Tests for Cointegration with Structural Breaks Based on Subsamples

James Davidson and Andrea Monticini

University of Exeter

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Models with Structural Shifts

I(1) vector $\boldsymbol{x}_t = (x_{1t}, \boldsymbol{x}_{2t}')'$ $(p \times 1)$, where

$$\boldsymbol{x}_t = \boldsymbol{x}_0 + \sum_{s=1}^{t} \boldsymbol{u}_s,$$

$E|\boldsymbol{x}_0| < \infty$, $E(\boldsymbol{u}_t) = \boldsymbol{0}$, $\boldsymbol{\Sigma} = E(\boldsymbol{u}_t\boldsymbol{u}_t')$ and $\boldsymbol{\Lambda} = \sum_{j=1}^{\infty} E(\boldsymbol{u}_t\boldsymbol{u}_{t-j}') < \infty$, such that

$$T^{-1}E(\boldsymbol{x}_T - \boldsymbol{x}_0)(\boldsymbol{x}_T - \boldsymbol{x}_0)' \to \boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\Lambda} + \boldsymbol{\Lambda}'.$$

- Cointegration is the case $\boldsymbol{\Omega}$ singular:
- Cointegration implies $\exists\, \boldsymbol{\beta}_0 \neq \boldsymbol{0}$ such that

$$z_t = x_{1t} - \alpha_0 - \boldsymbol{\beta}_0'\boldsymbol{x}_{2t} \sim \mathrm{I}(0), \qquad \text{where } \alpha_0 = x_{10} - \boldsymbol{\beta}_0'\boldsymbol{x}_{20}$$

- Cointegration exists in modified form if

$$z_t = x_{1t} - (\alpha_0 + \alpha_1\varphi_t) - (\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\varphi_t)'\boldsymbol{x}_{2t}$$

  where

  - $\varphi_t = 1$ if $t \in A$ and 0 otherwise
  - A is a specified subset of contiguous observations, of $O(T)$.
  - Leading case: $A = \{[Tr] + 1, \ldots, T\}$ for $0 < r < 1$.

Cases

1. *Regime Shift*: $\boldsymbol{\beta}_1 \neq \mathbf{0}$: shift in cointegrating relation.

$$([Tr_1])^{-1}E(\boldsymbol{x}_{[Tr]} - \boldsymbol{x}_0)(\boldsymbol{x}_{[Tr]} - \boldsymbol{x}_0)' \to \boldsymbol{\Omega}_1$$
$$(T - [Tr_1])^{-1}E(\boldsymbol{x}_T - \boldsymbol{x}_{[Tr]})(\boldsymbol{x}_T - \boldsymbol{x}_{[Tr]})' \to \boldsymbol{\Omega}_2.$$

   where $\boldsymbol{\Omega}_0$ and $\boldsymbol{\Omega}_1$ are both singular, but

$$T^{-1}E(\boldsymbol{x}_T - \boldsymbol{x}_0)(\boldsymbol{x}_T - \boldsymbol{x}_0)' \to \boldsymbol{\Omega} = r\boldsymbol{\Omega}_1 + (1-r)\boldsymbol{\Omega}_2$$

   is in general nonsingular. Also note,

$$\alpha_1 = -\boldsymbol{\beta}_1'\boldsymbol{x}_{20}.$$

2. *Level Shift*: $\boldsymbol{\beta}_1 = \mathbf{0}$, but $\alpha_1 \neq 0$:
   Interpretation: temporary break in the cointegrating covariance structure with permanent effect. If

$$\delta_t = \alpha_1 \text{ if } t = [Tr] + 1, \text{ and } 0 \text{ otherwise}$$

   and

$$u_{1t} = \boldsymbol{\beta}_0'\boldsymbol{u}_{2t} + \Delta z_t + \delta_t$$

   then

$$z_t = \begin{cases} x_{1t} - \alpha_0 - \boldsymbol{\beta}_0'\boldsymbol{x}_{2t} & t \leq [Tr] \\ x_{1t} - (\alpha_0 + \alpha_1) - \boldsymbol{\beta}_0'\boldsymbol{x}_{2t} & t > [Tr] \end{cases}$$

Subsample Tests for Noncointegration

Consider subsample statistics defined for $\lambda_1 \in [0,1)$ and $\lambda_2 \in (\lambda_1, 1]$

- Dickey-Fuller *t* statistic

$$t_T(\lambda_1,\lambda_2) = \frac{\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]} \hat{z}_{t-1}(\lambda_1,\lambda_2)\Delta\hat{z}_t(\lambda_1,\lambda_2)}{s(\lambda_1,\lambda_2)\left(\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]} \hat{z}_{t-1}(\lambda_1,\lambda_2)^2\right)^{1/2}}$$

where

$$s^2(\lambda_1,\lambda_2) = \frac{1}{[T\lambda_2]-[T\lambda_1]} \sum_{t=[T\lambda_1]+1}^{[T\lambda_2]} \Delta\widehat{z}_t(\lambda_1,\lambda_2)^2$$

$$\widehat{z}_t(\lambda_1,\lambda_2) = x_{1t}^*(\lambda_1,\lambda_2) - \hat{\boldsymbol{\beta}}(\lambda_1,\lambda_2)'\mathbf{x}_{2t}^*(\lambda_1,\lambda_2), \quad t = [T\lambda_1]+1,\ldots,[T\lambda_2].$$

$$\mathbf{x}_t^*(\lambda_1,\lambda_2) = \mathbf{x}_t - \frac{1}{[T\lambda_2]-[T\lambda_1]} \sum_{s=[T\lambda_1]+1}^{[T\lambda_2]} \mathbf{x}_s, \quad t = [T\lambda_1]+1,\ldots,[T\lambda_2].$$

$$\hat{\boldsymbol{\beta}}(\lambda_1,\lambda_2) = \left(\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]} \mathbf{x}_{2t}^*(\lambda_1,\lambda_2)\mathbf{x}_{2t}^*(\lambda_1,\lambda_2)'\right)^{-1} \sum_{t=[T\lambda_1]+1}^{[T\lambda_2]} \mathbf{x}_{2t}^*(\lambda_1,\lambda_2)x_{1t}^*(\lambda_1,\lambda_2).$$

- Phillips-Perron (1988) non-parametric autocorrelation correction:

HAC covariance estimator

$$C_l(\lambda_1,\lambda_2) = \frac{1}{[T\lambda_2]-[T\lambda_1]} \sum_{j=1}^{l(\lambda_1,\lambda_2)} w_{jl} \sum_{t=[T\lambda_1]+j}^{[T\lambda_2]} \widehat{z}_t(\lambda_1,\lambda_2)\widehat{z}_{t-j}(\lambda_1,\lambda_2) \qquad (*)$$

where (e.g.)

$$w_{lj} = 1 - j/(1 + l(\lambda_1,\lambda_2)), \qquad l(\lambda_1,\lambda_2) = O(([T\lambda_1]-[T\lambda_2])^{1/3})$$

$$\begin{aligned}\hat{Z}_T(\lambda_1,\lambda_2) &= \frac{\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]}(\hat{z}_{t-1}(\lambda_1,\lambda_2)\Delta\hat{z}_t(\lambda_1,\lambda_2) - C_l(\lambda_1,\lambda_2))}{S_l^2(\lambda_1,\lambda_2)\left(\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]}\hat{z}_{t-1}(\lambda_1,\lambda_2)^2\right)^{1/2}} \\ &= \frac{s^2(\lambda_1,\lambda_2)}{S_l^2(\lambda_1,\lambda_2)} t_T(\lambda_1,\lambda_2) - \frac{[T(\lambda_1-\lambda_2)]C_l(\lambda_1,\lambda_2)}{S_l^2(\lambda_1,\lambda_2)\left(\sum_{t=[T\lambda_1]+1}^{[T\lambda_2]}\hat{z}_{t-1}(\lambda_1,\lambda_2)^2\right)^{1/2}}\end{aligned}$$

where

$$S_l^2(\lambda_1,\lambda_2)^2 = s^2(\lambda_1,\lambda_2) + 2C_l(\lambda_1,\lambda_2).$$

  - Asymptotically equivalent to $t_T(\lambda_1,\lambda_2)$ in case $\boldsymbol{u}_t$ is serially uncorrelated.

- Use $q_T(\lambda_1,\lambda_2)$ in the sequel as generic notation for cointegration statistic,

Subsampling Schemes

We consider the class of tests based on extreme values of subsample statistics over alternative of sets of (λ_1, λ_2) values:

Split Sample:

$$\{\lambda_1, \lambda_2\} \in \Lambda_s = \{\{0, \tfrac{1}{2}\}, \{\tfrac{1}{2}, 1\}\}$$

$$\{\lambda_1, \lambda_2\} \in \Lambda_{0r}^* = \Lambda_{0r} \cup \{0, 1\}$$

Incremental Samples (λ_0):

Forwards:

$$\{\lambda_1, \lambda_2\} \in \Lambda_{0f} = \{0, [\lambda_0, 1]\}$$

Backwards

$$\{\lambda_1, \lambda_2\} \in \Lambda_{0b} = \{[0, 1 - \lambda_0], 1\}$$

Rolling Samples (λ_0) :

$$\{\lambda_1, \lambda_2\} \in \Lambda_{0r} = \{[0, 1 - \lambda_0], \lambda_0 + \lambda_1]\}$$

$$\{\lambda_1, \lambda_2\} \in \Lambda_s^* = \Lambda_s \cup \{0, 1\}$$

Subsample Extremum Tests

1. Split Sample Tests

$$Q_{sT} = \min_{\{\lambda_1,\lambda_2\}\in\Lambda_s} q_T(\lambda_1,\lambda_2)$$

$$Q^*_{sT} = \min_{\{\lambda_1,\lambda_2\}\in\Lambda^*_s} q_T(\lambda_1,\lambda_2).$$

2. Incremental Tests (λ_0) :

$$Q_{iT}(\lambda_0) = \inf_{\lambda\in\Lambda_{0f}\cup\Lambda_{0b}} q_T(\lambda_1,\lambda_2).$$

3. Rolling sample tests (λ_0) :

$$Q_{rT}(\lambda_0) = \inf_{\lambda\in\Lambda_{0r}} q_T(\lambda_1,\lambda_2)$$

$$Q^*_{rT}(\lambda_0) = \inf_{\lambda\in\Lambda_{0r}} q_T(\lambda_1,\lambda_2)$$

Asymptotic Analysis

Define, for $r \in [0,1]$:

$$\boldsymbol{X}_T(r) = T^{-1/2}\boldsymbol{x}_{[Tr]} \qquad \boldsymbol{B}(r),\ \text{BM with covariance matrix } \boldsymbol{\Omega}$$

$$\boldsymbol{J}_T(r) = T^{-1}\sum_{t=2}^{[Tr]} \boldsymbol{x}_{t-1}\boldsymbol{u}_t'. \qquad \boldsymbol{J}(r) = \int_0^r \boldsymbol{B}d\boldsymbol{B}' + r\boldsymbol{\Lambda}$$

Assumption 1

$$(\boldsymbol{X}_T, \mathrm{Vec}\,\boldsymbol{J}_T) \overset{d}{\to} (\boldsymbol{B}, \mathrm{Vec}\,\boldsymbol{J}) \tag{1}$$

where ‘ $\overset{d}{\to}$ ’ denotes joint weak convergence with respect to the Skorokhod metric on $D[0,1]^{p(1+p)}$.

Assumption 2

$$\frac{1}{T}\sum_{j=1}^{l(0,1)} w_{lj} \sum_{t=1+j}^{T} \boldsymbol{u}_t\boldsymbol{u}_{t-j}' \overset{pr}{\to} \boldsymbol{\Lambda}$$

where $l()$ and w_{lj} are defined in (*) and ‘ $\overset{pr}{\to}$ ’ denotes convergence in probability.

- Conditions suffcient for (1) given in (e.g.) Kurtz and Protter (1991), Hansen (1992).
- Limits lie in $C[0,1]$, so Skorkhod convergence equivalent to uniform convergence

Basic Idea: For given (λ_1, λ_2), subsample statistics are continuous functionals of $(\boldsymbol{B}, \boldsymbol{J})$.

Let $\boldsymbol{W}$ denote a standard p-vector BM having variance matrix $\boldsymbol{I}_p$.

Define:

$$\boldsymbol{W}^*(r) = \boldsymbol{W}(r) - \boldsymbol{W}(\lambda_1) - \int_{\lambda_1}^{\lambda_2} \boldsymbol{W} ds, \qquad \lambda_1 \leq r \leq \lambda_2$$

$$\boldsymbol{\zeta}(\lambda_1, \lambda_2) = \left(\int_{\lambda_1}^{\lambda_2} \boldsymbol{W}_2^* \boldsymbol{W}_2^{*\prime} dr \right)^{-1} \int_{\lambda_1}^{\lambda_2} \boldsymbol{W}_2^* W_1^* dr.$$

$$\boldsymbol{\xi}(\lambda_1, \lambda_2) = (1, \ -\boldsymbol{\zeta}(\lambda_1, \lambda_2)')' \quad (p \times 1)$$

Similarly, let

$$\int_{\lambda_1}^{\lambda_2} \boldsymbol{W}^* d\boldsymbol{W}' = \int_{\boldsymbol{\lambda_1}}^{\boldsymbol{\lambda_2}} (\boldsymbol{W}(r) - \boldsymbol{W}(\lambda_1)) d\boldsymbol{W}' - (\boldsymbol{W}(\boldsymbol{\lambda}_2) - \boldsymbol{W}(\boldsymbol{\lambda}_1)) \int_{\lambda_1}^{\lambda_2} \boldsymbol{W} ds.$$

If $\boldsymbol{W} = \boldsymbol{L}'^{-1}\boldsymbol{B}$ where $\boldsymbol{\Omega} = \boldsymbol{L}'\boldsymbol{L}$, further note

$$\int_{\boldsymbol{\lambda_1}}^{\boldsymbol{\lambda_2}} \boldsymbol{W} d\boldsymbol{W}' = \boldsymbol{L}^{-1}[\boldsymbol{J}(\lambda_2) - \boldsymbol{J}(\lambda_1) - (\lambda_2 - \lambda_1)\boldsymbol{\Lambda}]\boldsymbol{L}'^{-1}.$$

Under Assumptions 1and 2, CMT implies

$$\hat{Z}_T(\lambda_1,\lambda_2) \overset{d}{\to} \tau(\lambda_1,\lambda_2) = \frac{\boldsymbol{\xi}(\lambda_1,\lambda_2)' \int_{\lambda_1}^{\lambda_2} \boldsymbol{W}^* d\boldsymbol{W} \boldsymbol{\xi}(\lambda_1,\lambda_2)}{\sqrt{\boldsymbol{\xi}(\lambda_1,\lambda_2)'\boldsymbol{\xi}(\lambda_1,\lambda_2)}\sqrt{\boldsymbol{\xi}(\lambda_1,\lambda_2)' \int_{\lambda_1}^{\lambda_2} \boldsymbol{W}^* \boldsymbol{W}^{*\prime} dr \boldsymbol{\xi}(\lambda_1,\lambda_2)}}$$

Remark:
Marginal distribution of $\tau(\lambda_1,\lambda_2)$ is standard DF distribution, independent of λ_1 and λ_2.

Theorem

Under Assumptions 1 and 2, the weak convergence specified in

$$\inf_{\{\lambda_1,\lambda_2\}\in\Lambda} \hat{Z}_T(\lambda_1,\lambda_2) \overset{d}{\to} \inf_{\{\lambda_1,\lambda_2\}\in\Lambda} \tau(\lambda_1,\lambda_2)$$

holds for the cases $\Lambda = \Lambda_s$ and $\Lambda = \Lambda_s^*$, and also $\Lambda = \Lambda_{0f} \cup \Lambda_{0b}$, $\Lambda = \Lambda_{0r}$ and $\Lambda = \Lambda_{0r}^*$ for any $\lambda_0 > 0$.

Regressors	**Type**	λ_0	**50%**	**10%**	**5%**	**2.5%**	**1%**
1	Q_{sT}	–	–2.480	–3.336	–3.621	–3.835	–4.156
	Q^*_{sT}	–	–2.620	–3.464	–3.745	–3.992	–4.308
	$Q_{iT}(\lambda_0)$	0.5	–3.200	–4.049	–4.275	–4.530	–4.784
		0.35	–3.403	–4.206	–4.454	–4.670	–4.948
		0.2	–3.564	–4.329	–4.566	–4.78	–5.041
		0.1	–3.702	–4.433	–4.648	–4.863	–5.143
	$Q_{rT}(\lambda_0)$	0.5	–3.336	–4.168	–4.419	–4.621	–4.898
	$Q^*_{rT}(\lambda_0)$	0.5	–3.365	–4.167	–4.395	–4.624	–4.881
2	Q_{sT}	–	–2.895	–3.746	–4.010	–4.274	–4.568
	Q^*_{sT}	–	–3.041	–3.873	–4.129	–4.385	–4.641
	$Q_{iT}(\lambda_0)$	0.5	–3.636	–4.461	–4.717	–4.944	–5.200
		0.35	–3.801	–4.595	–4.842	–5.034	–5.281
		0.2	–3.949	–4.718	–4.957	–5.142	–5.407
	$Q_{rT}(\lambda_0)$	0.5	–3.729	–4.542	–4.794	–4.990	–5.242
	$Q^*_{rT}(\lambda_0)$	0.5	–3.757	–4.542	–4.794	–5.018	–5.270

Critical Values ($T = 1000$, 40,000 replications)

Monte Carlo Experiments

DGPs:

$$\Delta x_{1t} = \gamma z_{t-1} + u_{1t}, \qquad \Delta x_{2t} = -\gamma z_{t-1} + u_{2t}$$

where $\gamma > 0$, $u_{1t}, u_{2t} \sim NI(0,1)$ and (in any period t)

$$z_t = x_{2t} - \alpha - \beta x_{1t}.$$

- Note that

  $$z_t = (1 - \gamma(1+\beta))z_{t-1} + u_{1t} - \beta u_{2t}.$$

  The model is therefore cointegrating if

  $$\beta \neq 0, \quad 0 < \gamma(1+\beta) < 2$$

- In the case $\beta = 0$, x_{2t} is stationary subject to $0 < \gamma < 2$, while x_{1t} contains a unit root.

- In the experimental model, we allow structural change by replacing α and β by

  $$\alpha_T(r) = \alpha_1 \varphi_{[Tr]} \qquad \beta_T(r) = 1 + \boldsymbol{\beta}_1 \varphi_{[Tr]}$$

  where

  $$\varphi_{[Tr]} = 1 \text{ for } r_1 \leq r \leq r_2, \text{ 0 otherwise}$$

  Thus, the model allows one break ($0 < r_1 < 1$, $r_2 = 1$) or two breaks. ($0 < r_1 < r_2 < 1$).

Points

- "Rejection" in tables means

  1. Did not reject I(1) hypothesis at 5% level (ADF) for at least one variable in the set.
  2. Conducted cointegration test with normalized variable = variable with largest ADF.
  3. Rejected at 5% level.

- Break points r_1 and r_2 are randomized in replications: drawn from $U[0,1]$ distribution.

- Boldface in tables shows maximum of row.

	γ	ADF	Q_{sT}	Q^*_{sT}	$Q_{iT}(0.5)$	$Q_{rT}(0.5)$	$Q^*_{rT}(0.5)$	I(1)
$\boldsymbol{H_0}$	0	0.016	**0.052**	0.037	0.031	0.033	0.036	0.99
Case 1	0.05	**0.38**	0.14	0.28	0.19	0.08	0.12	0.95
No Break	0.1	**0.88**	0.53	0.84	0.78	0.35	0.64	0.90
	0.2	**0.84**	0.84	0.84	0.84	0.83	0.84	0.84
	0.5	**0.92**	0.92	0.92	0.92	0.92	0.92	0.92
Case 2	0.05	0.06	**0.12**	0.12	0.12	0.07	0.08	1
1 Intercept Break	0.1	0.15	0.41	0.37	**0.59**	0.31	0.35	0.99
$\alpha_1 = 10$	0.2	0.21	0.94	0.93	**0.95**	0.91	0.91	0.97
$r_1 \sim U[0,1]$	0.5	0.12	**0.98**	0.98	0.97	0.98	0.98	0.98
Case 3	0.05	0.06	0.08	**0.09**	0.07	0.05	0.05	0.99
2 Intercept Breaks	0.1	0.15	0.25	0.23	**0.27**	0.21	0.23	0.98
$\alpha_1 = 10$	0.2	0.18	0.57	0.54	0.55	**0.69**	0.67	0.96
$r_1, r_2 \sim U[0,1]$	0.5	0.14	0.60	0.60	0.60	**0.78**	0.78	0.98

$\cdots$Continued

	$\boldsymbol{\gamma}$	ADF	Q_{sT}	Q^*_{sT}	$Q_{iT}(0.5)$	$Q_{rT}(0.5)$	$Q^*_{rT}(0.5)$	I(1)
Case 4	0.05	0.04	**0.07**	0.07	0.07	0.06	0.06	0.99
1 Slope Break	0.1	0.09	0.24	0.24	**0.26**	0.15	0.18	0.98
$\beta_1 = -1$	0.2	0.11	**0.53**	0.52	0.51	0.49	0.49	0.97
$r_1 \sim U[0,1]$	0.5	0.09	0.54	0.55	0.55	0.55	**0.58**	0.98
Case 5	0.05	**0.32**	0.14	0.25	0.19	0.07	0.11	0.96
1 Slope break	0.1	0.70	0.55	**0.72**	0.70	0.36	0.56	0.93
$\beta_1 = 1$	0.2	0.73	0.88	**0.89**	0.89	0.87	0.88	0.90
$r_1 \sim U[0,1]$	0.5	0.49	0.85	0.85	**0.93**	0.89	0.91	0.96

	γ	ADF	$Q_{iT}(0.5)$	$Q_{iT}(0.35)$	$Q_{iT}(0.2)$	$Q_{iT}(0.1)$	I(1)
H_0	0	0.016	0.031	0.032	0.055	**0.10**	0.99
Case 2	0.05	0.06	**0.12**	0.10	0.11	0.15	1
1 Intercept Break	0.1	0.15	**0.59**	0.50	0.47	0.45	0.99
$\alpha_1 = 10$	0.2	0.21	**0.95**	0.95	0.96	0.95	0.97
$r_1 \sim U[0,1]$	0.5	0.12	0.97	**0.99**	0.98	0.98	0.98
Case 3	0.05	0.06	0.07	0.07	**0.08**	0.13	0.99
2 Intercept Breaks	0.1	0.15	0.27	**0.28**	0.26	0.28	0.98
$\alpha_1 = 10$	0.2	0.18	0.55	0.68	**0.71**	0.74	0.96
$r_1, r_2 \sim U[0,1]$	0.5	0.14	0.60	0.79	**0.92**	0.95	0.98