Testing for Multiple Structural Changes in Cointegrated Regression Models

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Objectives

- Study issues related to Structural Changes in Cointegrated Regression models.

- Provide a general regression framework that allows for both $I(1)$ and $I(0)$ regressors.

- Derive the limit distribution of the sup Wald and UDmax tests in a wide variety of models.

- Propose a sequential procedure which enables consistent estimation of the number of breaks as well as distinguish a cointegrated model from a spurious one.

- Propose a modified Wald Test to circumvent the problem of non-monotonic power while maintaining adequate size.

Outline of Presentation

- Motivation and Related Research
- Model and Assumptions
- Tests of Structural Change
- Monte Carlo Simulation
- Conclusion

Motivation and Related Research (1)

- Accounting for parameter shifts is crucial in cointegration analysis since it normally involves long spans of data which are more likely to be affected by structural breaks.

- Kejriwal and Perron (2006) study the properties of the estimates of the break dates and parameters in cointegrated regression models with multiple structural changes. In particular, it is shown that when the $I(1)$ coefficients are allowed to change, the estimates of break dates are asymptotically dependent and confidence intervals need to be constructed jointly.

- Hansen (1992) develops Sup and Mean LM tests of the null hypothesis of no change against an alternative of a one time change in all coefficients (pure structural change). He also considers a version of the LM test directed against the alternative that the coefficients are random walk processes.

Motivation and Related Research (2)

- Vogelsang (1999) shows that the LM tests suffer from the problem of non-monotonic power in finite samples. Moreover, Hansen (2000) shows that the LM test is quite poorly behaved in the presence of structural changes in the marginal distribution of the regressors.

- Multiple breaks are important in that single break tests can suffer from non-monotonic power when the alternative involves more than one break.

- Multiple breaks correspond to the idea of multiple equilibria in economic models-each cointegrating regime corresponding to a particular equilibrium.

- Bai and Perron (1998) provide a comprehensive treatment of issues relating to estimation and inference with multiple structural changes, occurring at unknown dates, in stationary regression models. We generalize their framework to allow for stationary as well as integrated regressors.

Model and Main Assumptions (1)

- Consider the following linear regression model with m breaks ($m+1$ regimes) :

  $$y_t = c_j + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t$$

  ($t = T_{j-1}+1, ..., T_j$) for $j = 1, ..., m+1$ where $T_0 = 0, T_{m+1} = T$ and T is the sample size.

  $$\begin{aligned} z_{ft} &= z_{f,t-1} + u_{zt}^f \\ z_{bt} &= z_{b,t-1} + u_{zt}^b \\ x_{ft} &= \mu_f + u_{xt}^f \\ x_{bt} &= \mu_b + u_{xt}^b \end{aligned}$$

- The break points $(T_1, ..., T_m)$ are treated as unknown so that the parameter estimates include estimates of the unknown regression coefficients as well as those of the break dates.

- This is a partial structural change model in which the coefficients of only a subset of the regressors are subject to change while the remaining coefficients are effectively estimated using the entire sample.

Model and Main Assumptions (2)

- A1: $\xi_t = (u_t,\, u_{zt}^{f\prime},\, u_{zt}^{b\prime},\, u_{xt}^{f\prime},\, u_{xt}^{b\prime})'$satisfies a multivariate invariance principle:

$$T^{-1/2}\sum_{t=1}^{[Tr]}\xi_t \Rightarrow B(r)$$

$B(r) = (B_1(r), B_z^f(r)', B_z^b(r)', B_x^f(r)', B_x^b(r)')'$ is a n vector Brownian motion with symmetric covariance matrix

$$\begin{aligned}\Omega &= \begin{pmatrix} \sigma^2 & \Omega_{1z}^f & \Omega_{1z}^b & \Omega_{1x}^f & \Omega_{1x}^b \\ \Omega_{z1}^f & \Omega_{zz}^{ff} & \Omega_{zz}^{fb} & \Omega_{zx}^{fb} & \Omega_{zx}^{ff} \\ \Omega_{z1}^b & \Omega_{zz}^{bf} & \Omega_{zz}^{bb} & \Omega_{zx}^{bf} & \Omega_{zx}^{bb} \\ \Omega_{x1}^f & \Omega_{xz}^{ff} & \Omega_{xz}^{fb} & \Omega_{xx}^{ff} & \Omega_{xx}^{fb} \\ \Omega_{x1}^b & \Omega_{xz}^{bf} & \Omega_{xz}^{bb} & \Omega_{xx}^{bf} & \Omega_{xx}^{bb} \end{pmatrix} \begin{matrix} 1 \\ q_f \\ p_f \\ q_b \\ p_b \end{matrix} \\ &= \lim_{T\to\infty} T^{-1}E(S_T S_T') = \Sigma + \Lambda + \Lambda'\end{aligned}$$

where $S_T = \sum_{t=1}^{T}\xi_t$, $\Sigma = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}E(\xi_t\xi_t')$ and

$$\Lambda = \lim_{T\to\infty} T^{-1}\sum_{j=1}^{T-1}\sum_{t=1}^{T-j}E(\xi_t\xi_{t+j}').$$

Model and Main Assumptions (3)

- A2: $T^{-1/2}\sum_{t=1}^{[Tr]}(u_{xt}^{f}, u_{xt}^{b})u_t \Rightarrow \sigma Q^{*1/2}W_x^*(r)$, where $W_x^*(r) = (W_{xf}^*(r)', W_{xb}^*(r)')'$ is a $(p_f + p_b)$ vector of independent Wiener processes. Note that $W_x^*(\cdot)$ is based on a quadratic function of the elements of u_t whereas $B(\cdot)$ depends on partial sums which are linear in u_t's. Hence, $W_x^*(\cdot)$ and $B(\cdot)$ are uncorrelated and, being Gaussian, are therefore independent.

- A3: $\begin{pmatrix} \Omega_{zz}^{bb} & \Omega_{zz}^{fb} \\ \Omega_{zz}^{bf} & \Omega_{zz}^{ff} \end{pmatrix} > 0$.

- A4: $T^{-1}\sum_{t=1}^{[Ts]} x_t x_t' \xrightarrow{p} sQ$ and $T^{-1}\sum_{t=1}^{[Ts]} u_{xt}u_{xt}' \xrightarrow{p} sQ^*$, uniformly in $s \in [0,1]$, for some positive definite matrices Q and Q^*.

Tests for Structural Change (1)

First we consider the case where the $I(1)$ regressors are strictly exogenous and the errors $\{u_t\}$ form an array of martingale differences relative to $\{\mathcal{F}_t\} = \sigma$-field$\{\xi^*_{t-s}, u_{t-1-s}; s > 0\}$.

- Tests of the null hypothesis of no structural change ($m = 0$) versus the alternative hypothesis that there are $m = k$ changes.

$$\begin{aligned} F_T(\lambda, k) &= \frac{SSR_0 - SSR_k}{k(q_b + p_b)\hat{\sigma}^2} \\ \sup\ F_T(k) &= \sup_{\lambda \in \Lambda_\epsilon} F_T(\lambda, k) \\ UD\max F_T(M) &= \max_{1 \le m \le M} sup_{\lambda \in \Lambda_\epsilon} F_T(\lambda, m) \end{aligned}$$

where for some arbitrary small positive number ϵ,

$$\Lambda_\epsilon = \{\lambda : |\lambda_{i+1} - \lambda_i| \ge \epsilon, \lambda_1 \ge \epsilon, \lambda_k \le 1 - \epsilon\}.$$

.

Tests for Structural Change (2)

- Sequential Test of k changes versus $k+1$ changes:

$$\begin{aligned} \text{Let } A_T(k) &= SSR_T(\hat{T}_1, ..., \hat{T}_k), \\ B_T(\tau, k) &= SSR_T(\hat{T}_1, ...\hat{T}_{j-1}, \tau, \hat{T}_j, ..., \hat{T}_k) \end{aligned}$$

$$\begin{aligned} & SEQ_T\,(k+1|k) \\ = \quad & \max_{1 \le j \le k+1} \sup_{\tau \in \Lambda_{j,\varepsilon}} \left\{ \frac{A_T(k) - B_T(\tau, k)}{T^{-1} B_T(\tau, k)} \right\} \end{aligned}$$

where

$$\Lambda_{j,\varepsilon} = \{\tau; \hat{T}_{j-1} + (\hat{T}_j - \hat{T}_{j-1})\varepsilon \le \tau \le \hat{T}_j - (\hat{T}_j - \hat{T}_{j-1})\varepsilon\}.$$

- The procedure amounts to performing a one break test for each of the $(k+1)$ segments defined by the partition $(\hat{T}_1, ..., \hat{T}_k)$ and assessing whether the maximum of the tests is significant.

Tests for Structural Change (3)

Theorem For models with $I(1)$ variables only, the limiting distributions of the tests under the null hypothesis of no structural change ($m = 0$) are independent of nuisance parameters. Moreover, the limiting distributions are different for the different cases. For instance, the limits vary according to whether the intercept is allowed to change or not.

Remark For the partial change models, we show that the limit distribution depends on the number of $I(1)$ coefficients that are not allowed to change. This is different from a stationary framework where the limit distribution is independent of the number of regressors whose coefficients are not allowed to change.

Remark For the pure structural change model involving a change in intercept and slope, the limit is given by

$$\sup_{\lambda \in \Lambda_\epsilon} \sum_{i=1}^{k} (S^*(\lambda_i, \lambda_{i+1})' V(\lambda_i, \lambda_{i+1})^{-1} S^*(\lambda_i, \lambda_{i+1}))$$

where $S^*(\lambda_i, \lambda_{i+1}) = S(\lambda_i) - M(\lambda_i) M(\lambda_{i+1})^{-1} S(\lambda_{i+1})$, $V(\lambda_i, \lambda_{i+1}) = M(\lambda_i) - M(\lambda_i) M(\lambda_{i+1})^{-1} M(\lambda_i)$, $S(\lambda_i) = \int_0^{\lambda_i} Z^* dW_1$, $M(\lambda_i) = \int_0^{\lambda_i} Z^* Z^{*\prime}$, $Z^* = (1, W_z^{b\prime})'$.

Tests for Structural Change (4)

Theorem In models involving both $I(1)$ and $I(0)$ variables, asymptotic inference is possible as long as the intercept is allowed to change across regimes. Otherwise, the limit distribution will depend on the means of the $I(0)$ regressors.

Corollary Including stationary regressors whose coefficients are not allowed to change leads to the same distribution as that obtained without these regressors under the null hypothesis of no structural change.

Implication Inference is difficult if the intercept is not allowed to change across regimes. When the stationary regressors have mean zero and only their coefficients are allowed to change, the limiting distribution that applies is the same as in the case when all regressors are stationary (Bai and Perron, 1998). It follows that the same limit distribution applies with mean zero stationary regressors and no intercept included.

Tests for Structural Change (5)

Remark For the pure structural change model in which the intercept, $I(1)$ and $I(0)$ coefficients are all allowed to change, the limit can be expressed as:

$$\sup_{\lambda\ \in \Lambda_\epsilon} \left\{ \begin{array}{c} \sum_{i=1}^{k} (S^*(\lambda_i, \lambda_{i+1})' V(\lambda_i, \lambda_{i+1})^{-1} S^*(\lambda_i, \lambda_{i+1})) \\ + \sum_{i=1}^{k} B^*(\lambda_i, \lambda_{i+1})' B^*(\lambda_i, \lambda_{i+1}) \end{array} \right\}$$

where $B^*(\lambda_i, \lambda_{i+1}) = \frac{\lambda_i W^*_{xb}(\lambda_{i+1}) - \lambda_{i+1} W^*_{xb}(\lambda_i)}{(\lambda_i \lambda_{i+1}(\lambda_{i+1} - \lambda_i))^{1/2}}$

Corollary For a particular testing problem considered, denote the limit distribution of the test $sup_{\lambda \in \Lambda_\epsilon} F_T(\lambda, k)$ by $sup_{\lambda \in \Lambda_\epsilon} F(\lambda, k)$, then

$$\begin{aligned} UD \max F_T(M) &= \max_{1 \le m \le M} sup_{\lambda \in \Lambda_\epsilon} F_T(\lambda, m) \\ &\Rightarrow \max_{1 \le m \le M} sup_{\lambda \in \Lambda_\epsilon} F(\lambda, m) \end{aligned}$$

and

$$\lim_{T \to \infty} P\left(SEQ_T\left(k+1|k\right) \le x\right) = G_\varepsilon\left(x\right)^{k+1}$$

with $G_\varepsilon(x)$ the distribution function of $\sup_{\lambda \in \Lambda_\varepsilon} F(\lambda, 1)$.

Serially Correlated Errors

- A problem with the usual Sup-Wald test is that with persistent errors, the size distortions are substantial especially with multiple breaks.

- But the LM test suffers from the problem of non-monotonic power in finite samples due to the estimation of the long run variance under the null hypothesis.

- Solution to this size-power trade-off: a new estimator of the long run variance constructed using a hybrid method that involves residuals computed under both the null and alternative hypotheses.

$$\hat{\sigma}^2 = T^{-1} \sum_{t=1}^{T} \tilde{u}_t^2 + 2T^{-1} \sum_{j=1}^{T-1} w(j/\hat{h}) \sum_{t=j+1}^{T} \tilde{u}_t \tilde{u}_{t-j}$$

- $\hat{h}$ is the bandwidth computed under the alternative while $\tilde{u}_t$ are the residuals under the null.

$$\begin{aligned} F_T^*(\lambda; k) &= \left(\hat{\sigma}_u^2 / \hat{\sigma}^2\right) F_T(\lambda, k) \\ sup_{\lambda \in \Lambda_\epsilon} F_T^*(\lambda, k) &= F_T^*(\hat{\lambda}, k) \end{aligned}$$

Endogenous I(1) Regressors

- Saikkonen's Dynamic OLS estimator :

$$\begin{aligned} y_t &= \hat{c}_i + z'_{ft}\hat{\delta}_f + x'_{ft}\hat{\beta}_f + z'_{bt}\hat{\delta}_{bi} + x'_{bt}\hat{\beta}_{bi} \\ &\quad + \sum_{j=-\ell_T}^{\ell_T} \Delta z'_{t-j}\hat{\Pi}_j + \hat{v}^*_t \end{aligned}$$

- a) *(Upper Bound Condition)* $\ell_T^2/T \to 0$, and b) *(Lower Bound Condition)* $\ell_T \sum_{|j|>\ell_T} \left\| \Pi_j \right\| \to 0$.

- These conditions are much weaker than those in Saikkonen (1991). The weaker lower bound condition allows for the use of data dependent rules (AIC/BIC) to select l_T. (Kejriwal and Perron, 2006).

- **Theorem** The limiting distributions of the tests based on the dynamic OLS regression are the same as those that apply under strict exogeneity.

Monte Carlo - Size

- DGP:

$$
\begin{aligned}
y_t &= z_t + u_t \\
z_t &= z_{t-1} + \eta_t \\
u_t &= \rho u_{t-1} + e_t - \text{AR}(1) \\
u_t &= e_t - \theta e_{t-1} - \text{MA}(1) \\
\eta_t, e_t &\sim i.i.d.\ N(0,1),\ \text{Cov}(\eta_t, e_t) = 0
\end{aligned}
$$

- The regressors are $\{1, z_t\}$: both coefficients are allowed to change. (Pure Structural Change)

- 1000 replications. 5% nominal level. The value of the trimming ϵ is set at .20.

Monte Carlo - Size

Table 1 - AR(1) Errors

	$T = 120$		$T = 240$	
	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.5$	$\rho = 0.7$
$SupF_T^*(1)$	.03	.01	.03	.02
$SupF_T^*(2)$	.02	.01	.03	.00
$SupF_T^*(3)$	.01	.01	.00	.00
$UDmax$	.03	.01	.02	.02

Table 2 - MA(1) Errors

	$T = 120$		$T = 240$	
	$\theta = 0.5$	$\theta = 0.7$	$\theta = 0.5$	$\theta = 0.7$
$SupF_T^*(1)$	.02	.04	.02	.02
$SupF_T^*(2)$	.05	.09	.02	.03
$SupF_T^*(3)$	.05	.09	.02	.04
$UDmax$	.03	.06	.02	.02

Monte Carlo - Spurious Regression

- If the regression is spurious, the sequential procedure, uncorrected for serial correlation in the errors, will tend to select the maximum number of breaks allowed. Thus, selecting the maximum permissible number of breaks can be indicative of the presence of $I(1)$ errors.

  DGP :

$$\begin{aligned} y_t &= y_{t-1} + e_t \\ z_t &= z_{t-1} + \eta_t \\ \eta_t, e_t &\sim i.i.d.N(0,1),\ Cov(\eta_t, e_t) = 0 \end{aligned}$$

Table 3: Probability of Selecting m Breaks

Max. Breaks	m	$T = 120$	$T = 240$
3	0	.00	.00
	1	.01	.00
	2	.15	.03
	3	.83	.97
4	0	.00	.00
	1	.02	.00
	2	.15	.03
	3	.46	.31
	4	.37	.66

Monte Carlo - Power

- DGP with one break:

$$\begin{aligned} y_t &= \delta_1 z_t + u_t \text{ if } t \leq [T/2] \\ y_t &= \delta_2 z_t + u_t \text{ if } t > [T/2] \end{aligned}$$

- DGP with two breaks:

$$\begin{aligned} y_t &= \delta_1 z_t + u_t \quad if \quad t \leq [T/3] \\ y_t &= \delta_2 z_t + u_t \quad if \quad [T/3] < t \leq [2T/3] \\ y_t &= \delta_3 z_t + u_t \quad if \quad [2T/3] < t \leq T \end{aligned}$$

- To illustrate the usefulness of the UDmax test, we consider the case where the first and third regimes are identical ($\delta_1 = \delta_3$).

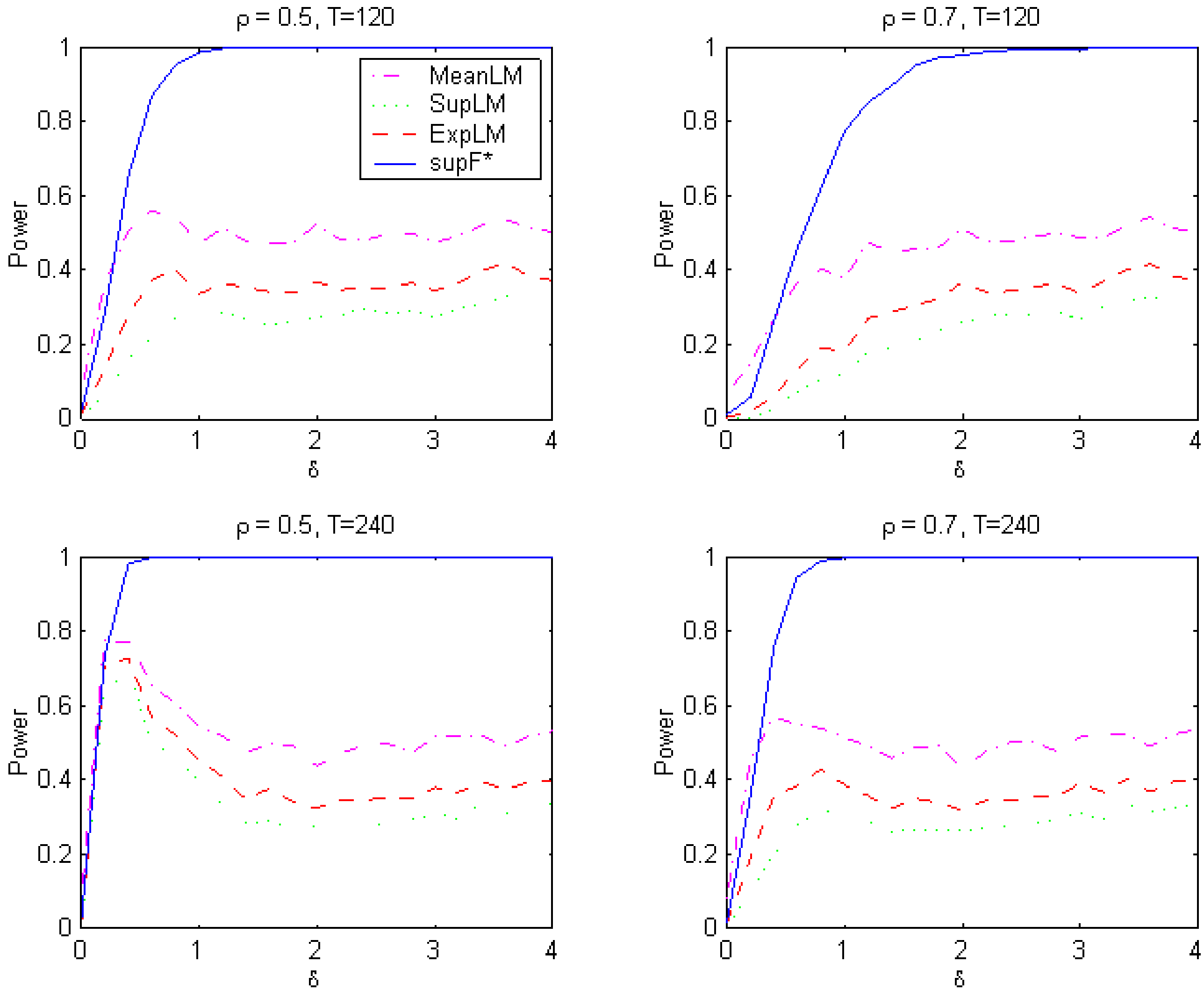

Figure 1: Power Functions with AR(1) Errors : DGP with one break

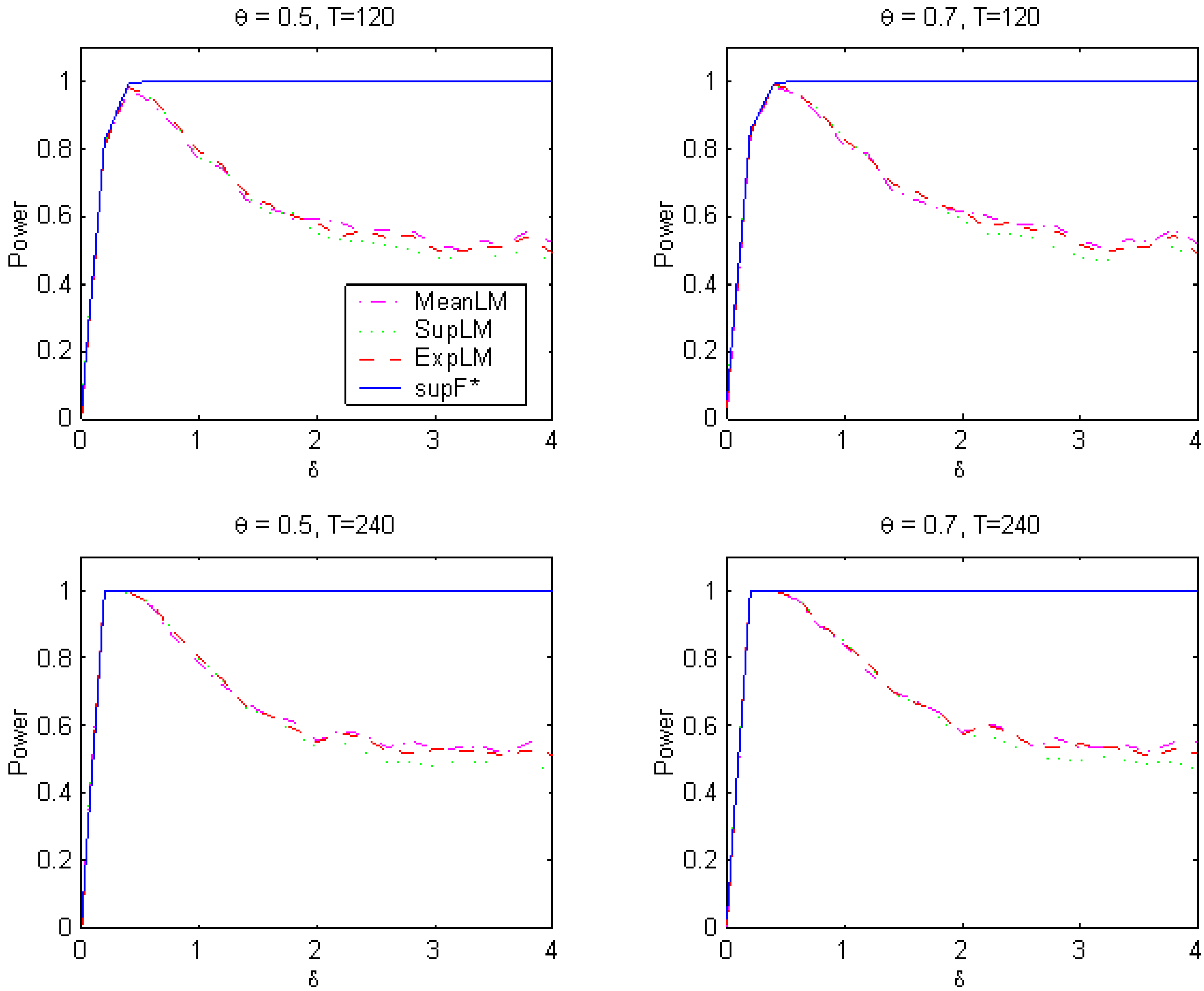

Figure 2: Power Functions with MA(1) Errors : DGP with one break

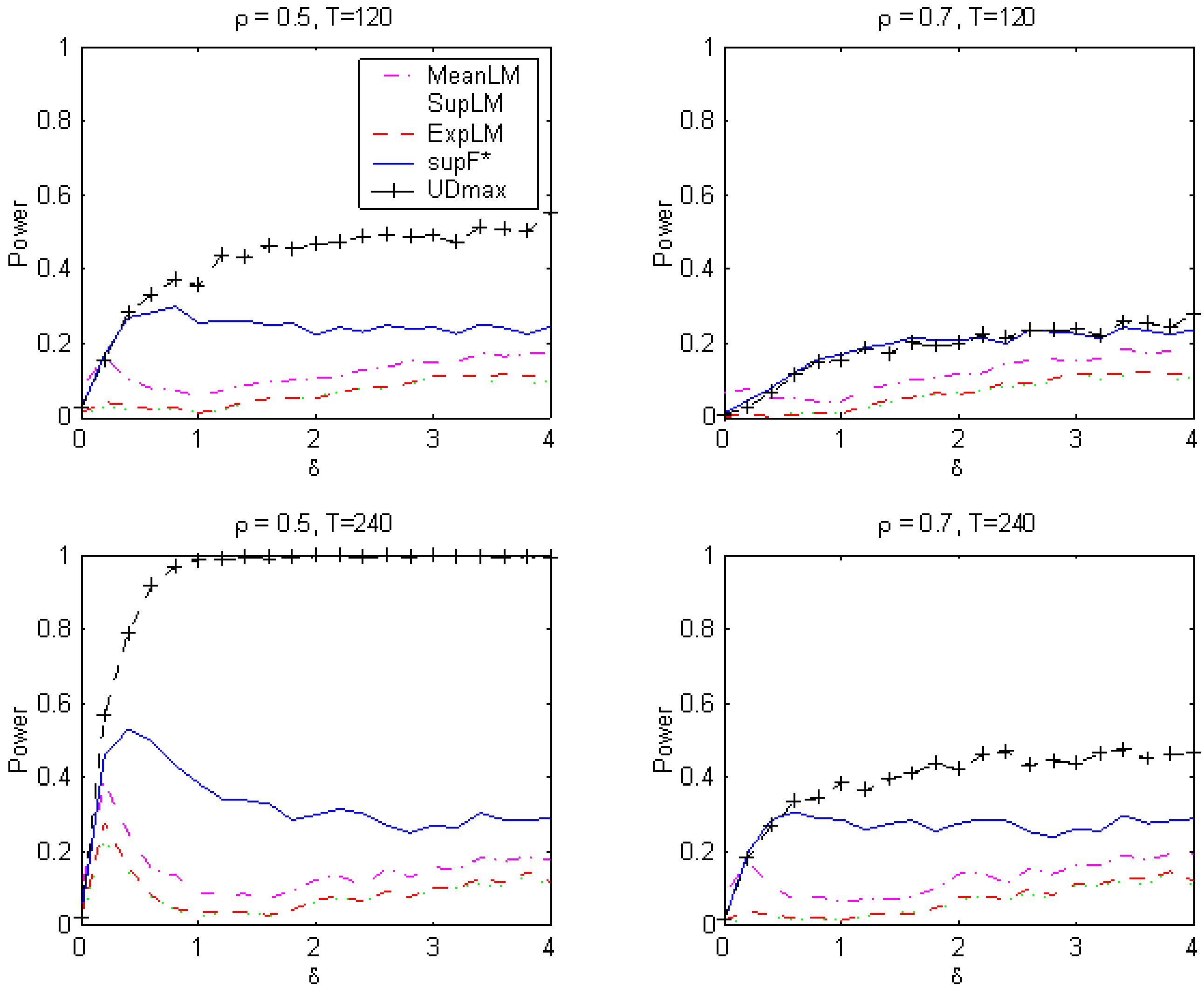

Figure 3: Power Functions with AR(1) Errors : DGP with 2 breaks

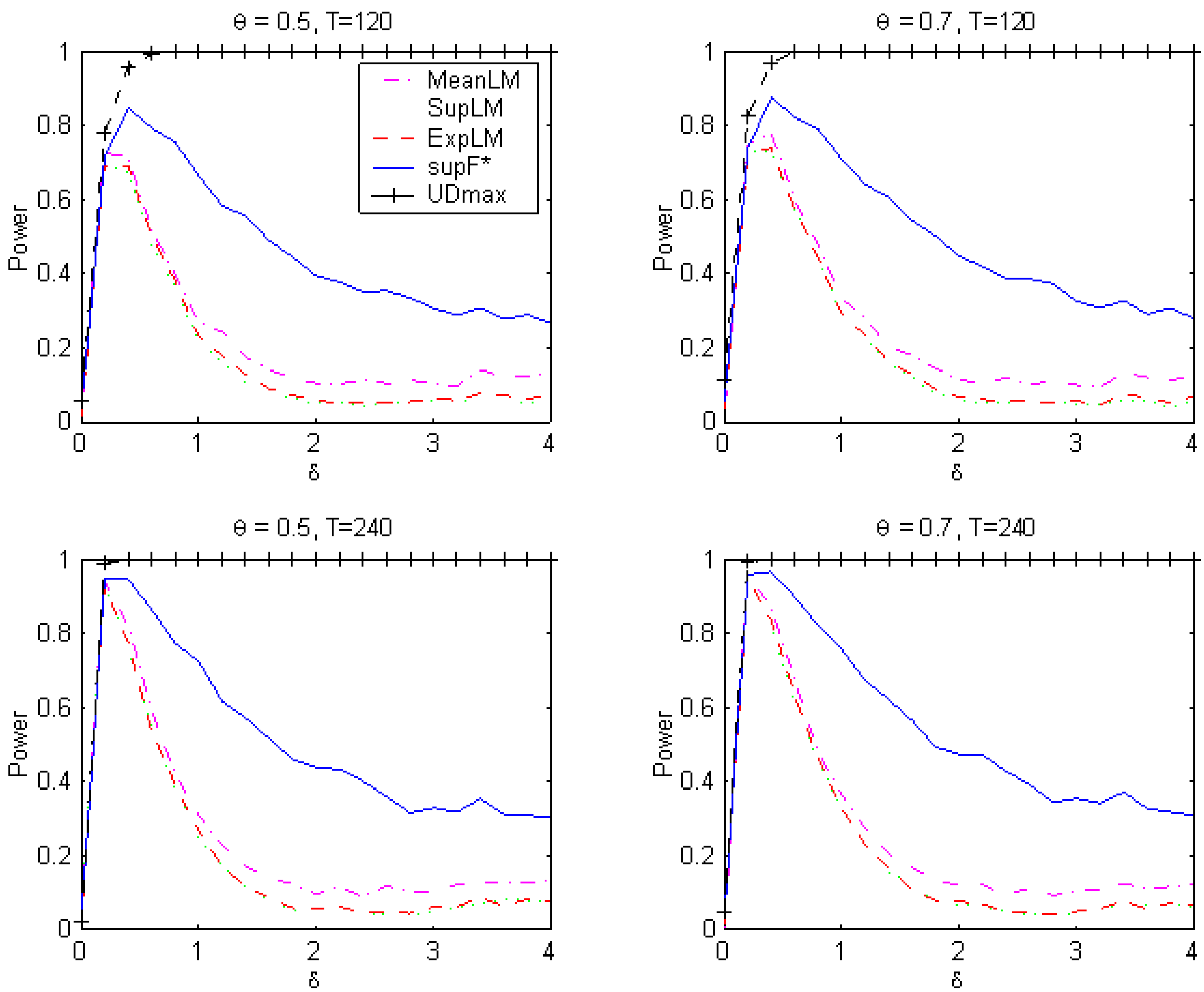

Figure 4: Power Functions with MA(1) Errors : DGP with 2 breaks

Conclusion

- Presented issues related to testing for multiple structural changes in cointegrated regression models.

- Provided a general framework which allows both $I(0)$ and $I(1)$ regressors.

- Proposed a sequential procedure to select the number of breaks.

- Proposed a modified Wald Test which can bypass the problem of non-monotonic power while retaining adequate size.

- Empirical Applications : The Feldstein-Horioka Puzzle (Kejriwal, 2006); Demand For Money; Taylor Rule; Term Structure of Interest Rates.