Mononotic power in tests

for structural change in the mean

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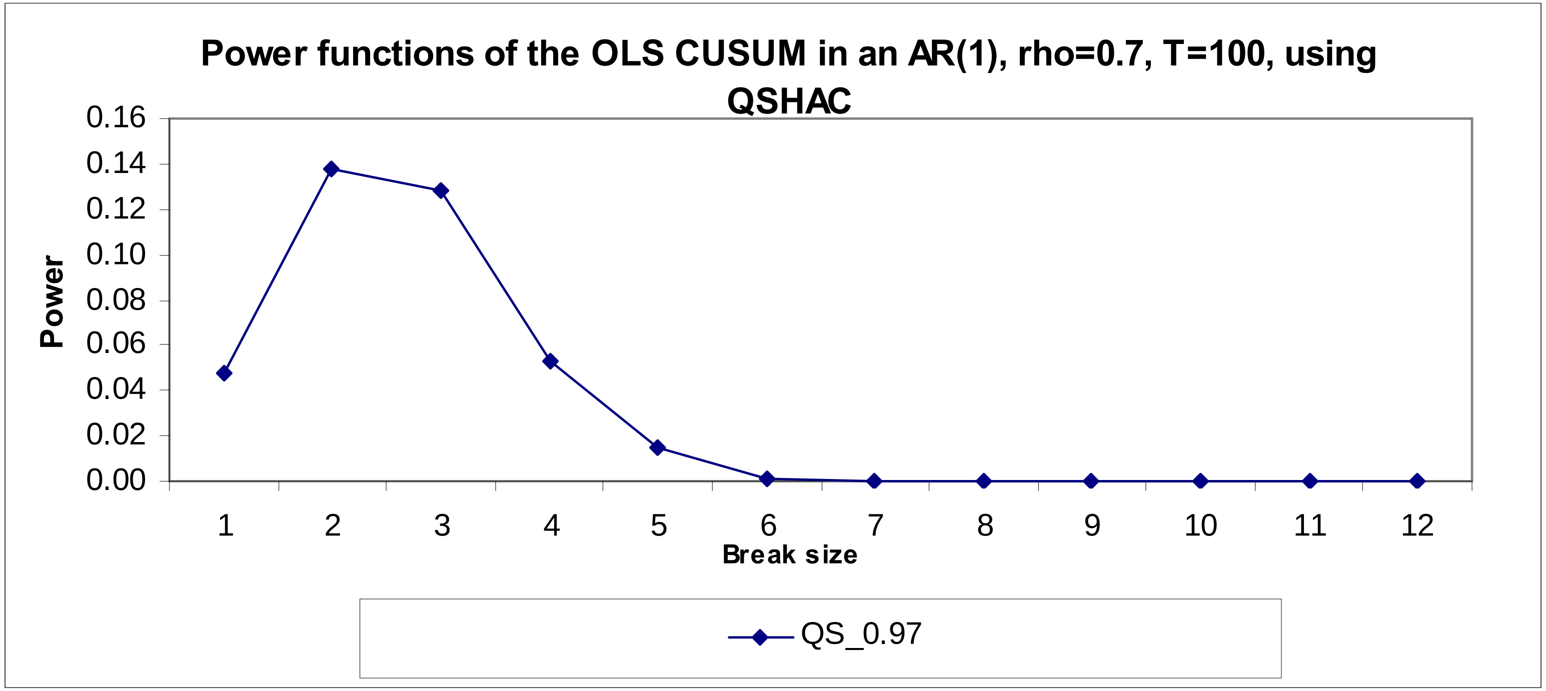
1 Introduction

A family of tests for structural changes in the mean of a temporally dependent process can exhibit non-monotonic power which can even go to zero as the alternative considered is further away from the null value (Perron, 1991 and Vogelsang, 1999).

Although most of these tests are consistent and have good local asymptotic properties for given fixed values in the relevant set of alternative hypotheses in finite samples they have non-monotone power.

Example: CUSUM test for an AR(1) with a break in the constant



Some structural break statistics with nonmonote power

Cumulative or Moving Sums (CUSUM and MOSUM) tests (Brown et al., 1975 and Chu et al., 1995)

Partial Sum test (Gardner, 1969, MacNeill, 1978, Perron, 1991)

Average Wald test (Andrews and Ploberger, 1994)

Supremum Wald test (Andrews, 1993)

Here we focus on the CUSUM test but our results apply to the other tests (e.g. Vogelsang, 1999).

What is the source of non-monotone power?

The non-monotone power is due to the variance estimate which scales the change-point statistic. For example, the HAC estimator in the presence of dependence.

The HAC is often evaluated under the null which assumes that the mean of the process is homogeneous. Hence under the unknown change-point alternative of a mean shift, the HAC

(i) yields an inflated variance estimate which hurts power

(ii) is inconsistent since it assumes a homogeneous mean.

Contribution

A method that restores the monotone power of the CUSUM test for a mean shift in a weakly dependent process.

The method refers to a simple near-stationarity boundary condition for the HAC inspired by Andrews (1991) and Sul, Phillips and Choi (2005).

We show that this boundary condition

(i) solves the overestimation problem of the variance under the single or multiple change-point alternatives and

(ii) preserves the $\sqrt{T}$ consistency of the HAC estimator.

Simulation and empirical evidence support this method.

2 HAC estimators for the CUSUM test

Consider the following stochastic process for a univariate time series, y_t :

$$y_t = \mu + u_t, \ t = 1, ..., T, \tag{1}$$

where u_t is a second-order stationary mean zero error process.

$S_t = \sum_{j=1}^{t} u_j$ satisfies the FCLT (for regularity conditions e.g. Herrndorf, 1984),

$$T^{-1/2} S_{[mT]} \to \sigma W(m),$$

where $W(m)$ is a standard Wiener process on $[0, 1]$ and

$\sigma^2 = \lim_{T\to\infty} E\left[T^{-1}\left(\sum_{t=1}^{T} u_t\right)^2\right].$

The CUSUM statistic for detecting structural changes in the mean of y_t in (1):

$$CUSUM = \left(\sigma\sqrt{T}\right)^{-1} \sup_{1\leq j\leq T} \left|\sum\nolimits_{t=1}^{j} y_t - \sum\nolimits_{t=1}^{T} y_t\right| \rightarrow \sup |B(m)|. \tag{2}$$

converges to the supremum of a BB, $B(m) = W(m) - mW(1)$.

Equivalently (2) can be expressed in terms of the OLS residuals

$$\widehat{u}_t^{OLS} = y_t - 1/T \sum\nolimits_{t=1}^{T} y_t$$

$$CUSUM = \left(\widehat{\sigma}\sqrt{T}\right)^{-1} \sup_{1\leq j\leq T} \left|\sum\nolimits_{t=1}^{j} \widehat{u}_t^{OLS}\right| \tag{3}$$

where $\widehat{\sigma}$ is a consistent estimator under the null of stability.

Traditional estimators of σ^2

Non-parametric spectral density estimators given by

$$\widehat{\sigma}^2 = \sum_{j=-(T-1)}^{T-1} K\left(j/s(T)\right)\widehat{\gamma}_j,$$

where $K(.)$ is the kernel function, $\widehat{\gamma}_j = T^{-1}\sum_{t=j+1}^{T}\widehat{u}_t\widehat{u}_{t-j}$, $s(T)$ is the bandwidth and $\widehat{\sigma}^2$ is consistent if $s(T)/T \to 0$ and $s(T) \to \infty$ as $T \to \infty$.

For instance, Andrews and Monahan (1992) propose the prewhitened estimator $\widehat{\sigma}^2_{PW}$ given by:

$$\widehat{\sigma}^2_{PW} = \widehat{\sigma}^2_{\varepsilon}/(1-\widehat{\rho}^2) \text{ and } \widehat{\rho} = \sum_{t=2}^{T} \widehat{u}_t \widehat{u}_{t-1} / \sum_{t=2}^{T} \widehat{u}^2_{t-1}, \text{ where} \tag{4}$$

$$\begin{gathered}\widehat{\sigma}^2_{\varepsilon} = \sum_{j=-(T-1)}^{T-1} K\left(j/\widehat{s}_{PW}(T)\right) \widehat{\gamma}^{\varepsilon}_j, \\ \widehat{\gamma}^{\varepsilon}_j = T^{-1} \sum_{t=j+1}^{T} \widehat{\varepsilon}_t \widehat{\varepsilon}_{t-j}, \ \widehat{\varepsilon}_t = \widehat{u}_t - \widehat{\rho}\widehat{u}_{t-1}.\end{gathered} \tag{5}$$

The bandwidth $\widehat{s}_{PW}(T)$ is based on the AR(1) plug-in method and depends on the parameter $\widehat{\alpha}_{PW}(1)$ given by:

$$\begin{gathered}\widehat{s}_{PW}(T) = 1.1447(\widehat{\alpha}_{PW}(1)T)^{1/3}, \\ \widehat{\alpha}_{PW}(1) = 4\widehat{\rho}^2_{\varepsilon}/(1-\widehat{\rho}^2_{\varepsilon})^2, \ \widehat{\rho}_{\varepsilon} = \sum_{t=2}^{T} \widehat{\varepsilon}_t \widehat{\varepsilon}_{t-1} / \sum_{t=2}^{T} \widehat{\varepsilon}^2_{t-1},\end{gathered} \tag{6}$$

which uses the autoregressive estimate $\widehat{\rho}_{\varepsilon}$ obtained from $\widehat{\varepsilon}_t$ instead of $\widehat{u}_t$.

Recoloring Procedure

The recoloring procedure in prewhitened HAC estimators (4) involves $\widehat{\rho}$.

* Andrews and Monahan (1992) suggest to replace any $\widehat{\rho}$ that exceeds 0.97 by 0.97 and is less than -0.97 by -0.97. Andrews (1991) suggests a boundary condition based on the idea of a confidence interval for $\widehat{\rho}$ which can lead to accurate size of a test and reduce its variance.

* Sul, Phillips and Choi (2005) propose another recoloring rule based on T given by the boundary condition $\widehat{\rho}' = \min[1 - 1/\sqrt{T}, \widehat{\rho}]$. This represents the maximum allowable value for ρ to be unity minus its asymptotic standard error, $1/\sqrt{T}$.

We generalize this boundary to represent deviations from unity by some fixed local coefficient c, in the spirit of the 'stationary order of magnitude' distance from unity in Sul et al (2005), so that any root preserves near-stationarity and recoloring is based on:

$$\widehat{\rho}' = \min[1 - c/\sqrt{T}, \widehat{\rho}]. \tag{7}$$

One could consider $c = 1, 1.28, 1.65$ in (7) where $c = 1$ is proposed by Sul et al (2005) and $c = 1.28$ and 1.65 are the one-sided confidence intervals values for near-stationarity deviations from $\rho = 1$ (given $c > 0$ such that $\rho < 1$) that correspond to the 10% and 5% standard normal probabilities, respectively (Andrews, 1991). On theoretical grounds any $c > 0$ in (7) preserves the consistency of the variance estimator under the null and alternative.

Consistency of σ^2 using the new boundary condition

Consider the process (1)

$$y_t = \mu + u_t,\ t = 1, ..., T,$$

where for simplicity we assume that u_t is an AR(1):

$$u_t = \rho u_{t-1} + \varepsilon_t, \varepsilon_t \sim NIID(0, \sigma_\varepsilon^2). \tag{8}$$

The limiting distribution of the CUSUM depends on $\widehat{\rho}$ and $\widehat{\sigma}_\varepsilon^2$ that define the long-run variance of u_t in (3).

Parametric model (8) the LS long-run variance estimator is $\widehat{\sigma}_u^2 = \widehat{\sigma}_\varepsilon^2/(1-\widehat{\rho}^2)$. In the non-parametric setting $1/(1-\widehat{\rho}^2)$ are used in the final stage of recoloring to obtain the HAC estimator (4).

Perron (1989) shows that neglected shifts in (8) cause $\widehat{\rho} \to 1$ which imply

$$\widehat{\rho} = 1 + O_p(T^{-1}), (1 - \widehat{\rho}^2) = O_p(T^{-2})$$

and

$$\widehat{\sigma}_\varepsilon^2 / (1 - \widehat{\rho}^2) = O_p(T^2).$$

Consequently, $\widehat{\sigma}_u$ does not satisfy the $\sqrt{T}$ consistency.

On the contrary, the proposed near-stationarity boundary

$$\widehat{\rho}' = \min[1 - c/\sqrt{T}, \widehat{\rho}].$$

controls the order of magnitude of the long-run variance under the alternative of a spurious unit root and yields $\widehat{\sigma}_\varepsilon^2/(1 - \widehat{\rho}^2) = O_p(T)$ which is bounded by $\widehat{\sigma}_u^2 = T\widehat{\sigma}_\varepsilon^2/c^2$.

Case 1: Prewhitened HAC estimators.

Under the alternative of large shifts u_t is $I(1)$ and ε_t is $I(0)$.

Hence $\widehat{\alpha}_{PW}(1) = O_p(T)$ which implies $\widehat{s}_{PW}(T) = O_p(T^{2/3})$.

This preserves the consistency of the non-parametric variance $\widehat{\sigma}^2_{\varepsilon}$ in (5) since

$$\widehat{s}_{PW}(T)/T = O_p(T^{-1/3}) \to 0 \text{ as } T \to \infty.$$

The problem arises at the recoloring stage since the PW HAC estimator (4) involves the spurious unit root $\widehat{\rho} = 1 + O_p(T^{-1})$ and thereby $\widehat{\sigma}^2_{PW} = O_p(T^2)$ which hurts power.

In contrast if we adopt the near-stationarity recoloring rule (7) then $\widehat{\sigma}^2_{PW} = O_p(T)$ which yields a $\sqrt{T}$ consistent $\widehat{\sigma}_{PW}$.

Case 2: HAC estimators without prewhitening.

When there is no prewhitening

$$\widehat{\alpha}(1) = 4\widehat{\rho}^2/(1-\widehat{\rho}^2)^2 \text{ where } \widehat{\rho} = \sum_{t=2}^{T} \widehat{u}_t \widehat{u}_{t-1} / \sum_{t=2}^{T} \widehat{u}_{t-1}^2$$

$\widehat{s}(T) = 1.1447(\widehat{\alpha}(1)T)^{1/3}$ and $\widehat{\sigma}^2 = \sum_{j=-(T-1)}^{T-1} K\left(j/\widehat{s}(T)\right)\widehat{\gamma}_j$.

Given the spurious unit root

$$\widehat{\alpha}(1) = O_p(T^2) \text{ and } \widehat{s}(T) = O_p(T)$$

which implies that $\widehat{s}(T)/T = O_p(1)$ and $\widehat{\sigma}^2$ is inconsistent.

Using the new boundary as the plug-in estimate in $\widehat{\alpha}(1)$ implies

$$\widehat{\alpha}(1) = O_p(T), \widehat{s}(T) = O_p(T^{2/3}) \text{ and}$$

$$\widehat{s}_{PW}(T)/T = O_p(T^{-1/3}) \to 0 \text{ as } T \to \infty.$$

Concluding the near-stationarity boundary

$$\widehat{\rho}' = \min\left[1 - c/\sqrt{T}, \widehat{\rho}\right]$$

yields $\sqrt{T}$ consistent long-run variance estimators

(i) when used as the *plug-in estimate* in $\widehat{\alpha}(1)$ for HAC estimators with no prewhitening and

(ii) when used as a *recoloring boundary* for prewhitened HAC estimators.

3 Simulation Results on Monotonic Power

Evaluate the effects of the near-stationarity boundary (7) on the finite sample power of the CUSUM test and on the properties of the HAC estimators for the

(i) single change-point

(ii) multiple change-points

along with various robustness checks.

3.1 For a single change-point alternative.

The Data Generating Process is:

$$y_t = \mu + \delta D_t + u_t,\ u_t = \rho u_{t-1} + \varepsilon_t,\ t = 1, ..., T, \tag{9}$$

where $\varepsilon_t \sim NIID(0,1)$, $D_t = 0$ for $t < \tau$ and 1 otherwise and $\tau = 0.5T$ is the change-point.

Break size is $\delta = 1, 2, ..., 12$ represents the alternative hypotheses and

$\delta = 0$ denotes the null hypothesis of stability.

Let $\rho = 0.5, 0.7, 0.9$ and $T = 100, 200$.

The $H_0 : \delta = 0$ is examined using the CUSUM statistic for alternative $\widehat{\sigma}^2_{PW}$. For $K(.)$ we use the Quadratic Spectral (QS).

Size of the OLS CUSUM

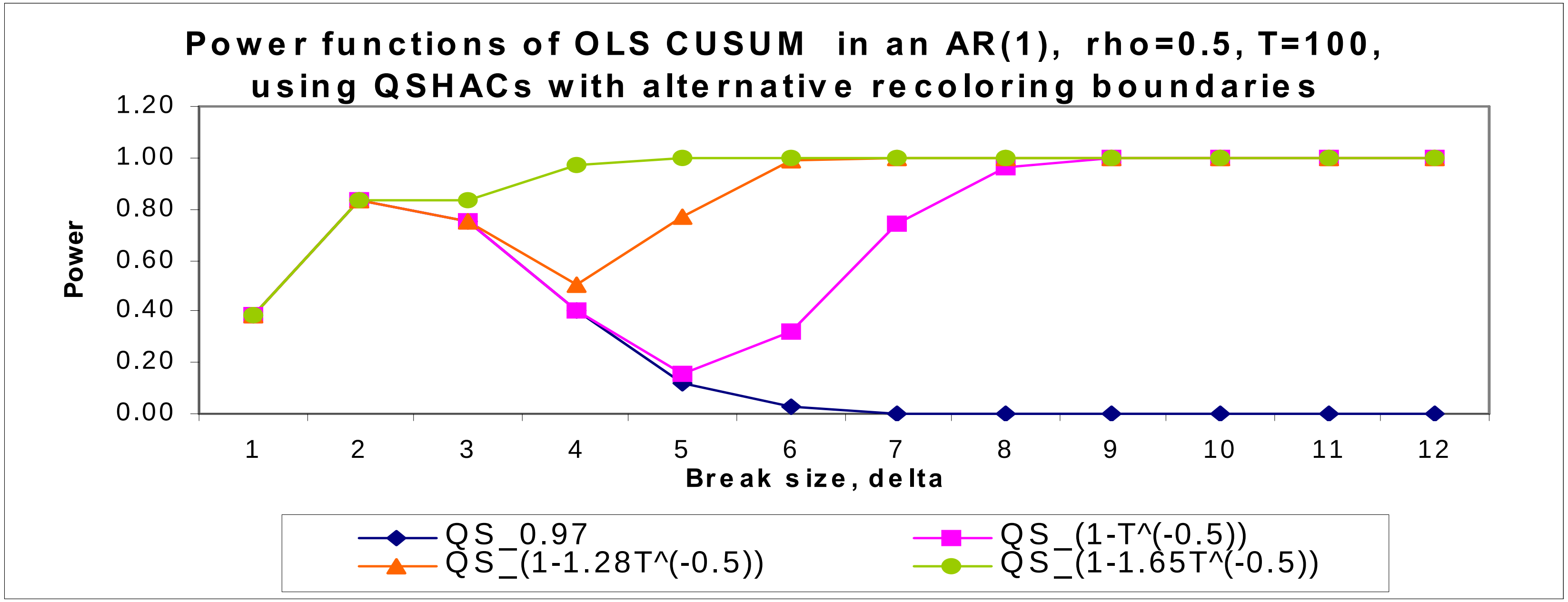
Table 1: Size of OLS CUSUM using QS_PW_HACs

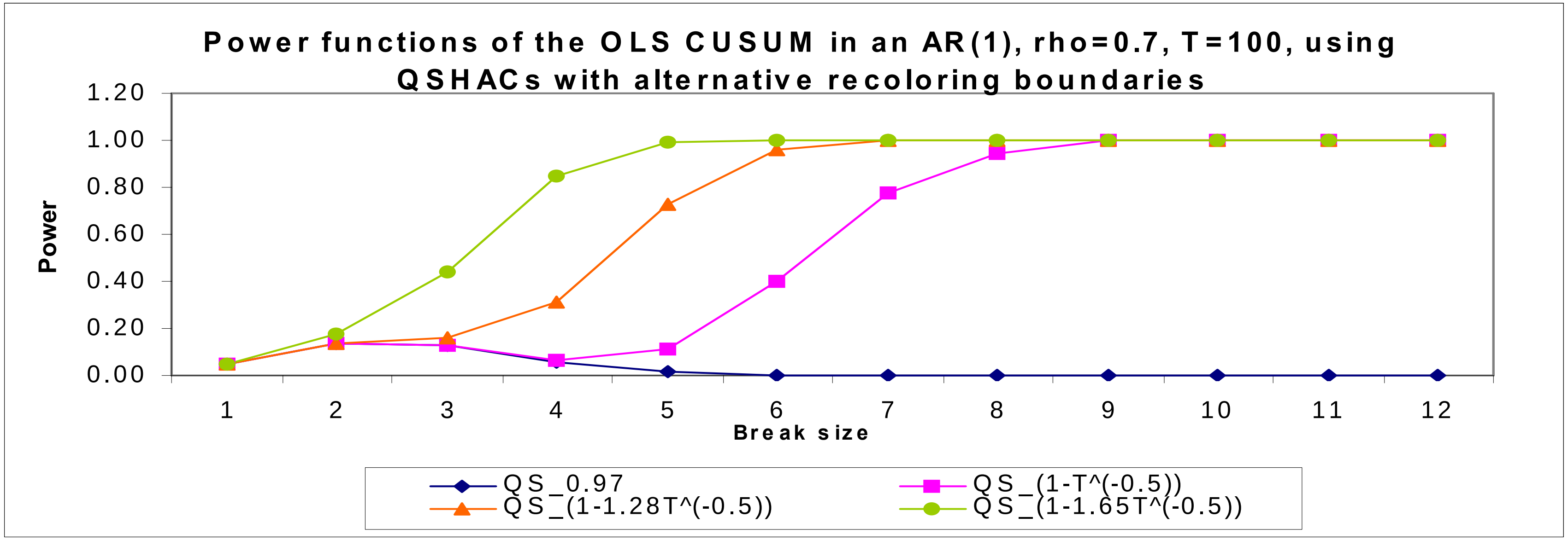
T	ρ	Recoloring boundaries 0.97	(1-1/$\sqrt{}$T)	(1-1.28/$\sqrt{}$T)	(1-1.65/$\sqrt{}$T)
100	0.5	0.025	0.025	0.025	0.025
	0.7	0.012	0.012	0.012	0.012
	0.9	0.001	0.004	0.018	0.086
200	0.5	0.032	0.032	0.032	0.032
	0.7	0.023	0.023	0.023	0.023
	0.9	0.001	0.001	0.003	0.037

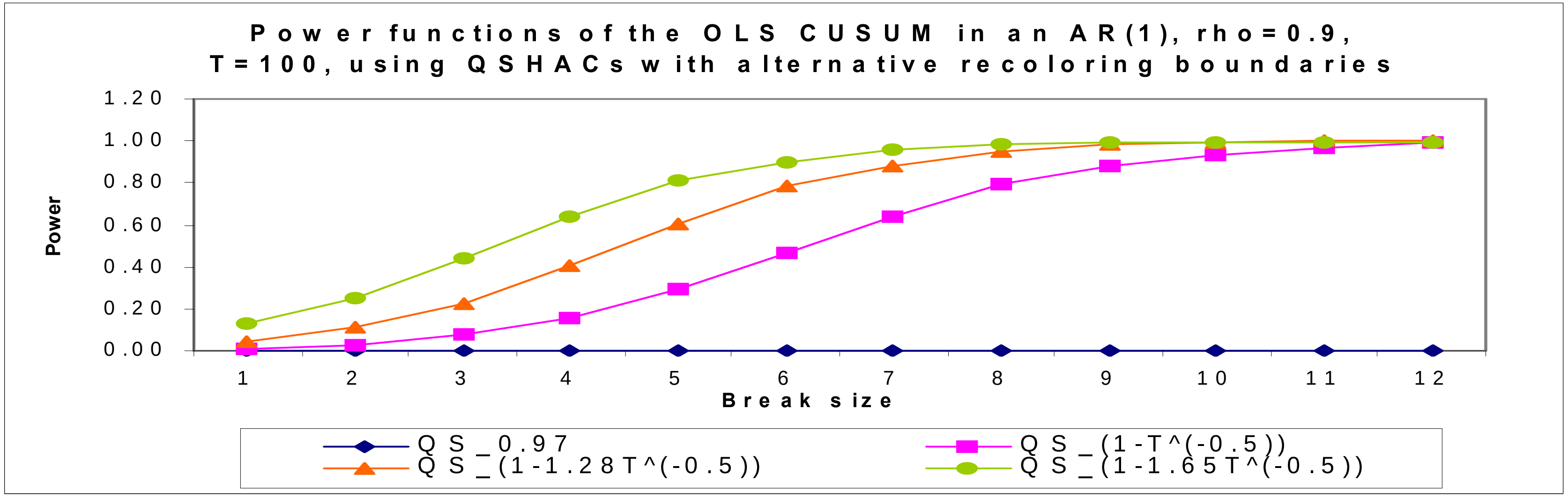
$\widehat{\sigma}^2_{PW}$ with the near-stationarity boundary condition $\widehat{\rho}' = \min[1 - c/\sqrt{T}, \widehat{\rho}]$ when $c = 1.65$ yields relatively better size properties for the CUSUM test for alternative ρ and T.

Power of the OLS CUSUM

Power functions of the CUSUM test Each figure presents four power function that refer to each of the four recoloring rules, 0.97 and $c = 1, 1.28, 1.65$, in (7) for $\widehat{\sigma}^2_{PW}$.




Power functions of the OLS CUSUM in an AR(1), rho=0.7, T=100, using QSHACs with alternative recoloring boundaries
Power
1.20
1.00
0.80
0.60
0.40
0.20
0.00
1 2 3 4 5 6 7 8 9 10 11 12
Break size
QS_0.97
QS_(1-T^(-0.5))
QS_(1-1.28T^(-0.5))
QS_(1-1.65T^(-0.5))


Power functions of the OLS CUSUM in an AR(1), rho=0.9, T=100, using QSHACs with alternative recoloring boundaries
Power
1.20
1.00
0.80
0.60
0.40
0.20
0.00
1 2 3 4 5 6 7 8 9 10 11 12
Break size
QS_0.97
QS_(1-T^(-0.5))
QS_(1-1.28T^(-0.5))
QS_(1-1.65T^(-0.5))

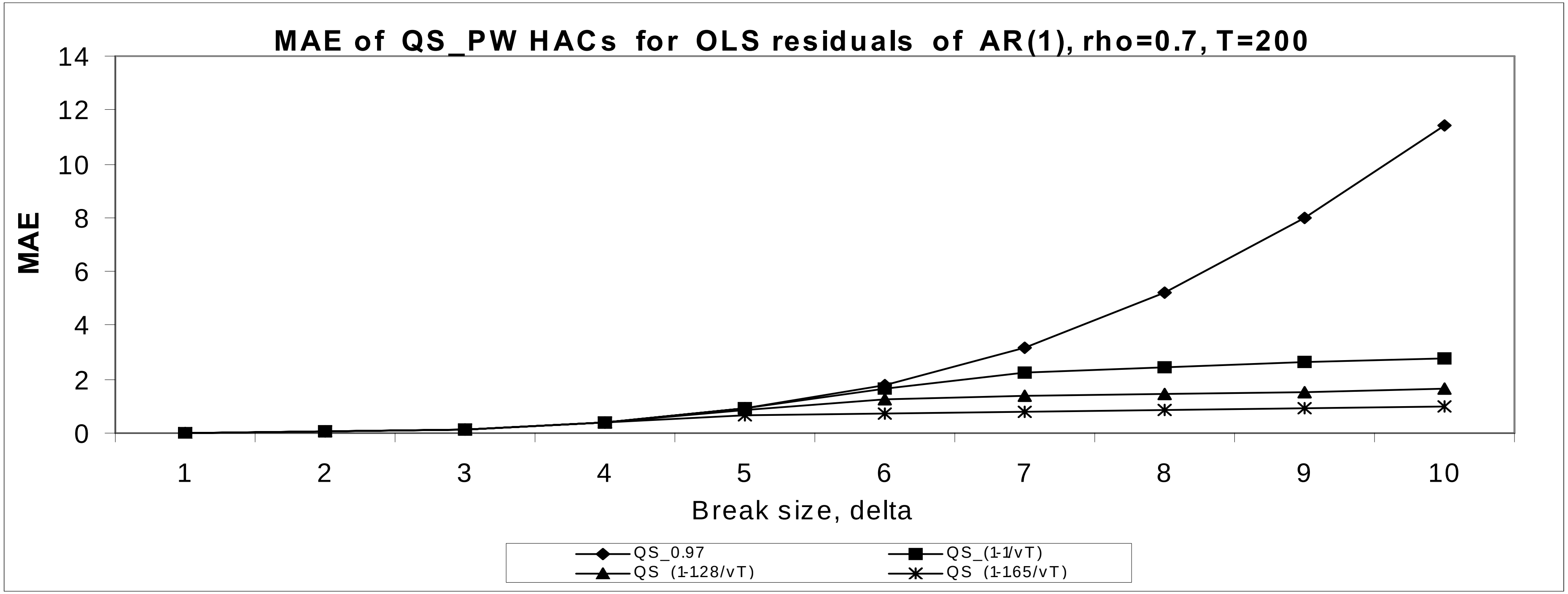
Three broad results on the power of the CUSUM test:

(i) As the size of the break δ increases the power functions of the CUSUM for *all* $c = 1, 1.28, 1.65$ in $\widehat{\rho}' = \min[1 - c/\sqrt{T}, \widehat{\rho}]$ approach one, irrespective of ρ and T.

(ii) For alternative c in $\widehat{\rho}' = \min[1 - c/\sqrt{T}, \widehat{\rho}]$ there is still some weak evidence of non-monotone power in the CUSUM for $c = 1$ when $\rho = 0.5, 0.7$ and $T = 100, 200$, which disappears when $\rho = 0.9$. Similarly $c = 1.28$ yields non-monotonic power functions.

(iii) In general, $c = 1.65$ for the near-stationarity boundary condition (7) in $\widehat{\sigma}_{PW}$ yields not only monotone power functions but also relatively better power compared to $c = 1$ and 1.28.

MAE of $\widehat{\sigma}^2_{PW}$ as a function of the break, δ



Inflated MAEs for $\widehat{\sigma}_{PW}$ with the 0.97 recoloring under the alternatives are contrasted to MAE of $\widehat{\sigma}_{PW}$ with $1 - c/\sqrt{T}$ and especially with $c = 1.65$ that yields the lowest relative MAE.

Robustness checks

The above results are robust to

- Parametric AR model: $y_t = \beta_0 + \beta_1 y_{t-1} + e_t,\ e_t \sim WN$ with plug-in $\beta_1' = \min[1 - c/\sqrt{T}, \widehat{\beta}_1]$ and OLS CUSUM for e_t.

- Other change-point locations ($\tau = 0.25T, 0.75T$)

- Other kernels with data-dependent bandwidths (e.g. Bartlett kernel)

- Other ρ and T ($\rho = 0, -0.5, 0.95$ and $T = 300, 500$)

- Under the alternative of a deterministic trend and/or a unit root

Robustness checks (continued)

- The initial conditions, y_1: (i) $y_1 \sim N(0, 1)$ and (ii) $y_1 \sim N(0, (1 - \rho^2)^{-1})$

Similar power functions are found for $\widehat{\sigma}_{PW}$ and $\widehat{\sigma}_{NPW}$

Similar results are found for the Recursive Least Squares CUSUM test.

3.2 Monotonic power for multiple change-points.

Can the near-stationarity boundary condition restore monotone power of the CUSUM test under the multiple change-point alternative?

Under the alternative of 2 breaks the DGP is:

$$y_t = \mu + \delta_1 D_t^2 + u_t,\ u_t = \rho u_{t-1} + \varepsilon_t,\ \varepsilon_t \sim NIID(0,1),\ t = 1, ..., T, \quad (10)$$

where $D_t^2 = \begin{cases} 0 & t < \tau_1 \\ 1 & \tau_1 \leq t \leq \tau_2 \\ 0 & t > \tau_2 \end{cases}$

s.t. $(\tau_1, \tau_2) = (0.4T,\ 0.9T)$ and $(0.1T, 0.5T)$ are the change-points.

Break sizes are $\delta_1 = 1, 2, ..., 12$.

Under the alternative of 3 breaks the DGP is:

$$y_t = \mu + \delta_1 D_t^3 + u_t,\ u_t = \rho u_{t-1} + \varepsilon_t,\ \varepsilon_t \sim NIID(0,1),\ t = 1, ..., T, \quad (11)$$

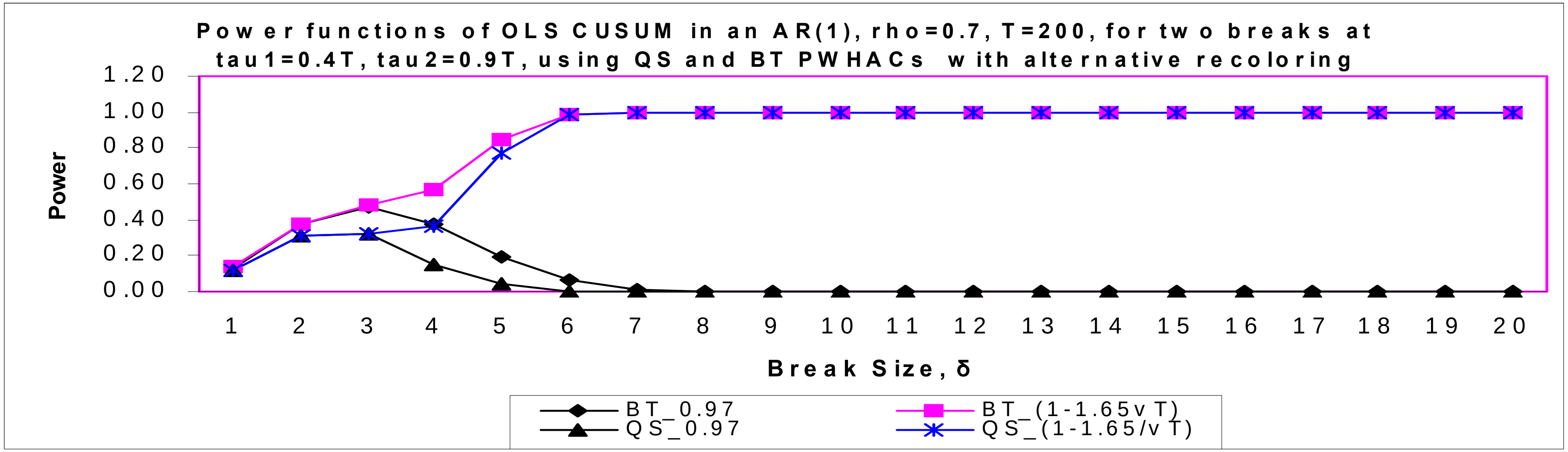
where $D_t^3 = \begin{cases} 0 & t < \tau_1 \\ 1 & \tau_1 \leq t < \tau_2 \\ 0 & \tau_2 \leq t \leq \tau_3 \\ 1 & t > \tau_3 \end{cases}$ s.t.

$(\tau_1, \tau_2, \tau_3) = (0.3T,\ 0.4T, 0.6T),\ (0.5T, 0.7T, 0.8T),\ (0.5T, 0.8T, 0.9T)$.

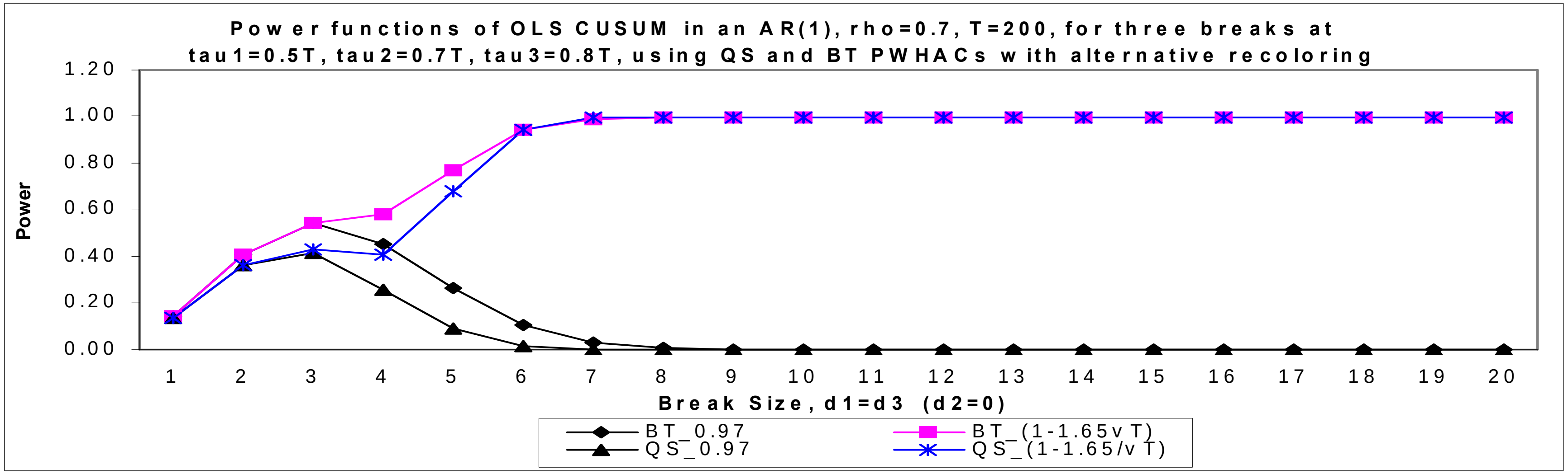
Break sizes are $\delta_1 = 1, 2, ..., 12$.

Evidence of monotonic power functions for the alternative of 2 and 3 breaks.

TWO BREAKS RESULTS



THREE BREAKS RESULTS



4 Empirical analysis

Two empirical examples

(1) Single break in the money market rate

(2) Multiple breaks in the inflation rate

of some emerging markets.

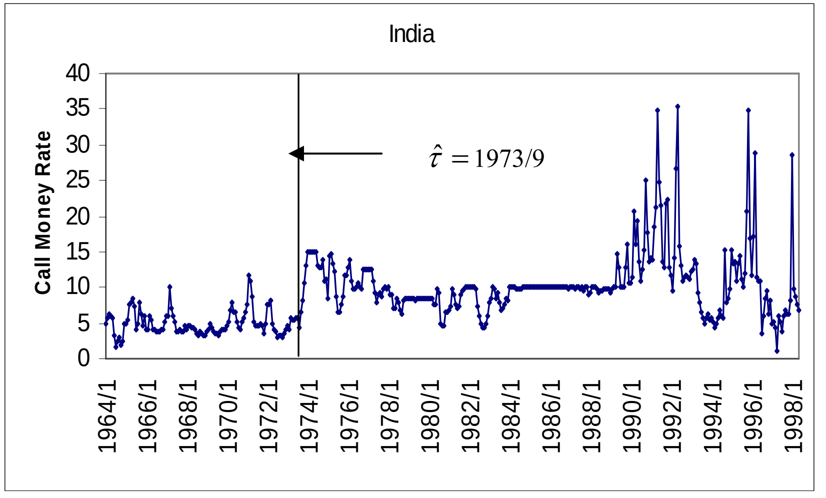
Data Source: IFS. Monthly data since the late 70s with $T = 440$ for the inflation and $T \approx 350$ for the money market rate series.

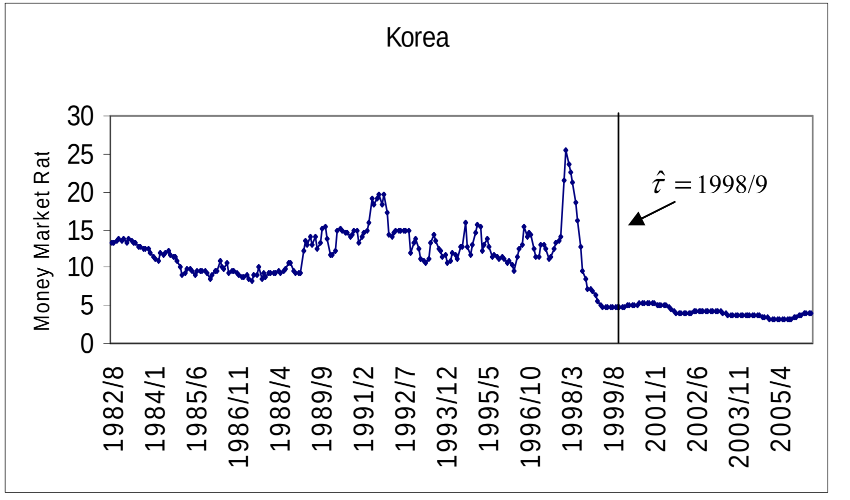
Money market rate in four emerging markets

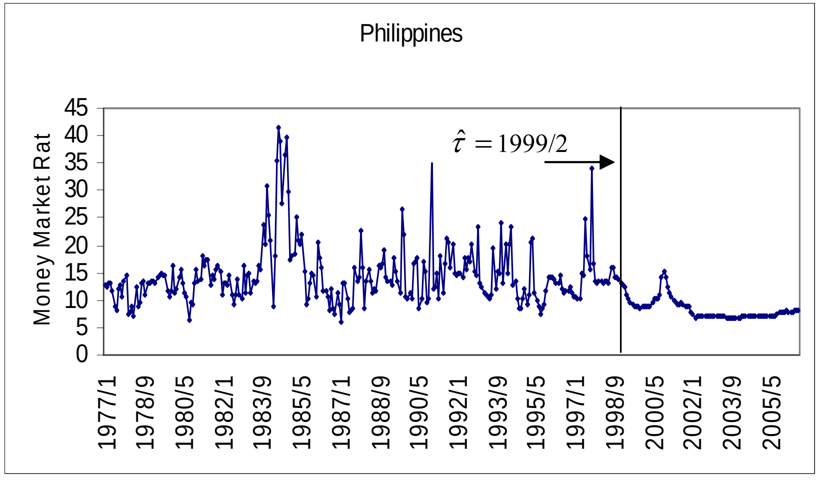
(I) OLS CUSUM & BAI and PERRON TESTS, $R_t = \beta_0 + u_t, u_t \sim AR$

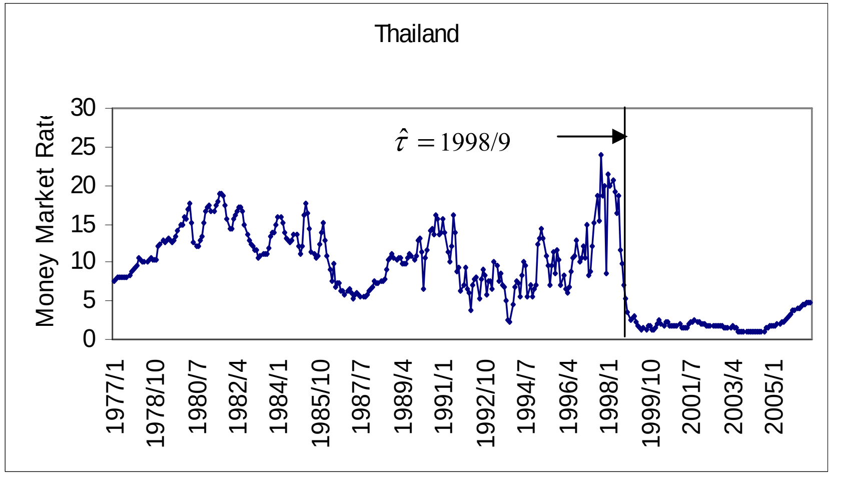
	QSHAC, $(1 - c/\sqrt{T})$						SupF test	
	$\widehat{\rho}$ $se(\widehat{\rho})$	0.97	c=1	c=1.28	c=1.65	Break Date	supF(2/1)	Break Date
Country								
India	0.67 (0.01)	1.65*	1.65*	1.65*	1.65*	73/9	1.26	73/9
Korea	0.98 (0.02)	1.24	1.64*	1.95*	1.84*	98/9	2.61	98/12
Philippines	0.70 (0.07)	1.89*	1.89*	1.89*	1.89*	99/2	3.48	99/3
Thailand	0.94 (0.02)	1.03	1.81*	2.29*	2.85*	98/9	7.03	98/10

MONEY MARKET RATES









Inflation rate in four emerging markets

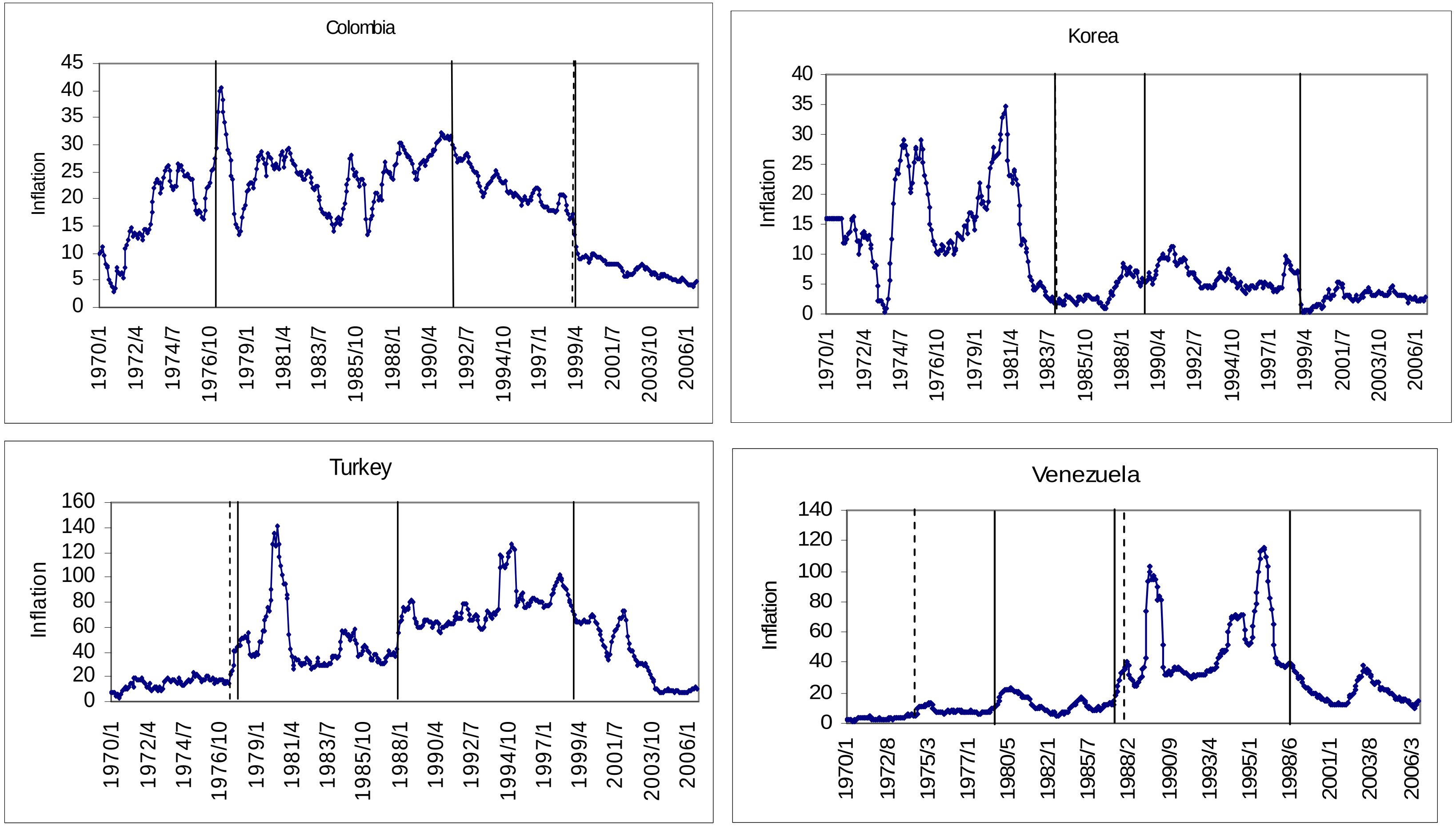
(I) OLS CUSUM, $\Delta P_t = \beta_0 + u_t, u_t \sim AR$

QSHAC	$\widehat{\rho}$ $se(\widehat{\rho})$	0.97	$(1 - \frac{1}{\sqrt{T}})$	$(1 - \frac{1.28}{\sqrt{T}})$	$(1 - \frac{1.65}{\sqrt{T}})$	Break Date
Country						
Colombia	0.992	0.95	1.26	1.36*	1.34	99/1
	(0.006)					
Korea	0.988	0.86	1.19	1.46*	1.79*	82/6
	(0.011)					
Turkey	0.986	0.96	1.30	1.35	1.52*	77/8
	(0.011)					
Venezuela	0.989	1.31	1.71*	1.79*	1.67*	87/2
	(0.013)	-	1.49*	1.59*	1.52*	74/6

(II) BAI & PERRON TEST, $\Delta P_t = \beta_0 + u_t, u_t \sim AR$

Country	Sequential SupF test			Estimated mean regimes			
	supF(2/1)	supF(3/2)	Breaks	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\beta}_3$	$\widehat{\beta}_4$
Colombia	45.35	24.33	77/4	1.19	1.38	1.33	0.83
			91/10	(0.03)	(0.01)	(0.01)	(0.01)
			99/2				
Korea	77.4	86.50	82/5	1.15	0.53	0.78	0.39
			89/9	(0.03)	(0.02)	(0.01)	(0.03)
			98/12				
Turkey	80.14	192.17	77/7	1.14	1.64	1.88	1.39
			87/12	(0.02)	(0.02)	(0.01)	(0.04)
			99/4				
Venezuela	315.11	156.94	79/2	0.72	1.08	1.66	1.27
			87/2	(0.02)	(0.02)	(0.02)	(0.02)
			99/2				

INFLATION



Note: The solid line refer to the structural breaks detected by the Bai & Perron test and the dashed line to the CUSUM test found in previous Table.

5 Conclusions

The paper proposes a method that restores monotone power of structural change tests in a weakly dependent process.

In the nonparametric case of dealing with dependence it annihilates the divergence of the HAC estimator (caused by $\widehat{\rho} \longrightarrow 1$ due to the mean shift) via the near-stationarity boundary condition (7). This condition also retains $\sqrt{T}$ consistency of the HAC under the alternative hypothesis of a single or multiple breaks. At the same time this is a simple procedure that yields relatively more efficient HAC estimators both under the null and the change-point alternative.

Future work

1. Multivariate regression model

2. Tests when the errors are ARCH

3. CUSUM for cointegration