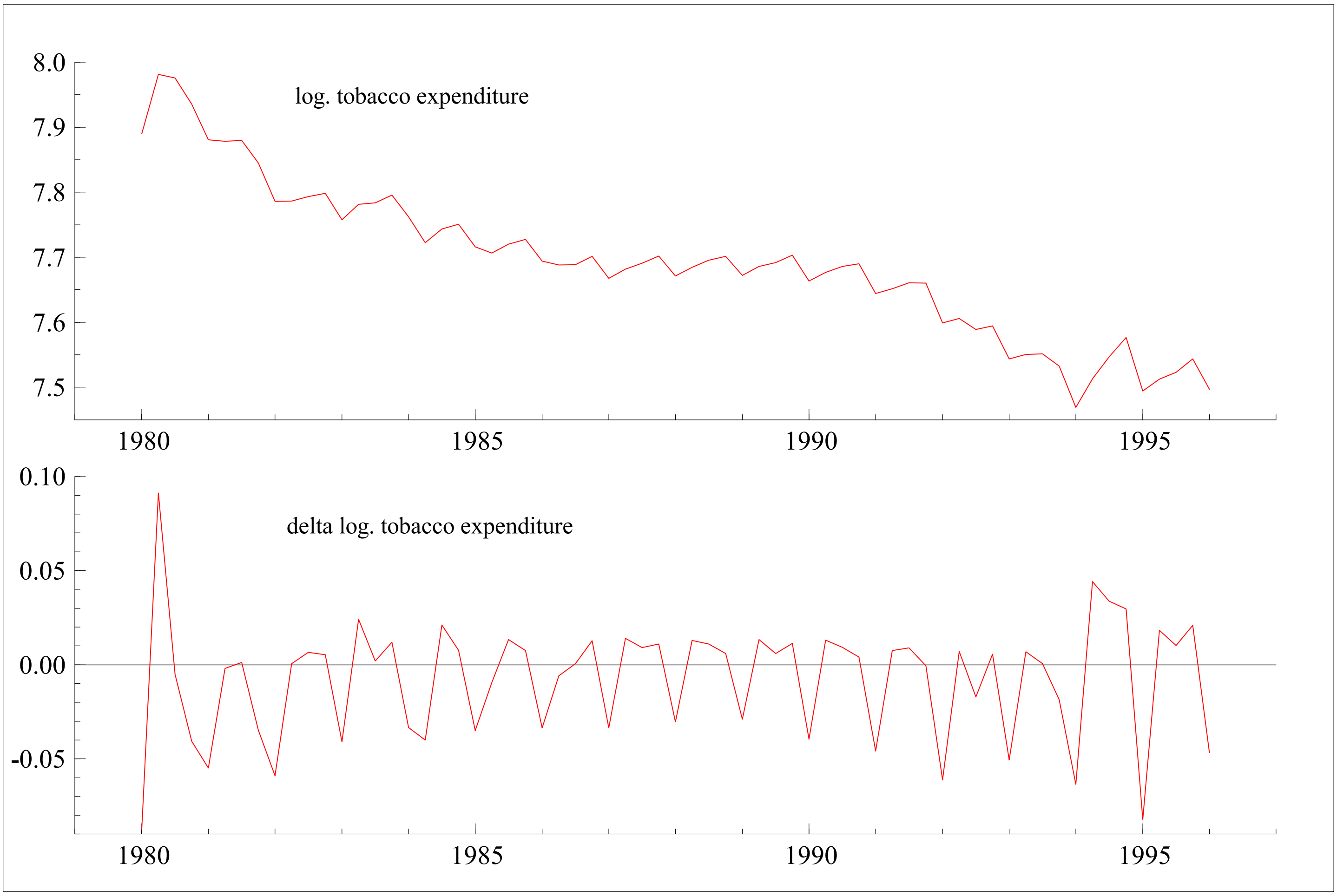

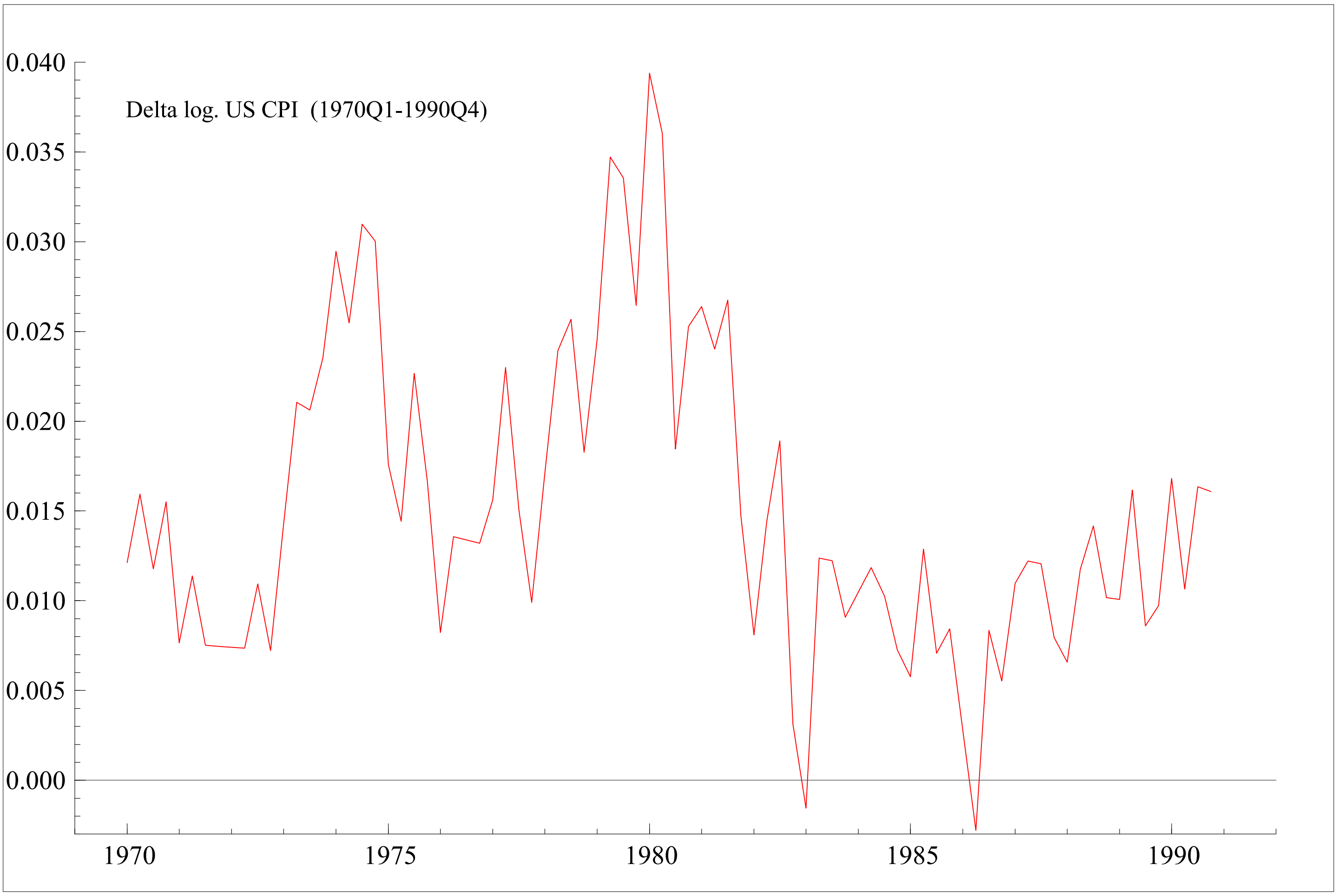
Delta log. US CPI (1970Q1-1990Q4)
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
1970
1975
1980
1985
1990