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Estimating the Joint Probability of Default using CDS and Bond Data

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Estimating the joint probability of default using CDS and bond data

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Abstract

Systemic default risk -i.e. the risk of simultaneous default of multiple institutions- has caused great concern in the recent past. The aim of this paper is to estimate the joint probability of default for couples of financial institutions. Both bond and credit derivative markets convey information on the default process: the former provides information on the *marginal* whilst the latter on the *joint* default probabilities. In this paper we will consider the corporate bond and the credit default swap (CDS) markets. The OTC nature of the CDS market implies the presence of counterparty risk, i.e. the risk that the protection seller will fail to fulfill its obligations. The counterparty risk is reflected in the CDS price through the joint default probability of the reference entity and the protection seller. Applying a no-arbitrage argument, we extract from market data forward-looking joint default probabilities of financial institutions operating in the CDS market during the period 03-Jan-2005–15-Mar-2010.

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1 Introduction

The determination of joint default probabilities is of extreme importance, having become more and more popular since the appearance on markets of credit derivatives and credit portfolio models. Bank supervision, i.e. the monitoring of financial system's health, has been playing an important role, being aimed to guard against systemic consequences of financial crises. Indeed, a joint default probability assessment is crucial for the pricing of credit derivatives, the valuation of credit risk exposure and the determination of systemic risk.

The valuation of credit derivatives strongly depends on the default correlation: the price is influenced by the interdependent default structures of the basket in multi-name products, but also by counterparty risk both in multi-name and single-name ones (see Schönbucher [31]).

To measure their risk exposure, large banks adopt portfolio credit risk models, which they provide the probability distribution of overall losses[1]. Correlated defaults are modeled via the common dependence on industry and country factors. If obligors have a strong dependency on the same systematic risk factors then they have a higher default probability.

Furthermore, the joint default probability provides a measure of systemic risk i.e. the risk of a trigger event, such as an economic shock or institutional failure, that causes a chain of bad economic consequences - domino effect - that could lead to significant losses or collapse of an entire financial system or market (see [15]). In other words, it refers to the risks imposed by interlinkages and interdependencies in a system or a market, where the failure of a single entity or a cluster of entities can cause a cascading failure, which could potentially bankrupt the entire system or market. The empirical relevance of this stylized fact has been clearly shown by Das et al. [10].

Controlling systemic risk is a major concern for governments and international organizations, particularly because consolidation in banking has led to the creation of some extremely large financial institutions. FED, ECB and the IMF have likewise expressed concern about potential systemic effects (see Board of Governors of the Federal Reserve System [5], European Central Bank [13] and IMF [21]). While contamination effects may also occur in other sectors of the economy, the likelihood and severity in the financial system is often regarded as considerably higher; a meltdown in it may have strong adverse consequences for the real economy and the general economic welfare.

It is worth mentioning that a clear distinction can be drawn among the previous extents. Pricing techniques are proper of the risk-neutral probability environment, while risk measurement is typically carried out with respect to the real-world probability measure. Our contribution lies within the former area of finance, being based on the assumption of absence of arbitrage in the market.

The reminder is as follows. Section 2 provides a review of the relevant contributions on our theme. Section 3 explores the relation between CDS and

[1] Popular credit risk models are Credit-Metrics^{TM} (JP Morgan), CreditPortfolioView^{TM} (McKinsey), PortfolioManager^{TM} (KMV), CreditRisk+^{TM} (CreditSuisse).

bond spreads. In Section 4 we derive the formula for the joint probability of default. Section 5 proposes an empirical application and Section 6 concludes.

2 Literature review

Different methods for the modeling of default dependency in both firm's value based and intensity based modeling framework have been developed[2].

Defaults in structural models are triggered by the change in the value of the assets of the firm. The firm asset values are correlated with each other and driven by common systemic factors. Conditional on the realization of the factors, the firm values and the defaults are independent. An interesting extension of Merton's framework [28] is the paper by Cathcart and El-Jahel [8], which proposes a close form solution to compute single as well as joint default probabilities in the case of more than two defaultable firms.

Credit risk dependence among firms within the framework of reduced-form models is incorporated through the conditionally independent defaults (CID) model where the default is prompted by the first jump of a multidimensional Cox process with correlated intensities. Conditional on the realization of the intensity processes, the single components of the Cox process are independent. Extensions of the this framework have been proposed by Davis and Lo [11] and Jarrow and Yu [23], which account for default clustering and default contagion.

Kunisch and Uhrig-Homburg [24] provide a new hybrid approach, in which the default of the obligors is mapped by using a marked Poisson process, while the credit event is based on a Merton-type condition on the value of the firms.

Finally, a copula approach (see [25] and [9]) is suitable for both type of model separating individual probability of default from the credit risk dependence structure. See Giesecke [16] as an example of reduced-form copula approach. Goodhart and Segoviano [32] extract from market data marginal probabilities of default for a pool of banks and employ a nonparametric copula approach to derive the probability of all the banks in the pool going into default, providing at the same time a measure of the tail risk of the system.

Indeed, it is possible to extract the implied joint default probability from market prices [30]. Both bond and credit derivative markets convey information on the default process: the former provides information on the *marginal* default probabilities whilst the latter on the *joint* default probabilities. Two contributions on this line of research are Brigo and Masetti [7] and Brigo and Chourdakis [6]. Brigo and Masetti [7] present a general pricing formula and derive the value of a defaultable claim as the sum of the value of the corresponding default-free claim minus a call option with zero strike on the residual value, called counterparty-risk credit valuation adjustment (CR-CVA). Brigo

[2] A fundamental distinction must be kept in mind talking about credit risk models: structural models and intensity or reduced-form models. The former introduced by Merton's seminal work [28], and further developed, among the others, by Black and Cox [3]; Longstaff and Schwart [26] relate credit risk to the capital structure of the firm. Reduced form models focus on the default intensity dynamic (see Duffie and Singleton [12] and Jarrow, Lando and Turnbull [22]).

and Chourdakis [6] investigate the impact of default correlation and credit spread volatility on the credit valuation adjustment, i.e. the quantity that one needs to subtract from the default-free price of a CDS to take into account the counterparty risk. To this purpose, they model the stochastic intensities of the counterpart and the reference entity through a CIR++ model and defaults are connected through a copula function on the exponential triggers of the default times. Finally, they compute by simulation credit valuation adjustments for different levels of correlations and intensity.

Giglio [17] observes the CDS price on the reference entity α with protection seller β and the price of bond issued by α to extract marginal and joint default probabilities for the couple (α, β). Finally, he proposes a linear programming model to derive upper and lower bounds for the joint default probability on N institutions. In line with this approach, applying a no-arbitrage argument, we derive a formula for the joint probability of default and extract from market data forward-looking joint default probabilities of institutions operating in the CDS market. The analysis of the dynamic of the joint default probability can provide clear signals of an increase in systemic risk and danger of contagion.

3 Cds premia and bond spreads

As a first approximation, CDS prices reflect the expected loss of the reference entity given by its default probability and the recovery rate. Additional risk premia are required to compensate for an unexpected default. See Amadei et al. [2] for a detailed discussion. These factors are actually the same that influence bond spreads: theoretically bond spreads should be equal to CDS premia for the same reference entity. For the sake of illustration, consider two financial agents α and β and let:

- $r(t, T)$ be the risk-free rate in t for the maturity T.
- $y_\alpha(t, T)$ be the yield at time t on a zero-coupon bond (ZCB) issued by α with maturity date T.
- $s_\alpha(t, T) \equiv y_\alpha(t, T) - r(t, T)$ be the spread over the risk-free rate of the issuance cost of α, prevailing in t and referred to the maturity T.
- $w_{\alpha,\beta}(t, T)$ be the periodic CDS premium to insure against the default of α within the period $[t, T]$ with β as the protection seller.

In equilibrium, a portfolio composed by a zero coupon bond with maturity T and a CDS on that same bond with the same maturity, should replicate a synthetic risk-free asset. Hence the ZCB yield $y_\alpha(t, T)$ minus the CDS premium $w_{\alpha,\beta}(t, T)$ should be exactly equal to the risk-free rate $r(t, T)$. The invoked equilibrium is ensured by the two following arbitrage strategies[3]:

[3]The portfolio payoffs are guaranteed for each strategy if and only if the positions are kept until bond maturity or until the credit event occurs. Otherwise the strategy faces a roll over risk in the financing/investing positions linked to the volatility of $r(t, T)$.

1. *Case* $w_{\alpha,\beta}(t,T) < s_\alpha(t,T)$: the arbitrage strategy in this case consists in buying the bond, financing at the risk-free rate $r(t,T)$ and then buying the CDS by paying the premium $w_{\alpha,\beta}(t,T)$. The portfolio return is $y_\alpha(t,T) - r(t,T) - w_{\alpha,\beta}(t,T) = s_\alpha(t,T) - w_{\alpha,\beta}(t,T) > 0$.

2. *Case* $w_{\alpha,\beta}(t,T) > s_\alpha(t,T)$: the arbitrage strategy in this case consists in short selling the bond, investing the proceeds at the risk-free rate of return $r(t,T)$ and selling protection in the CDS market to get the premium $w_{\alpha,\beta}(t,T)$. The portfolio return is $w_{\alpha,\beta}(t,T) + r(t,T) - y_\alpha(t,T) = w_{\alpha,\beta}(t,T) - s_\alpha(t,T) > 0$.

However, in reality $w_{\alpha,\beta}(t,T) \neq s_\alpha(t,T)$ and the basis is rarely zero. We then define as "basis" the difference $w_{\alpha,\beta}(t,T) - s_\alpha(t,T)$. From 2007, the basis for corporate debt has been mainly negative for reference entities rated BBB and below and moderately positive for high-quality reference entities (Amadei et al. [2]). The persistence of negative basis can be motivated by the failure in implementing the arbitrage strategy 1 due to difficulties in:

1. buying the bond and financing the position at the risk-free rate, in the presence of liquidity problems and high tensions in the interbank market.

2. buying the CDS, either for the lack of protection sellers or for the perception of a high counterparty risk linked to this contracts.

Hence, the reasons for the basis to be non-zero are mainly linked to differences in the liquidity of CDS and corporate bond markets, to counterparty risk and other market imperfections. As an example of the latter we can mention the different reactivity of CDS and bond markets to new information on a issuer. A negative or positive basis can reflect a different degree of adjustment between the two markets that arbitrage strategies correct only in the long run. A large body of literature has shown that CDS corporate markets have a leading position in the price discovery process i.e. the CDS prices variation anticipate the variations in bond prices which react with a temporary lag.

In the following, we concentrate our attention on counterparty risk, which is not present in bond markets, but crucially affects CDS markets; when counterparty risk increases the CDS premium decreases. CDS market is superior to bond and stock market in signaling early warnings of future changes in the creditworthiness of a reference entity. Furthermore, CDS quotes provide accurate estimates of the reference entity's credit risk over short time horizons ([4] [29] [20] [30]).

In order to take in account the liquidity premium influencing the basis, instead of modeling an unobservable liquidity process with strong assumptions (see [17]), we measure liquidity on the market by considering the spread between short-term interbank rates and the overnight rate. Such a spread is effectively an indicator of the liquidity available to the high-standing operators in the interbank markets, since a widening difference between the two testifies a lack of confidence in lending money on the market even over short-term maturities,

together with a flight to security in the form of overnight deposits at the lender of last resort.

4 A formula for the joint probability of default

In the previous section we have stated that in equilibrium the value of the basis should be zero. This holds when the counterparty risk is not explicitly taken into consideration. In the following, we relax this assumption: a negative basis, representing the counterparty cost, can still be consistent with a arbitrage-free valuation. Negative basis is typical of financial crisis periods and we will extract from it information on the joint default probability. A formula for measuring the joint probability of default of two financial institutions is hereby derived. This is first pursued in a one-period framework.

4.1 The one-period case

Consider two risky financial institutions and, for illustrative purpose, denote them as α and β. Consider a third party called γ and, for the sake of argument, assume that it can not go bankrupt. Imagine that at time $t = 0$, the riskless entity γ builds a portfolio, according to the following uniperiodal strategy:

Strategy 1 (*One-period case*):

- Buy a 1-year zero coupon bond (ZCB) issued by α,
- buy a 1-year CDS from β, the protection seller, on the reference entity α,
- finance the positions on the market with a 1-year loan.

All the contracts have a face value of \$1. Since we assume that γ is risk-free, it can finance its positions at the current 1-year risk-free rate. On the contrary the interest rate offered by α on the bond issue is increased by a spread related to its creditworthiness. Now let:

- RR_α, RR_β be the recovery rates of α and β, respectively.
- Π_t be the value at time t of the portfolio built by γ according to Strategy 1.

The present value of the bond issued by α is $e^{-r(0,1)-s_\alpha(0,1)} \approx 1 - r(0,1) - s_\alpha(0,1)$ and the amount of money been borrowed by γ is $e^{-r(0,1)} \approx 1 - r(0,1)$.

In Table 1 we report the value of the portfolio in two different points in time, $t = 0$ and $t = 1$, when all the cash flows are exchanged. The second column summaries, at time $t = 0$, the present value of the financial instruments constituting the portfolio held by γ. The portfolio value at time $t = 0$ is therefore:

$$\Pi_0 = w_{\alpha,\beta}(0,1) - s_\alpha(0,1) \tag{1}$$

The other columns of Table 1 report the cash flows in the different states of the world at time $t = 1$. The dash on the name of a financial institution stands for the institution being in default.

	$t = 0$	$t = 1$			
		α, β	$\overline{\alpha}, \beta$	$\alpha, \overline{\beta}$	$\overline{\alpha}, \overline{\beta}$
Loan	$-(1 - r(0,1))$	-1	-1	-1	-1
ZCB	$1 - r(0,1) - s_\alpha(0,1)$	1	RR_α	1	RR_α
CDS	$w_{\alpha,\beta}(0,1)$	0	$1 - RR_\alpha$	0	$(1 - RR_\alpha)RR_\beta$
Π_t	$w_{\alpha,\beta}(0,1) - s_\alpha(0,1)$	0	0	0	$-(1 - RR_\alpha)(1 - RR_\beta)$

Table 1: Portfolio value of the uniperiodal Strategy 1 at time $t = 0$ and $t = 1$.

If α survives, independently from β, γ repays its loan using the money stemming from the zero coupon bond, while the CDS expires. If α defaults and β survives, the position of γ is hedged by the CDS and the portfolio value is null.

A non-zero cash flow is generated only when both α and β default, that is with a probability equal to the joint default probability of α and β. If the probability is null, i.e. there is no counterparty risk, then the basis is zero. Under the hypothesis that $0 \leq RR_\alpha < 1$ and $0 \leq RR_\beta < 1$, this would result in a negative present value of the portfolio in $t = 0$, i.e. $\Pi_0 < 0$. If we think in term of cash flows, γ implement such a strategy whenever the basis is negative, since he will receive $-\Pi_0$ from this strategy. The negative basis correspond to the premium required for the potential loss due to the joint default of α and β. Thus within our framework we admittedly exclude the presence of a positive basis, since in that case γ would not implement such strategy. A positive basis typically conveys information linked to market imperfections, here not explicitly modeled (see Section 3).

Hence, the expected value of the portfolio at $t = 1$ is:

$$\mathbf{E}[\Pi_1] = -P_{\overline{\alpha},\overline{\beta}}(0,1)(1 - RR_\alpha)(1 - RR_\beta) \tag{2}$$

where we denote $P_{\overline{\alpha},\overline{\beta}}(0,1)$ as the risk-neutral 1-year joint default probability of α and β.

Recall that in an arbitrage-free world, for every security with value f_t at time $t \geq 0$, it must hold that:

$$f_t = \mathbf{E}_Q[f_T]e^{-r(t,T)(T-t)} \qquad \forall t < T \tag{3}$$

where Q is the risk-neutral probability measure.

That is, excluding arbitrage possibilities, in our case we must have:

$$\Pi_0 = \mathbf{E}[\Pi_1]e^{-r(0,1)}, \quad \Pi_0 < 0 \tag{4}$$

Combining Eq. 1, 2 and 4, we obtain:

$$P_{\overline{\alpha},\overline{\beta}}(0,1) = \frac{\Big(s_\alpha(0,1) - w_{\alpha,\beta}(0,1)\Big)^+}{(1 - RR_\alpha)(1 - RR_\beta)} e^{r(0,1)} \tag{5}$$

where $(\cdot)^+ \equiv \max(\cdot, 0)$.

According to Eq. 5 and given RR_α and RR_β, the risk neutral joint probability of default is explained by the negative basis. In absence of counterparty risk and arbitrage opportunities, this difference is supposed to be zero. The presence of counterparty risk reduces the premium of the CDS and motivates a negative basis (see Section 3). The wider is this difference, the higher is the counterparty risk and the higher is the joint probability of default of the bond issuer α and the protection seller β. If the basis is positive, we set the joint default probability equal to zero since we attribute a positive basis to market imperfections.

For further development, it can be useful to give a marginal default probability formula, which is coherent with our framework. First define $P_{\overline{\alpha}}$ and $P_{\overline{\beta}}$ as the risk-neutral probabilities of default of α and β, accordingly.

Applying the risk-neutral principle stated in Eq. 3 to the case of the CDS considered, we have[4]:

$$w_{\alpha,\beta}(0,1) = \Big(P_{\overline{\alpha},\beta}(0,1)(1-RR_\alpha) + P_{\overline{\alpha},\overline{\beta}}(0,1)(1-RR_\alpha)RR_\beta\Big)e^{-r(0,1)} \tag{6}$$

from which we get:

$$P_{\overline{\alpha},\beta}(0,1) = \frac{w_{\alpha,\beta}(0,1)e^{r(0,1)}}{1-RR_\alpha} - P_{\overline{\alpha},\overline{\beta}}(0,1)RR_\beta \tag{7}$$

Thus, using the trivial equality $P_{\overline{\alpha}}(\cdot,\cdot) = P_{\overline{\alpha},\overline{\beta}}(\cdot,\cdot) + P_{\overline{\alpha},\beta}(\cdot,\cdot)$, together with Eq. 5 and 7, after some rearranging, we get:

$$P_{\overline{\alpha}}(0,1) = \frac{s_\alpha(0,1)e^{r(0,1)}}{1-RR_\alpha} \tag{8}$$

With a similar argument we can state:

$$P_{\overline{\beta}}(0,1) = \frac{s_\beta(0,1)e^{r(0,1)}}{1-RR_\beta} \tag{9}$$

Note how the previous two formulas are consistent with the literature[5].

4.2 Multi-period case

To extend the formula in Eq. 5 over the generic time sequence $0 = t_0 < t_1 < \ldots < t_n$, we consider the case in which γ pursues the following multi-period rolling strategy:

Strategy 2 (*Multi-period case*): At time t_i with $i = 0, \ldots, n-1$, if both α and β are alive, then:

- Buy a $(t_{i+1} - t_i)$-year ZCB issued by α,

[4]See the fifth row in Table 1.

[5]Ignoring the discount factor we are exactly in line with Hull [19, Chap. 20].

- buy a $(t_{i+1} - t_i)$-year CDS[6] from β on α,
- finance the positions on the market with a $(t_{i+1} - t_i)$-year loan,

otherwise close the position.

On the basis of the usual no-arbitrage argument, we state that the actual expected cost of the strategy must equate the present value of its outcomes, that is:

$$\Big(s_\alpha(0,t_1) - w_{\alpha,\beta}(0,t_1)\Big)^+ t_1 +$$

$$+ \sum_{i=1}^{n-1} P_{\alpha,\beta}(t_{i-1},t_i)\Big(s_\alpha(t_i,t_{i+1}) - w_{\alpha,\beta}(t_i,t_{i+1})\Big)^+ (t_{i+1}-t_i)e^{-r(0,t_i)t_i} =$$

$$= (1-RR_\alpha)(1-RR_\beta)\sum_{i=1}^{n} P_{\overline{\alpha},\overline{\beta}}(t_{i-1},t_i)e^{-r(0,t_i)t_i} \tag{10}$$

The left-hand side of Eq. 10 refers to the expected premia of rolling Strategy 2, while the right-hand side is the expected present value of the flows generated in the case both α and β go bankrupt. According to Strategy 2, the amount $\Big(s_\alpha(t_i,t_{i+1}) - w_{\alpha,\beta}(t_i,t_{i+1})\Big)$ has to be understood as the premium required for assuming the risk of the joint default of α and β in the time interval $[t_i, t_{i+1}]$. As before, we argue that, since the recovered amounts RR_α and RR_β are positive, γ won't implement such a strategy whenever $s_\alpha(t_i,t_{i+1}) \leq w_{\alpha,\beta}(t_i,t_{i+1})$ [7]. This is the reason why in Eq. 10 we consider negative bases only.

From Eq. 10 we get:

$$\begin{aligned} P_{\overline{\alpha},\overline{\beta}}(t_{n-1},t_n) &= \tilde{P}_{\overline{\alpha},\overline{\beta}}(0,t_1)e^{r(t_1,t_n)(t_n-t_1)} + \\ &\sum_{i=1}^{n-1}\Big[P_{\alpha,\beta}(t_{i-1},t_i)\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i,t_{i+1})e^{r(t_{i+1},t_n)(t_n-t_{i+1})} + \\ &- P_{\overline{\alpha},\overline{\beta}}(t_{i-1},t_i)e^{r(t_i,t_n)(t_n-t_i)}\Big] \end{aligned} \tag{11}$$

in which we define:

$$\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i,t_{i+1}) \equiv \frac{\Psi(t_i,t_{i+1})}{(1-RR_\alpha)(1-RR_\beta)} \quad i = 0,\ldots,n-1 \tag{12}$$

where:

$$\Psi(t_i,t_{i+1}) \equiv \Big(s_\alpha(t_i,t_{i+1}) - w_{\alpha,\beta}(t_i,t_{i+1})\Big)^+ (t_{i+1}-t_i)e^{r(t_i,t_{i+1})(t_{i+1}-t_i)} \tag{13}$$

[6]The quotes $w_{\alpha,\beta}(t_{i+1},t_i)$ of the CDS involved in Strategy 2 are expressed in annual terms. Imagine that they are single-premium contracts and let this unique premium to be paid in t_i, at the beginning of the life of the contract. Thus the premium due in t_i is given by $w_{\alpha,\beta}(t_{i+1},t_i)(t_{i+1}-t_i)$.

[7]This corresponds to the case of positive basis. We assume that investors do not pay for market imperfections

and $s_\alpha(t_i, t_{i+1})$ and $w_{\alpha,\beta}(t_i, t_{i+1})$ might be understood as on forward basis (for further details see Section 5 and Appendix A). Thus, within our framework, $\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i, t_{i+1})$ corresponds to the joint default probability for the time interval $[t_i, t_{i+1}]$, conditionally to the survive of both α and β up to time t_i (compare Eq. 12-13 with Eq. 5). Henceforth we will refer to $\tilde{P}_{\overline{\alpha},\overline{\beta}}(\cdot,\cdot)$ as the conditional risk-neutral probability of joint default.

To grasp a better insight of the formula provided, notice that Eq. 11 leads to the following recursive relation:

$$P_{\overline{\alpha},\overline{\beta}}(t_i, t_{i+1}) = \begin{cases} \tilde{P}_{\overline{\alpha},\overline{\beta}}(0, t_1) & \text{for } i = 0 \\ P_{\alpha,\beta}(t_{i-1}, t_i)\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i, t_{i+1}) & \text{for } i = 1, \ldots, n-1 \end{cases} \tag{14}$$

which corresponds to the standard definition of unconditional probability. Thus, using Eq. 14 we can build a time structure of the risk-neutral probability of joint default between α and β. Notice that, in order to make Eq. 14 feasible in practice we need to use the trivial relation:

$$P_{\alpha,\beta}(\cdot,\cdot) = 1 - P_{\overline{\alpha}}(\cdot,\cdot) - P_{\overline{\beta}}(\cdot,\cdot) + P_{\overline{\alpha},\overline{\beta}}(\cdot,\cdot) \tag{15}$$

where:

$$P_{\overline{\alpha}}(t_i, t_{i+1}) = \begin{cases} \tilde{P}_{\overline{\alpha}}(0, t_1) & \text{for } i = 0 \\ P_{\alpha}(t_{i-1}, t_i)\tilde{P}_{\overline{\alpha}}(t_i, t_{i+1}) & \text{for } i = 1, \ldots, n-1 \end{cases} \tag{16}$$

with $P_\alpha(t_{i-1}, t_i) \equiv 1 - P_{\overline{\alpha}}(t_{i-1}, t_i)$ and in line with Eq. 8 we define:

$$\tilde{P}_{\overline{\alpha}}(t_i, t_{i+1}) \equiv \frac{s_\alpha(t_i, t_{i+1})}{1 - RR_\alpha}(t_{i+1} - t_i)e^{r(t_i, t_{i+1})(t_{i+1} - t_i)} \tag{17}$$

as the marginal conditional default probability within t_i and t_{i+1}. The same holds for the marginal probabilities referred to β.

4.3 The impact of the recovery risk

All the variables entering our definition of joint default probability stated in Eq. 14 are directly observable on the market, with the only exception of the recovery rates RR_α and RR_β. It would be desirable for our estimates not to be affected by arbitrary assumptions on them. The only hypothesis we are going to embrace is a widely accepted one, that is $0 \leq RR_\alpha < 1$ and $0 \leq RR_\beta < 1$.

Notice that in order for Eq. 12 to assume a probability meaning, we need to ensure that:

$$\Psi(t_i, t_{i+1}) \leq (1 - RR_\alpha)(1 - RR_\beta) \tag{18}$$

being the right-hand side unknown. Thus we seek a logistic transform $\mathcal{L}(\cdot)$ of $\Psi(t_i, t_{i+1})$ of the type:

$$\mathcal{L}\Big(\Psi(t_i, t_{i+1})\Big) \equiv \frac{a}{1 + \exp\{-\Psi(t_i, t_{i+1})\}} + b \tag{19}$$

where a and b are chosen such that:

$$\begin{cases} \mathcal{L}(0) = 0 \\ \lim\limits_{\Psi(t_i,t_{i+1})\to+\infty} \mathcal{L}\Big(\Psi(t_i,t_{i+1})\Big) = (1-RR_\alpha)(1-RR_\beta) \end{cases}$$

Hence we get $a = 2(1-RR_\alpha)(1-RR_\beta)$ and $b = -(1-RR_\alpha)(1-RR_\beta)$. If we substitute $\Psi(t_i,t_{i+1})$ with its logistic transform in Eq. 12, we get:

$$\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i,t_{i+1}) = \frac{2}{1+\exp\{-\Psi(t_i,t_{i+1})\}} - 1 \tag{20}$$

as our ultimate formula for the conditional risk-neutral probability of joint default within $[t_i, t_{i+1}]$.

Recalling Eq. 14, we can state that the these logistic transformations are sufficient to guarantee the consistency of the unconditional probability of joint default $P_{\overline{\alpha},\overline{\beta}}(t_i,t_{i+1})$, since $0 \le P_{\alpha,\beta}(t_i,t_{i+1}) \le 1$, $\forall i = 0,\ldots,n-1$.

For the same reasoning as above, we apply the logistic transform even to the marginal probabilities $\tilde{P}_{\overline{\alpha}}(t_i,t_{i+1})$ and $\tilde{P}_{\overline{\beta}}(t_i,t_{i+1})$ to get:

$$\tilde{P}_{\overline{\alpha}}(t_i,t_{i+1}) = \frac{2}{1+\exp\{-\Psi_\alpha(t_i,t_{i+1})\}} - 1 \tag{21}$$

$$\tilde{P}_{\overline{\beta}}(t_i,t_{i+1}) = \frac{2}{1+\exp\{-\Psi_\beta(t_i,t_{i+1})\}} - 1 \tag{22}$$

where

$$\Psi_\alpha(t_i,t_{i+1}) \equiv s_\alpha(t_i,t_{i+1})(t_{i+1}-t_i)e^{r(t_i,t_{i+1})(t_{i+1}-t_i)} \tag{23}$$

$$\Psi_\beta(t_i,t_{i+1}) \equiv s_\beta(t_i,t_{i+1})(t_{i+1}-t_i)e^{r(t_i,t_{i+1})(t_{i+1}-t_i)} \tag{24}$$

5 Empirical application

In this section we present the empirical application of the methodology outlined above. We present the case of two generic investment-grade financial institutions. We consider US data for bond and CDS markets. Our sample consists of daily observations form 03-Jan-2005 to 15-Mar-2010 (1279 observations), choosing 5 and 10 years as reference maturities. In Table 2 we report descriptive statistics for the variables employed in the estimation. Data series are plotted in Fig. 1-4. Details are reported in Appendix B.

We adjust the bond spreads to role out liquidity effects. Henceforth, we denote with $s_\alpha(0,5)$ and $s_\alpha(0,10)$ the liquidity-adjusted 5 and the 10 year bond spreads. For further details refer to Appendix B. The liquidity process and the bases are displayed in Fig. 5-7.

[Fig. 1-7 about here]

Variable	Mean	Standard deviation	Min	Max
$s_\alpha(0,5)$	1.400%	1.520%	0.230%	5.940%
$s_\alpha(0,10)$	1.540%	1.400%	0.350%	5.790%
$w_{\alpha,\beta}(0,5)$	0.870%	0.570%	0.290%	2.790%
$w_{\alpha,\beta}(0,10)$	0.960%	0.390%	0.530%	2.470%
$r(0,1)$	3.500%	1.760%	0.430%	5.760%
$r(0,5)$	4.160%	1.030%	1.910%	5.770%
$r(0,10)$	4.570%	0.790%	2.330%	5.930%

Table 2: Descriptive statistics.

Our goal is to estimate risk-neutral joint default probabilities for the periods $[0,5]$ and $[5,10]$. We then fix $t_0 = 0$, $t_1 = 5$, $t_2 = 10$ and apply the formula for conditional probabilities stated in Eq. 14, using the logistic transform in Eq. 20:

$$\tilde{P}_{\overline{\alpha},\overline{\beta}}(t_i, t_{i+1}) = \frac{2}{1 + \exp\{-\Psi(t_i, t_{i+1})\}} - 1 \quad i = 0, 1 \tag{25}$$

In order for the previous formula to stand, we have assumed that a unique CDS premium is paid in t_i for ensuring against the default of the reference entity over the period $[t_i, t_{i+1}]$. However, the CDS indexes in the dataset embed contracts in which a premium is paid annually. Thus we need to rewrite Eq. 13 as[8]:

$$\begin{aligned}\Psi(t_i, t_{i+1}) \equiv \Big(& s_\alpha(t_i, t_{i+1})(t_{i+1} - t_i) + \\ & - w_{\alpha,\beta}(t_i, t_{i+1}) A_{t_i}(t_i, t_{i+1} - 1)\Big)^+ e^{r(t_i, t_{i+1})(t_{i+1} - t_i)}\end{aligned} \tag{26}$$

where we define $A_{t_i}(t_i, t_{i+1} - 1)$ as the value in t_i of a stream of annual unit payments over the period $[t_i, t_{i+1} - 1]$, that is:

$$A_{t_i}(t_i, t_{i+1} - 1) \equiv \sum_{\tau = t_i}^{t_{i+1} - 1} e^{-r(t_i, \tau)(\tau - t_i)} \tag{27}$$

This corresponds to the case in which a unique discounted CDS premium is exchanged in t_i, while all the contracts are settled in t_{i+1}.

The default correlation between two financial institutions might be estimated via the standard binomial correlation approach[9]:

$$\rho_{\alpha,\beta} = \frac{P_{\overline{\alpha},\overline{\beta}} - P_{\overline{\alpha}} P_{\overline{\beta}}}{\sqrt{P_{\overline{\alpha}}(1 - P_{\overline{\alpha}}) P_{\overline{\beta}}(1 - P_{\overline{\beta}})}} \tag{28}$$

The results of the estimations are reported in Fig. 8-9. In the former graph we can see the 5-year and the 5-to-10-year probability having quite a similar

[8]The formula for $w_{\alpha,\beta}(5,10)$ is derived in Appendix A.

[9]See Lucas [27].

dynamic evolution. In particular they are almost null up to the first semester of the 2007. The negligibility of the joint default probability amongst investment-grade institutions over short-term maturities was empirically established by Erturk [14]. The fact that such a probability is negligible over both short and medium term maturities, as in the first part of our sample, is typical of the situation in which the economic and financial system is well functioning, with wide possibilities of diversification.

Afterwards, the estimated probabilities started to follow a marked upward trend to peak right in the middle of the recent US financial crisis, namely in the second half of the 2008. This peak witnesses the worsening of the credit conditions that affected even the high-standing financial institutions. In this sense the case of Lehman Brothers is always taken as an emblem. A joint default of two investment-grade financial institutions became less and less likely starting from mid-2009 as the contagion effect tends to disappear from the markets. The 5-to-10-year probability seem to be more persistent, being well above the 5-year estimates throughout the 2009, as to anticipate the enduring of the crisis. Similar considerations come from the analysis of Fig. 9, in which we can see the 5-year default correlation reaching a level of 50% during the most troublesome period, while the 5-to-10-year correlation peaked just below 80%.

[Fig. 8-9 about here]

We compare our results with a well-accepted reduced-form approach for estimating probabilities of joint default. According to Schönbucher [31] the joint default probability between two entities α and β over the generic time interval $[0, T]$ is given by:

$$P_{\overline{\alpha},\overline{\beta}}(0,T) = P_{\overline{\alpha}}(0,T) + P_{\overline{\beta}}(0,T) + \mathbf{E}\left[\exp\left\{-\int_0^T (\lambda_\alpha + \lambda_\beta)dt\right\}\right] - 1 \quad (29)$$

where λ_α and λ_β are the instantaneous default intensity of α and β accordingly. We then plug:

$$P_{\overline{\alpha}}(0,T) = 1 - \exp\left\{-\overline{\lambda}_\alpha T\right\} \quad (30)$$

considering the standard definition for $\overline{\lambda}_\alpha$ as[10]:

$$\overline{\lambda}_\alpha \equiv \frac{s_\alpha(0,T)}{1 - RR_\alpha} \quad (31)$$

and the same for $\overline{\lambda}_\beta$. Furthermore, we substitute:

$$\int_0^T (\lambda_\alpha + \lambda_\beta)dt = (\overline{\lambda}_\alpha + \overline{\lambda}_\beta)T \quad (32)$$

We implement this model using our dataset, assuming together with Altman and Vellore [1] a recovery rate of $RR_\alpha = RR_\beta = 41\%$ [11]. This value is very

[10] See Hull [19, Chap. 20].

[11] Altman and Vellore [1] show that "original rating, at least in terms of the broad investment-grade versus junk-bond categories, has no effect on recoveries once seniority is taken into account". Thus we use the mean of their overall sample.

close to what is usually assumed in the literature[12]. A comparison of the two approaches is provided in Fig. 10-11.

[Fig. 10-11 about here]

The two estimates are very close to each other. For both the probabilities the alternative method shows even more marked picks in correspondence to the recent financial meltdown. On the other hand, our estimates seem to be more reactive to changes in market conditions. Moreover, they seem to anticipate the benchmark at the beginning of the crisis. An explanation of this feature can be motivated by a more efficient use of the market information. The benchmark ignores the information of the CDS market which are extremely liquid and anticipate the bond market (see discussion in Section 3).

In Fig 12 we compare our estimates of $P_{\overline{\alpha},\overline{\beta}}(0,5)$ and the benchmark with the S&P500, that we consider as a good indicator for the timing of the financial crisis. The S&P fell from an all-time high of 1565 on October 9th, 2007 to a low of 667 on March 9th, 2009, a drop of 56.8%. In August 2007 the joint default probabilities started reacting, anticipating the subsequent fall in the stock index. Only few days after March 6th, 2009 we observe a rapid decrease in the joint probabilities, indicating the vanishing of market fear. It is clear a specular behaviour of the joint default probability with respect to the dynamic of the S&P500.

[Fig. 12 about here]

Another shortcoming of the alternative estimates is that they are highly sensitive to the recovery risk. In order to emphasize this aspect, we assume, again together with Altman and Vellore [1], that the standard deviation of the recovery rates is 25.56%. We then assume the recovery rates to be beta-distributed. This is coherent with what is usually done in the empirical literature on this topic (Gupton and Stein [18]). We estimate parameters using the method of moments and we built 95% confidence intervals for the benchmark probabilities. From Fig. 13-14 we can see how the estimates change depending on the assumption on the recovery rates.

[Fig. 13-14 about here]

6 Conclusions

Our methodology offers a reliable estimate of the joint default probability amongst different financial institutions. The joint default probability provides evidence of increasing systemic risk and danger of contagion from mid-2007 on. It is based on a strong argument under the hypothesis of no-arbitrage, it provides estimates that are coherent with a benchmark method and it shows to be very reactive to changes in market conditions. Moreover, we remove the dependence from recovery rate assumptions.

[12] See Hull [19, Chap. 20].

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7 Appendix

A Derivation of forward spreads

In the definition provided in Eq. 12-13 we introduced $s_\alpha(t_i, t_{i+1})$ and $w_{\alpha,\beta}(t_i, t_{i+1})$. The former is given by the standard relation:

$$s_\alpha(t_i, t_{i+1}) = \frac{t_{i+1}}{t_{i+1} - t_i} s_\alpha(0, t_{i+1}) - \frac{t_i}{t_{i+1} - t_i} s_\alpha(0, t_i) \tag{33}$$

For the latter, consider for simplicity the case of annual time instants $t = 0, 1, \ldots, T$ and CDS contracts in which a premium is paid annually. Thus in a free arbitrage world the following relation can be stated:

$$w_{\alpha,\beta}(0,T)\left(1 + \sum_{t=1}^{T-1} P_{\alpha,\beta}(t-1,t)e^{-r(0,t)t}\right) = \sum_{t=1}^{T} w_{\alpha,\beta}(t-1,t)e^{-r(0,t-1)(t-1)} \tag{34}$$

that is the expected actual value of the payments in a T-year contract must equate the actual value of T annual forward CDS with one-year maturity. From Eq. 34 we get:

$$\begin{aligned} w_{\alpha,\beta}(t,t+1) = {} & w_{\alpha,\beta}(0,t+1)e^{r(0,t)t} + \\ & \sum_{\tau=1}^{t} \Big[w_{\alpha,\beta}(0,t+1)P_{\alpha,\beta}(\tau-1,\tau)e^{-r(\tau-1,\tau)} + \\ & - w_{\alpha,\beta}(\tau-1,\tau)e^{-r(0,\tau-1)(\tau-1)} \Big] \end{aligned} \tag{35}$$

with $t = 0, \ldots, T-1$.

In the empirical application (see Section 5) we allow for a unique discounted CDS premium to be exchanged in every time tick $t = 0, 5, 10$. Thus in this case Eq. 34 becomes:

$$\begin{aligned} & w_{\alpha,\beta}(0,10)A_0(0,4) + w_{\alpha,\beta}(0,10)P_{\alpha,\beta}(0,5)A_0(5,9) = \\ & \quad = w_{\alpha,\beta}(0,5)A_0(0,4) + w_{\alpha,\beta}(5,10)A_0(5,9) \end{aligned} \tag{36}$$

Finally, we get:

$$\begin{aligned} & w_{\alpha,\beta}(5,10) = \\ & \quad \Big(w_{\alpha,\beta}(0,10) - w_{\alpha,\beta}(0,5) \Big) \frac{A_0(0,4)}{A_0(5,9)} + w_{\alpha,\beta}(0,10)P_{\alpha,\beta}(0,5) \end{aligned} \tag{37}$$

where the annuity present value $A_\cdot(\cdot,\cdot)$ is defined as in Eq. 27.

B Data Appendix

This appendix provides all the details about the time series considered in the empirical application in Section 5. We consider data for the US bond and CDS

Name	Description	Source	Ticker
Bond yields			
BOFA ML US CORP A-AAA 3-5Y ($)	3 to 5 yr YTM on A-AAA US corporate bond	Datastream^{TM}	MLA3A35(RY)
BOFA ML US CORP A-AAA 5-7Y ($)	5 to 7 yr YTM on A-AAA US corporate bond	Datastream^{TM}	MLC310$(RY)
BOFA ML US CORP A-AAA 7-10Y ($)	7 to 10 yr YTM on A-AAA US corporate bond	Datastream^{TM}	MLC410$(RY)
BOFA ML US CORP A-AAA 10-15Y ($)	10 to 15 yr YTM on A-AAA US corporate bond	Datastream^{TM}	MLCA115(RY)
BOFA ML US CORP BBB 3-5Y ($)	3 to 5 yr YTM on BBB US corporate bond	Datastream^{TM}	ML3B3T5(RY)
BOFA ML US CORP BBB 5-7Y ($)	5 to 7 yr YTM on BBB US corporate bond	Datastream^{TM}	ML3B5T7(RY)
BOFA ML US CORP BBB 7-10Y ($)	7 to 10 yr YTM on BBB US corporate bond	Datastream^{TM}	ML3B710(RY)
BOFA ML US CORP BBB 10-15Y ($)	10 to 15 yr YTM on BBB US corporate bond	Datastream^{TM}	ML3B115(RY)
CDS spreads			
CDX IG CDSI GENERIC 5Y Corporate	5 yr CDS spread on US IG corporate	Bloomberg^{TM}	CDX IG CDSI GENERIC 5Y Corp
CDX IG CDSI GENERIC 10Y Corporate	10 yr CDS spread on US IG corporate	Bloomberg^{TM}	CDX IG CDSI GENERIC 10Y Corp
Libor rates			
US00O/N Index	BBA LIBOR USD Overnight	Bloomberg^{TM}	US00O/N Index
US0003M Index	BBA LIBOR USD 3 month	Bloomberg^{TM}	US0003M Index
USSW1 Curncy	USD SWAP 1YR	Bloomberg^{TM}	USSW1 Curncy
USSW5 Curncy	USD SWAP 5YR	Bloomberg^{TM}	USSW5 Curncy
USSW10 Curncy	USD SWAP 10YR	Bloomberg^{TM}	USSW10 Curncy

Table 3: List of the time series considered in the application.

markets. Our sample consists of daily data form 03-Jan-2005 to 15-Mar-2010 (1279 observations). In Table 3 we report all the time series considered.

The bond spreads are computed as the difference between bond yields and risk-free rates.

As for the bond yields we consider the average maturities of the available bond data and we compute the bond yields in 5 and 10 years' time using linear interpolation. So, for instance, the 5-year A-AAA bond yield is computed as the interpolation between the BOFA ML US CORP A-AAA 3-5Y ($) and the BOFA ML US CORP A-AAA 5-7Y ($) yields, using their average maturity, that is 4 and 6 years accordingly. We do the same for the 10 years and the BBB data. To get the investment-grade yields, we then take a weighted average of the A-AAA and the BBB data. The weights used were 0.75 for the former and 0.25 for the latter, on the basis of the number of the ratings classes the two groups of data account for. We then subtract the 5 and the 10 year Libor accordingly, so that we get an estimate of the 5 and the 10 year bond spreads.

In order to take into account the impact of the liquidity on the basis (see Section 3), we subtract from $s_\alpha(0,5)$ and $s_\alpha(0,10)$ the spread between the 3 month USD Libor rate and the overnight USD Libor rate. We apply a Hodrick-Prescott filter to the rate series, in order to role out excess volatility and get a trend measure of the liquidity process. Since we are dealing with daily data we use a smoothing parameter $\lambda_{HP} = 6,030,364$, following the rule 100(number of periods in a year)^2.

As for the risk-free rate, we consider Libor rates. We use 1, 5 and 10 year Libor data series and we do interpolation for the intermediate maturities, when needed.

CDS data are referred to investment-grade reference entities. We don't have visibility on the rating class of the protection seller. Anyway it seems to be common sense to assume that the protection seller is investment-grade, too.

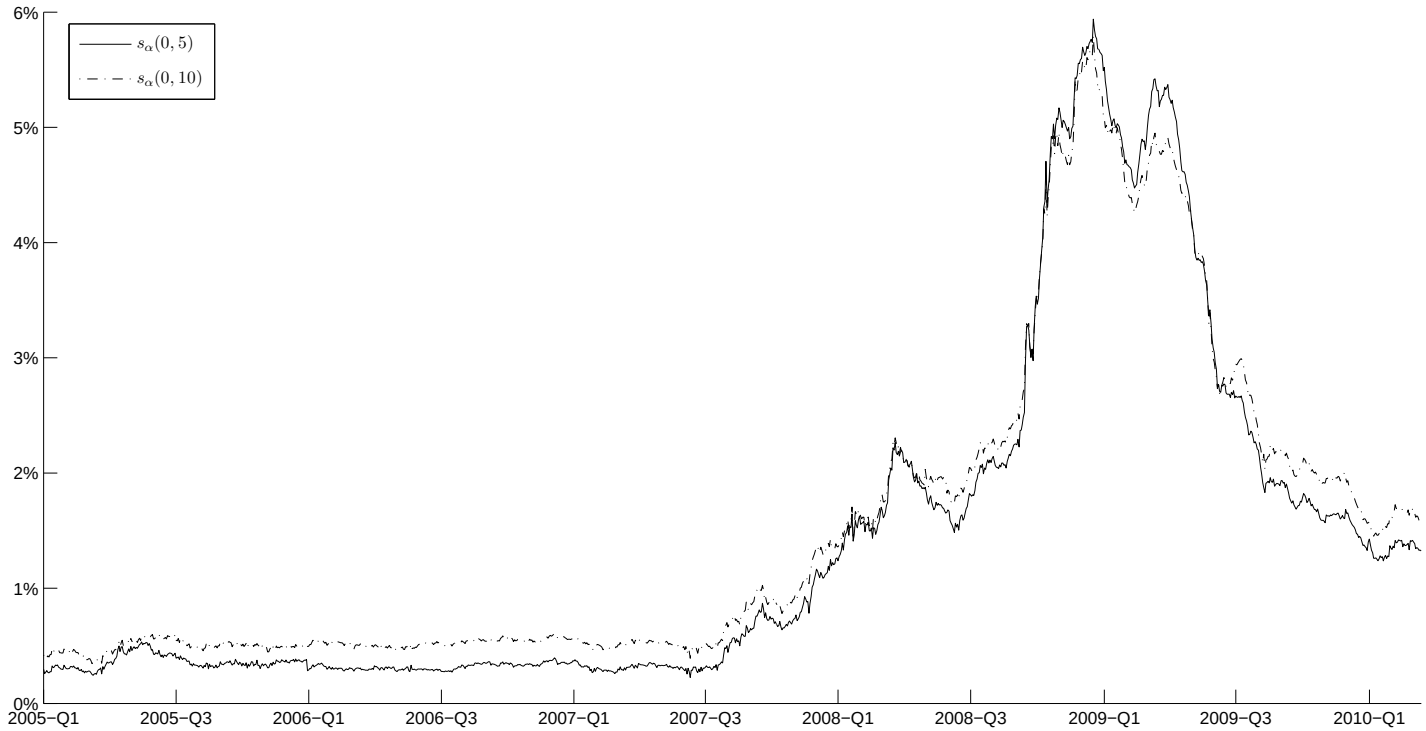

Figure 1: Data plot: bond spreads.

Figure 2: Data plot: CDS spreads.

Figure 3: Data plot: interest rates.

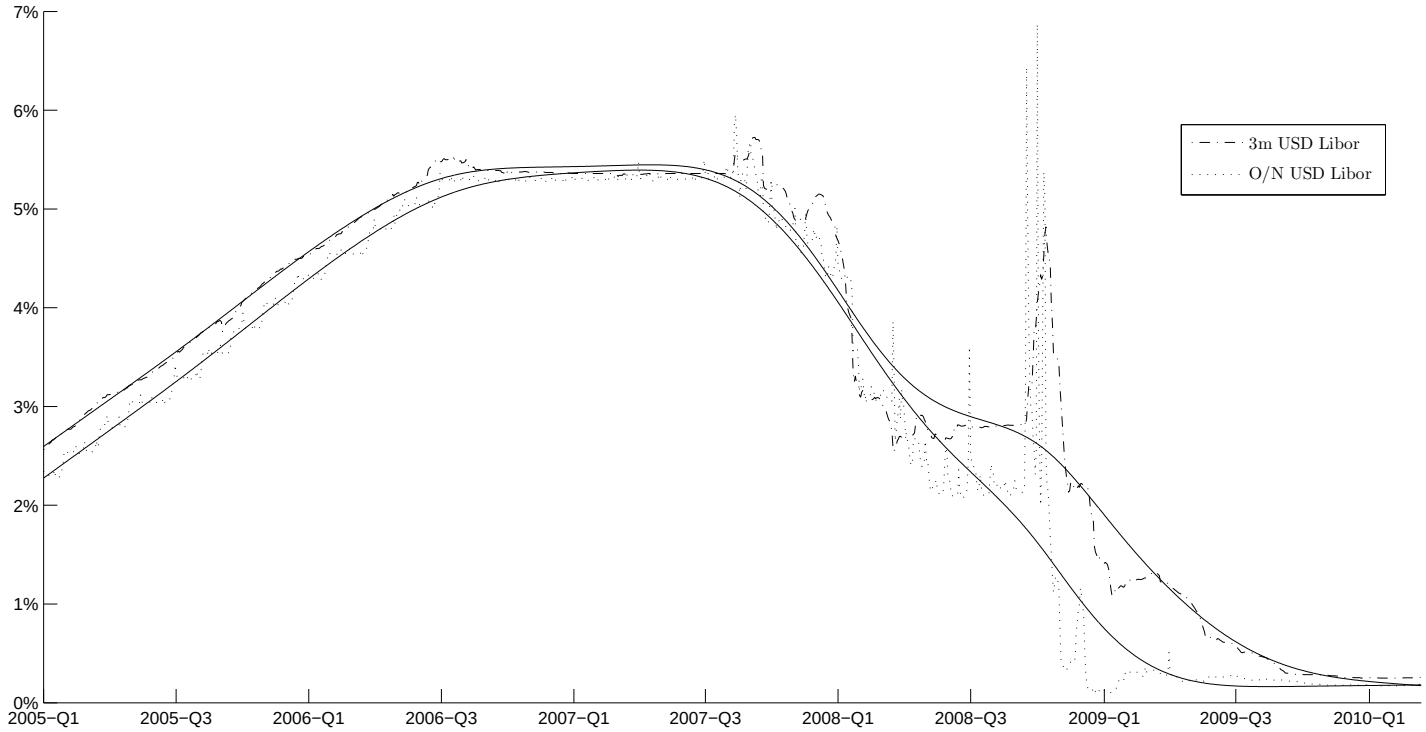

Figure 4: Data plot: 3 month USD Libor rate and overnight USD Libor rate (solid lines are the trend components extracted applying the Hodrick-Prescott filter with smoothing parameter $\lambda_{HP} = 6,030,364$).

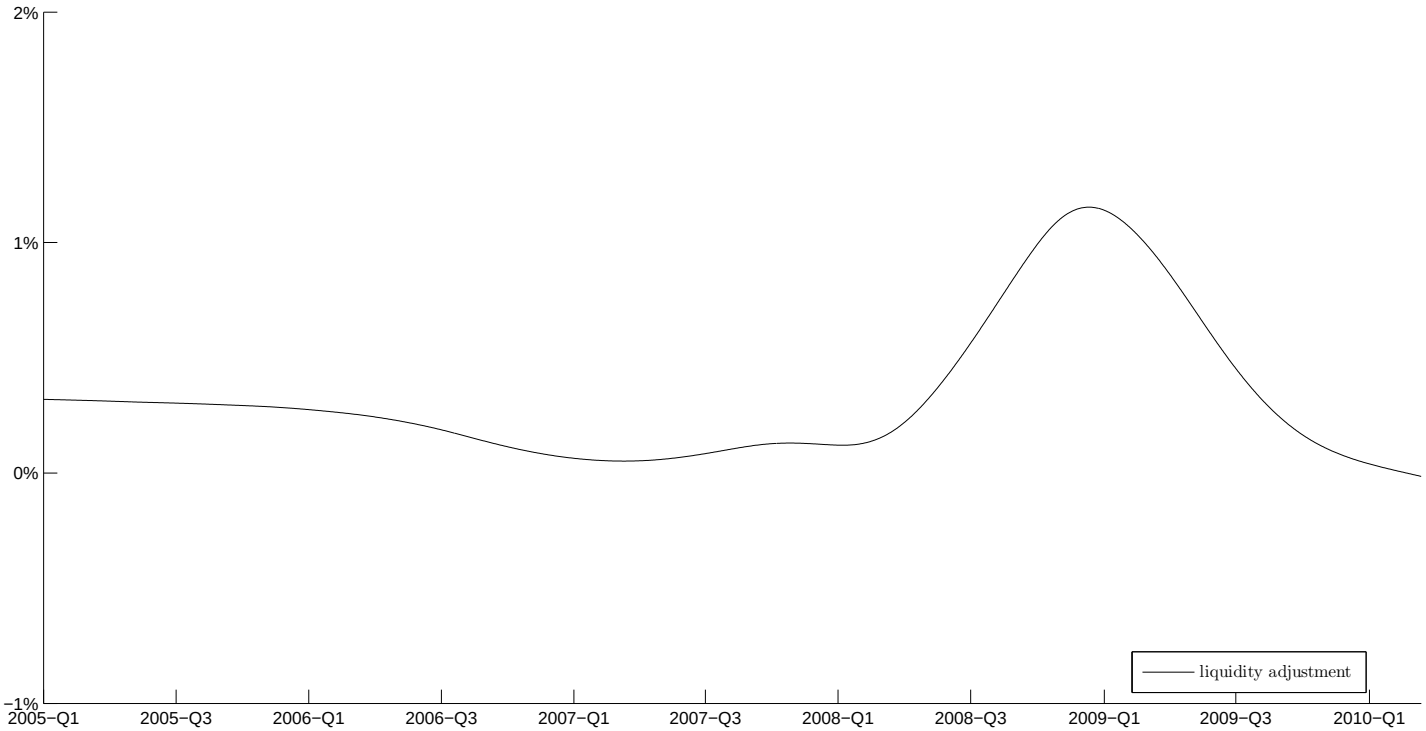

Figure 5: Liquidity adjustment.

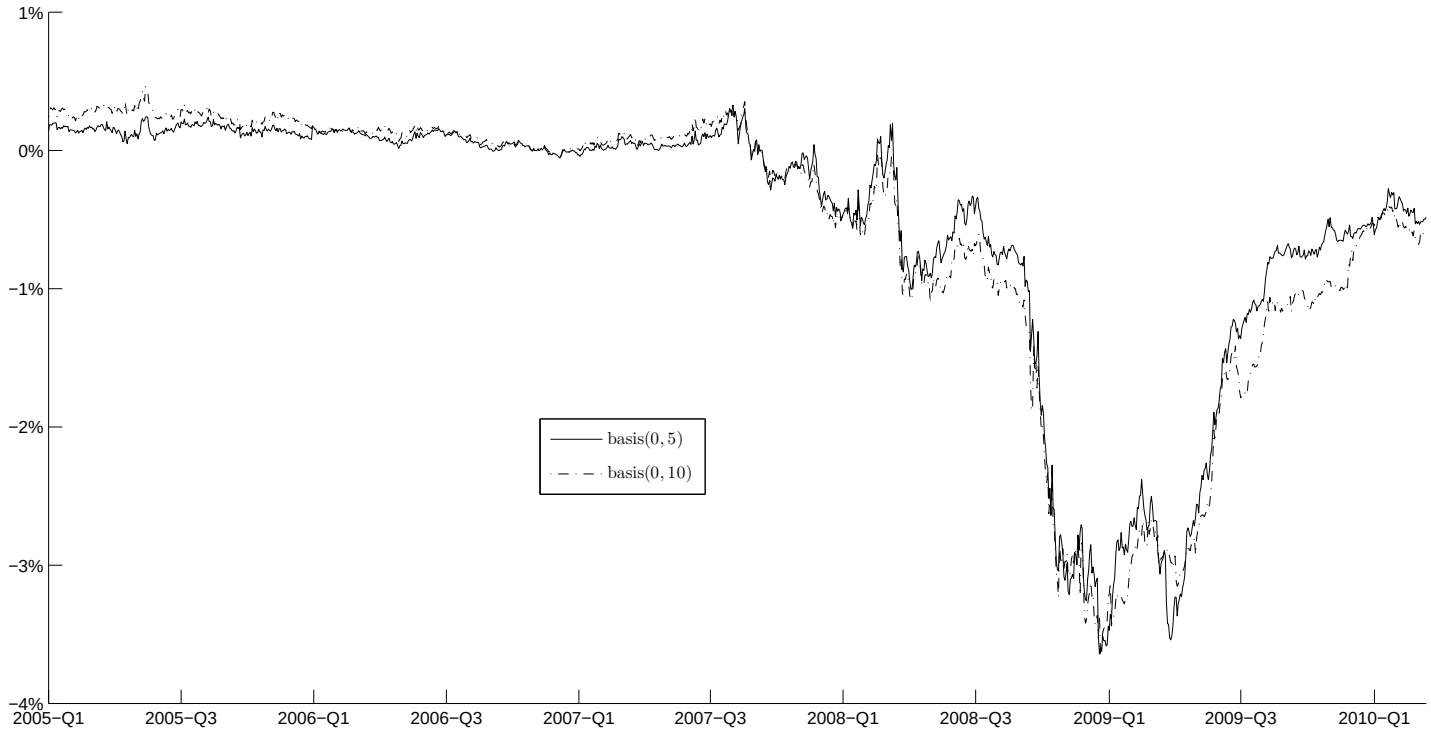

Figure 6: Unadjusted bases.

Figure 7: Liquidity-adjusted bases.

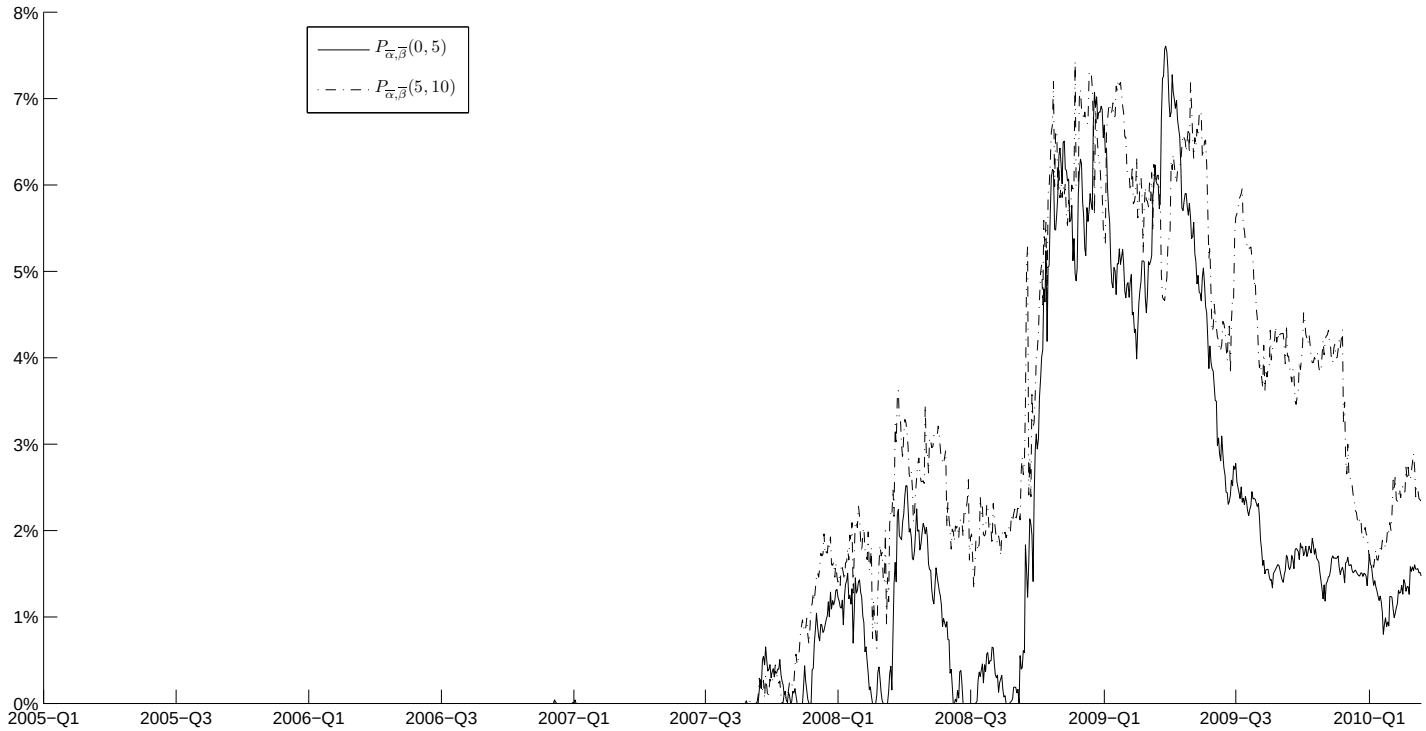

Figure 8: Estimates of the unconditional risk-neutral joint probabilities of default.

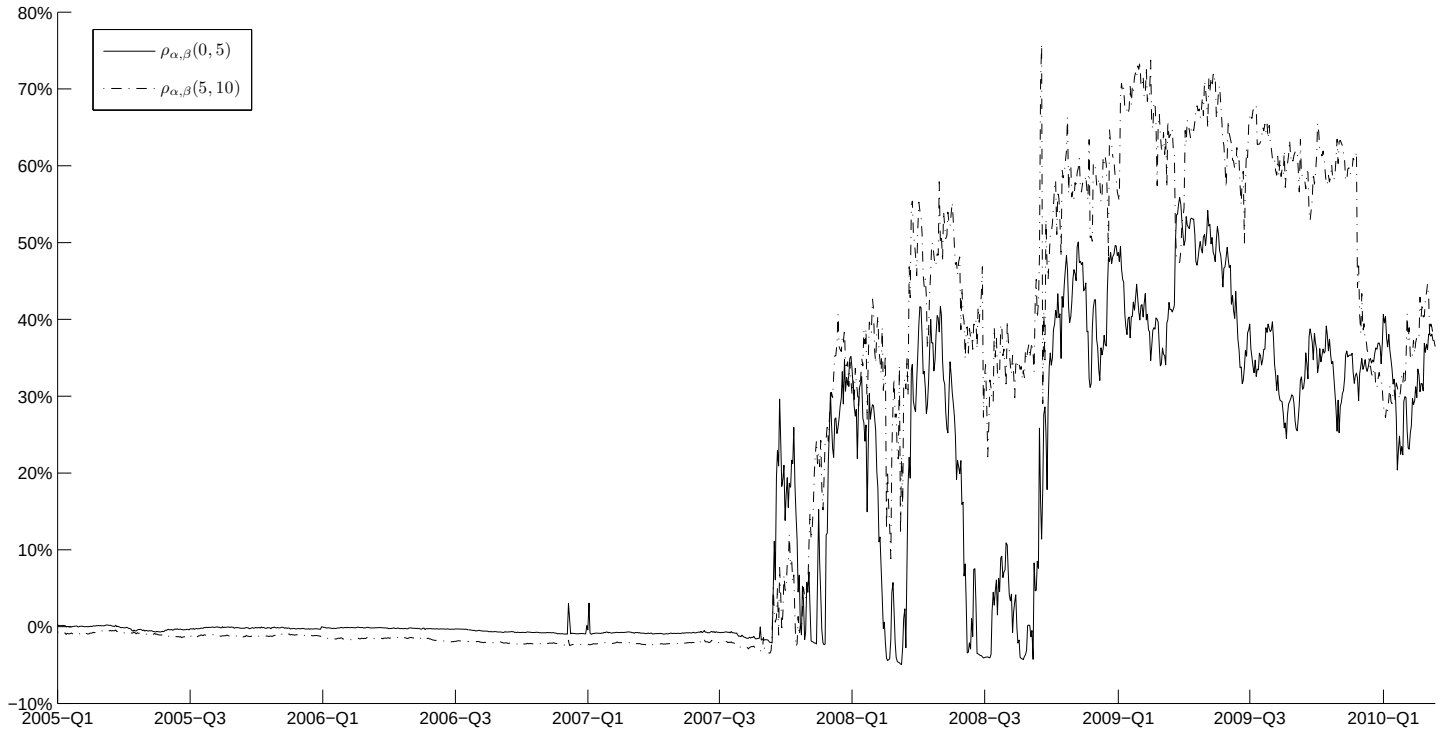

Figure 9: Estimates of the correlation of joint default.

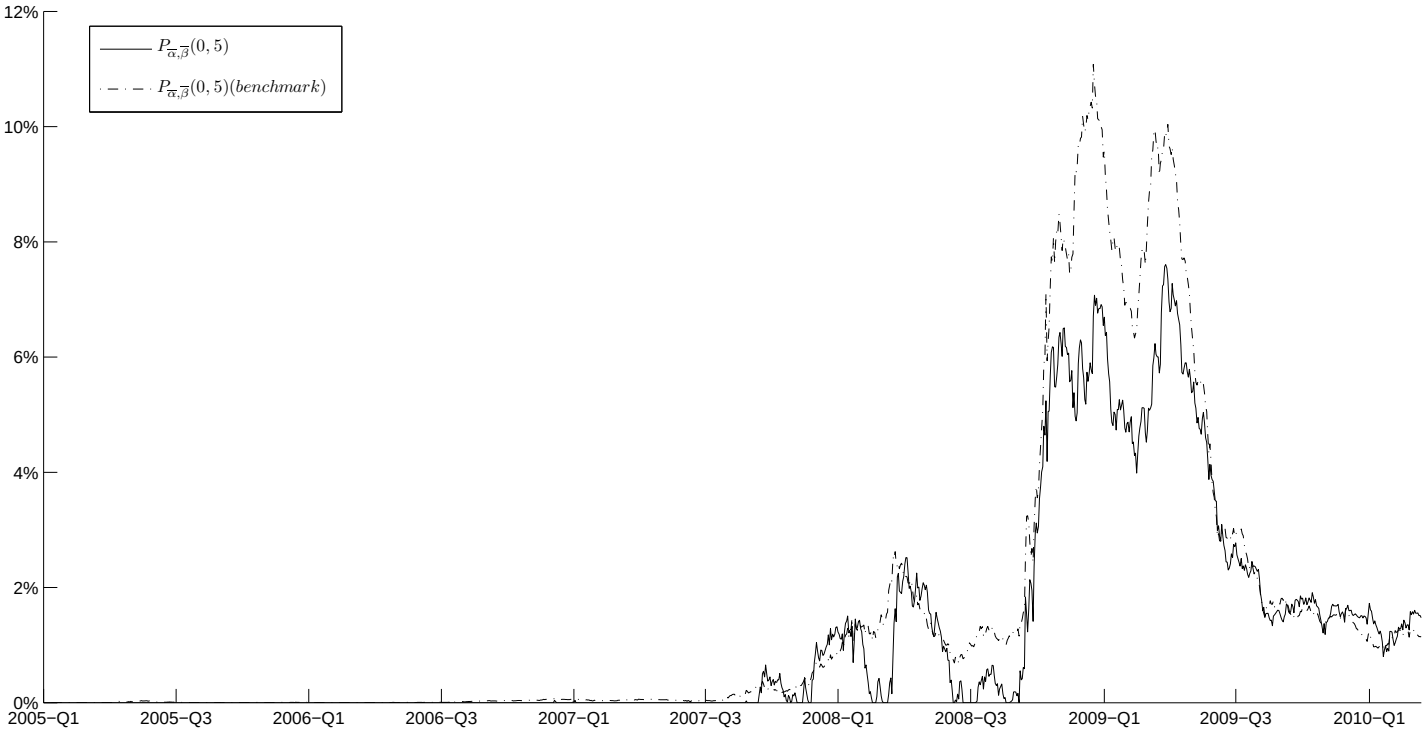

Figure 10: Estimates of the 5-year risk-neutral joint probability of default (comparison with the benchmark approach).

Figure 11: Estimates of the unconditional 5-to-10-year risk-neutral joint probability of default (comparison with the benchmark approach).

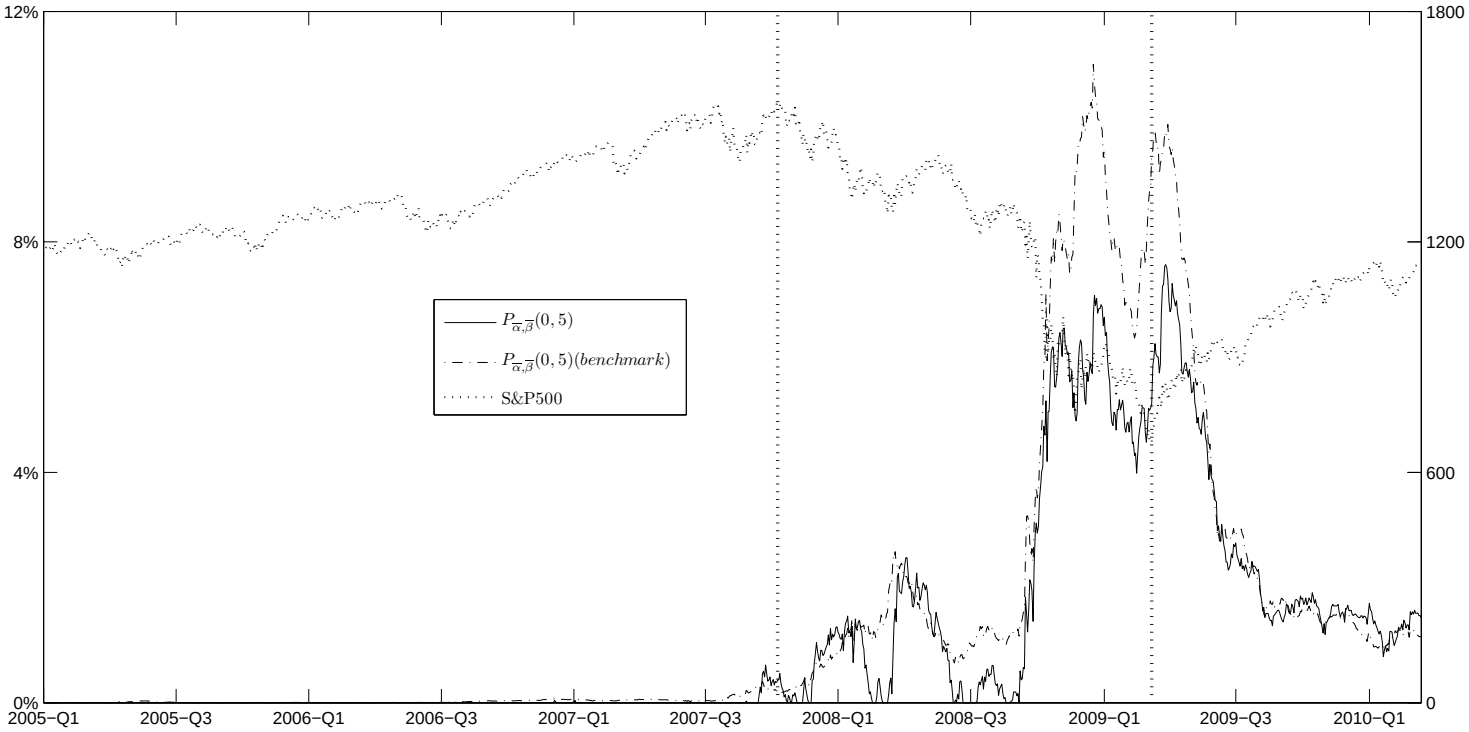

Figure 12: Comparison of our estimates and the benchmark estimates of $P_{\overline{\alpha},\overline{\beta}}(0,5)$ (left axis) with the S&P500 index (right axis). Vertical bars mark the peak and the bottom of the index.

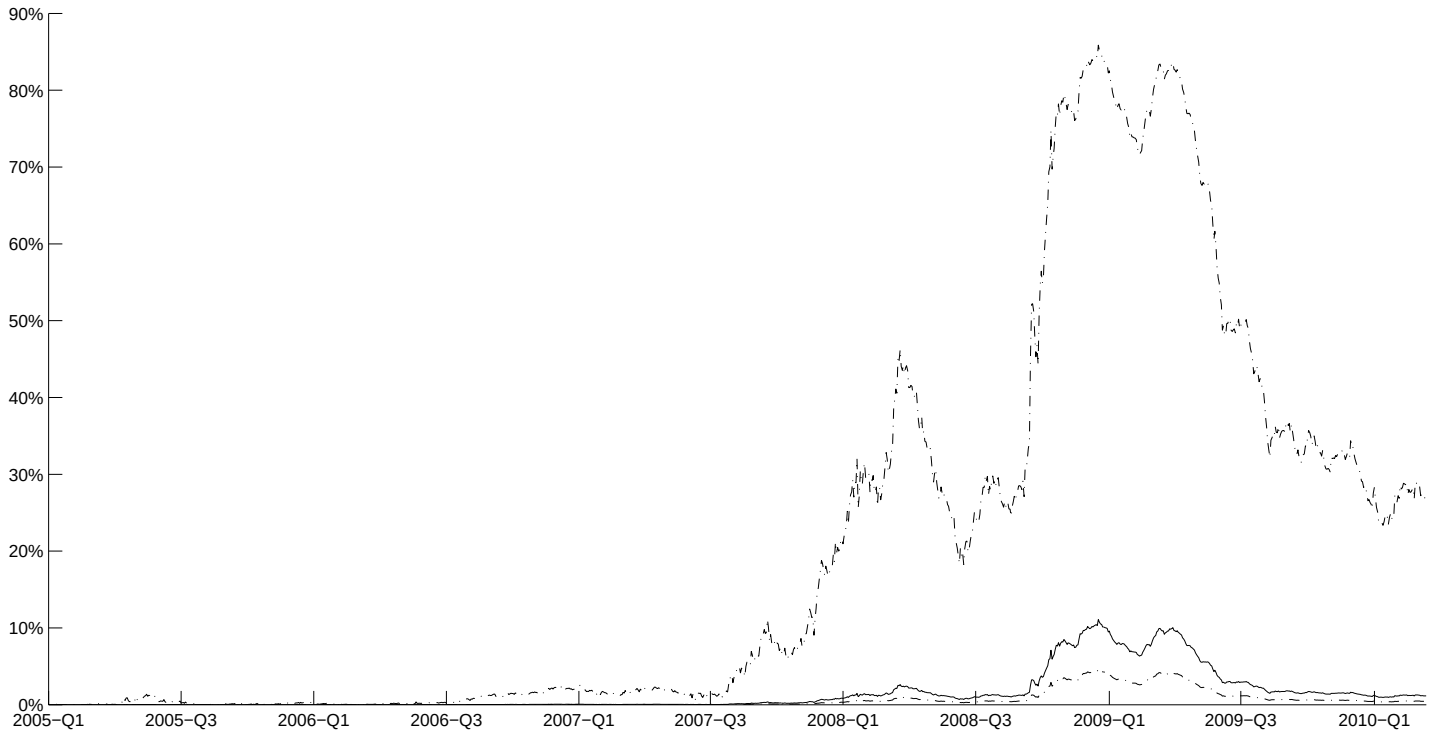

Figure 13: Estimates of the 5-year risk-neutral joint probability of default with 95% confidence interval (benchmark approach).

Figure 14: Estimates of the unconditional 5-to-10-year risk-neutral joint probability of default with 95% confidence interval (benchmark approach).