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Biffis E. and Denuit.

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Cass Business School
106 Bunhill Row
London EC1Y 8TZ
T +44 (0)20 7040 8470
www.cass.city.ac.uk

LEE-CARTER GOES RISK-NEUTRAL

AN APPLICATION TO THE ITALIAN ANNUITY MARKET

Enrico Biffis

Faculty of Actuarial Science and Statistics

Cass Business School

London EC1 Y8TZ, United Kingdom

E-mail: enrico.biffis.1@city.ac.uk

Michel Denuit

Institut de Statistique

Université Catholique de Louvain

Voie du Roman Pays, 20

B-1348 Louvain-la-Neuve, Belgium

E-mail: denuit@stat.ucl.ac.be

Abstract. We consider a class of stochastic intensities of mortality that generalizes the model proposed by Lee and Carter (1992), allowing general diffusions to drive the mortality time-trend. We analyze the stability of such class of intensities under measure changes and show how a risk-neutral version of the generalized Lee-Carter model can be employed for fair valuation purposes. We provide an example of model calibration based on the Italian annuity market.

Keywords: stochastic mortality, Lee-Carter model, mortality projections, fair valuation, longevity risk.

1. Introduction and Motivation

In the last decade, the model proposed by Lee and Carter (1992) (referred to as LC henceforth) has acquired increasing relevance among demographers and insurance practitioners. In particular, its description of the secular change of mortality as a function of a single time-index has proved effective for mortality projections (see, for example: Lee, 2000; Pitacco, 2004; Wong-Fupuy and Haberman, 2004). The last few years have also witnessed a considerable effort to

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take up the concept of no-arbitrage in actuarial valuations, partly prompted by the proposals of the International Accounting Standards Board (IASB), partly favored by the convergence of methodologies employed in finance and insurance (see Milevsky and Promislow, 2001; Biffis, 2004; Dahl, 2004, and references therein).

In this work, we bridge the gap between demographic and actuarial analyses based on the LC model and stochastic models aimed at fair valuation. We do so by consistently specifying the behavior of the LC model both in the physical world and in the so-called risk-neutral world, i.e. the world in which insurance security prices grow on average at the risk-free rate, thus requiring an adjustment in the intensity of mortality to reflect the investors' risk-aversion toward mortality risk.

The work is organized as follows: In Sec. 2, we first describe a stochastic mortality model based on an information structure quite common in the credit-risk literature. We then consider a continuous-time version of the LC model that allows the intensity of mortality to be driven by general diffusions. Following Biffis, Denuit and Devolder (2005), we describe a class of measure changes under which stochastic intensities of mortality remain of the LC type. Sec. 3 shows how the mortality setup of Sec. 2 can be nested into a stochastic model for the risk-neutral valuation of mortality-contingent securities. We show how the dynamics of insurance securities are affected by the measure changes introduced above and describe a rich class of mortality risk-premiums accounting for different sources of risk, in the spirit of IASB (2004). In Sec. 4, we focus on the behavior of LC intensities in the risk-neutral world. We show how to mimic the market practice of specifying risk-loaded intensities by suitably adjusting a reference physical intensity of mortality, eventually identifying the adjustments by calibration to observed security prices. A numerical example concerning the Italian annuity market is provided in Sec. 5. We calibrate risk-neutral LC intensities and estimate the margins to be incorporated into a classical LC model fitted to Italian population data. Finally, Sec. 6 summarizes our findings.

2. A Generalized Lee-Carter Model

In this section, we describe the information structure of the mortality model, which is based on the subfiltration approach of Jeanblanc and Rutkowski (2000) (see also Lando, 1998). We then consider a generalized version of the LC model, which allows the secular change in mortality to be described as a function of a diffusion process. A class of equivalent probability measures is then described, preserving the LC form of the intensity and allowing a rich class of mortality-risk premiums to be introduced in Sec. 3.2.

2.1. **Mortality Model.** Given a filtered probability space[1] $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T^*]}, \mathbb{P})$, we consider the $\mathbb{F}$-stopping times $(\tau^x)_{x\in I}$ representing the random lifetimes of individuals with ages in $I = \{x^1, \dots, x^n\}$ at the reference time 0. In other words, for each t and x, the σ-field $\mathcal{F}_t$ carries enough information to tell whether τ^x has occurred or not by time t. For each t, we take $\mathcal{F}_t$ to be structured as $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, with $\mathcal{G}_t$ strictly included in $\mathcal{F}_t$ and $\mathcal{H}_t = \vee_{x\in I}\mathcal{H}_t^x$, with $\mathcal{H}_t^x = \sigma(1_{\tau^x \le s} : 0 \le s \le t)$ for each t and x. In what follows, s, t and T will denote times in the compact interval $[0, T^*]$, while the filtrations $(\mathcal{F}_t)_{t\in[0,T^*]}$, $(\mathcal{G}_t)_{t\in[0,T^*]}$ and $(\mathcal{H}_t^x)_{t\in[0,T^*]}$ will be denoted by $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{H}^x$ respectively.

We see $\mathbb{G}$ as a filtration carrying the *available* information about relevant factors driving the evolution of mortality. Since it is a strict subfiltration of $\mathbb{F}$, we have that the $(\tau^x)_{x\in I}$ are *not* stopping times with respect to the filtration $\mathbb{G}$, which can thus be seen as carrying information about the likelihood of deaths happening, but not about their actual occurrence. For simplicity, we will take $\mathbb{G}$ to be the augmented filtration generated by a d-dimensional Brownian motion W.

For each x, we consider the jump process $N_t^x = 1_{\tau^x \le t}$ associated with τ^x. We say that τ^x admits the intensity μ^x if the compensated jump process $M_t^x = N_t^x - \int_0^{t\wedge\tau^x} \mu_s^x \mathrm{d}s$ is an $\mathbb{F}^x$-martingale, where we set $\mathbb{F}^x \doteq \mathbb{G}\vee\mathbb{H}^x$ and $t\wedge\tau^x = \min(t, \tau^x)$. To achieve greater tractability, we take each μ^x to be $\mathbb{G}$-predictable and set

$$\tau^x = \inf\left\{t : \int_0^t \mu_s^x \mathrm{d}s > \Theta^x\right\} \qquad \text{for} \quad x \in I \tag{1}$$

[1]In the sequel, all filtrations are assumed to satisfy the usual conditions, i.e. right-continuity and $\mathbb{P}$-completeness (see Protter, 2004, p. 3).

with $(\Theta^x)_{x\in I}$ independent unit exponential random variables, also independent of $\mathcal{G}_{T^*}$.

The main consequence of such setup is that the stopping times $(\tau^x)_{x\in I}$ are conditionally independent given $\mathcal{G}_{T^*}$ and that for all t and each x, τ^x is conditionally independent of $\mathcal{G}_{T^*}$, given $\mathcal{G}_t$. As a result, tractability can be achieved at expense of generality, in that every $\mathbb{G}$-martingale is also an $\mathbb{F}$-martingale. In particular, W is not just a Brownian motion with respect to $\mathbb{G}$, but also with respect to $\mathbb{F}$. Similarly, each M^x is an $\mathbb{F}$-martingale and not just an $\mathbb{F}^x$-martingale.

Given the above assumptions, one can show (e.g. Lando, 1998) that for all $T \geq t$ the following expression holds for the $\mathcal{F}_t$-conditional survival probability of each stopping-time τ^x

$$\mathbb{P}(\tau^x > T|\mathcal{F}_t) = 1_{\tau^x > t} E\left[\exp\left(-\int_t^T \mu_s^x \mathrm{d}s\right)\Big|\mathcal{G}_t\right], \tag{2}$$

where we have used the fact that the sigma-fields $\mathcal{F}_t^x$ and $\mathcal{G}_t$ agree on $\{\tau^x > t\}$. Furthermore, the following expression is available for the $\mathcal{F}_t$-conditional density of τ^x, on $\{\tau^x > t\}$:

$$\frac{\partial}{\partial s}\mathbb{P}(\tau^x \leq s|\mathcal{F}_t) = E\left[\exp\left(-\int_t^s \mu_u^x \mathrm{d}u\right)\mu_s^x\Big|\mathcal{G}_t\right]. \tag{3}$$

From the above expressions we recognize the doubly stochastic or Cox setting (see Duffie, 2001, App. I, and references therein), according to which, for all $T \geq t$ and conditional on $\mathcal{G}_T \vee \mathcal{F}_t$, the coordinate processes $(N^x)_{x\in I}$ are independent and such that for each $x \in I$ the increment $N_T^x - N_t^x$ is Poisson distributed with parameter $\int_t^T 1_{\tau^x > s}\mu_s^x \mathrm{d}s$.

2.2. **Classical LC Model.** Lee and Carter (1992) proposed a simple model for describing the secular change of mortality as a function of a single time index. This model is fitted to historical data and the resulting estimate of the time-varying parameter is then forecast as a stochastic time series using standard Box-Jenkins methods. From this forecast of the general level of mortality, the actual age-specific rates are derived using the estimated age effects. For a review of recent applications of the Lee-Carter methodology, we refer the interested readers to Lee (2000), as well as to the couple of review papers by Pitacco (2004) and Wong-Fupuy and Haberman (2004).

Let us introduce the set of dates $\mathcal{T} = \{t_1, \dots, t_m\}$ and denote by $m_x(t)$ the central death rate relative to age $x \in I$ and date $t \in \mathcal{T}$. In particular, for each i and j, $m_{x_i}(t_j)$ is the average number of deaths between ages x_i and x_{i+1} to the exposed to risk between t_j and t_{j+1}. Under the assumption of a piecewise constant intensity between ages x_i+t and x_i+t+1, we have that $\mu_t^{x_i} \cong m_{x_i+t}(t)$. The LC approach is in essence a relational model of the form

$$\ln \widehat{m_x}(t) = \alpha_x + \beta_x \kappa_t + \epsilon_x(t), \tag{4}$$

where $\widehat{m_x}(t)$ is the unconstrained maximum likelihood estimator of $m_x(t)$, the $\epsilon_x(t)$'s are homoskedastic centered error terms and where the parameters are subject to the constraints

$$\sum_{t \in \mathcal{T}} \kappa_t = 0 \text{ and } \sum_{x \in I} \beta_x = 1 \tag{5}$$

ensuring model identification. The model (4) is fitted to a matrix of age-specific observed forces of mortality using singular value decomposition (SVD). The resulting estimate of the time-varying parameter κ_t is then forecast as a stochastic time series using standard Box-Jenkins methods. For example, in the classical paper by Lee and Carter (1992), κ_t is modeled as a random walk with drift.

Interpretation of the parameters is quite simple: $\exp \alpha_x$ is the general shape of the mortality schedule and the actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x (the shape of the β_x profile tells which rates decline rapidly and which slowly over time in response of change in κ_t).

2.3. **Generalized Lee-Carter.** In light of the setup described in Sec. 2.1, we can see the LC approach as a discretized version of a general continuous-time intensity of the form

$$\mu_t^x = \exp(\alpha(x+t) + \beta(x+t) \cdot \kappa_t), \tag{6}$$

for some functions α and β and where κ is an $\mathbb{R}^d$-valued $\mathbb{G}$-predictable process. For simplicity, we focus here on diffusion processes and assume κ to have dynamics described by the stochastic differential equation

$$\mathrm{d}\kappa_t = \delta(t, \kappa_t)\mathrm{d}t + \sigma(t, \kappa_t)\mathrm{d}W_t, \tag{7}$$

for a d-dimensional standard Brownian motion W and some continuous vector and matrix-valued functions δ and σ ensuring that the solution to (7) is unique and strong (see Protter, 2004, Ch. 5).

Care must be taken in reconciling (4) with (6). We stress that the dependence of functions α and β on time is due to the fact that each process μ_t^x describes the evolution of the intensity of mortality of an individual aged $x+t$ at each time t. Thus, the α_x's and β_x's of expression (4) must be seen as the pointwise estimates of the functions $\alpha(\cdot)$ and $\beta(\cdot)$ at each age in I.

When $d=1$, $\delta(\cdot)\equiv\overline{\delta}\in\mathbb{R}$ and $\sigma(\cdot)\equiv\overline{\sigma}\in\mathbb{R}$, κ is a Brownian motion with drift and after discretization we are back to the model originally proposed by Lee and Carter (1992). Since κ in (6) can be a fairly general multidimensional diffusion, we refer to (6) as to a continuous-time *generalized* LC model. We note that several forms of stochastic Gompertz intensities are also encompassed by (6)-(7). In the one-dimensional case, for example, if $\delta(\cdot)=-a\kappa_t$ (with $a>0$) and $\sigma(\cdot)\equiv\overline{\sigma}\in\mathbb{R}$, we are back to the so called Mean Reverting Brownian Gompertz model introduced by Milevsky and Promislow (2001).

2.4. **Measure Changes.** According to Biffis, Denuit and Devolder (2005), a class of equivalent probability measures can be specified ensuring the stability of both the framework described in Sec. 2.1 and the class of generalized LC intensities. Specifically, let us consider for each $T>0$ a strictly positive random variable ξ_T such that $E^{\mathbb{P}}[\xi_T]=1$. Then, a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ can be defined on $(\Omega,\mathcal{F}_T)$ by setting its density equal to $\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}=\xi_t$. A suitable martingale representation theorem (see Jeanblanc and Rutkowski, 2000, and references therein) allows us to write ξ as

$$\mathrm{d}\xi_t=\xi_{t-}(-\eta_t\mathrm{d}W_t+\sum_{x\in I}(\phi_t^x-1)\mathrm{d}M_t^x), \tag{8}$$

with $\phi^x>0$ for all x, and to obtain a factorization $\xi=\xi'\xi''$ with factors having the following explicit expression

$$\xi_t' = \exp\left(-\int_0^t \eta_s\mathrm{d}W_s-\int_0^t ||\eta_s||^2\mathrm{d}s\right) \tag{9}$$

$$\xi_t'' = \prod_{x\in I}\exp\left(\int_0^t \ln\phi_s^x\mathrm{d}N_s^x-\int_0^{t\wedge\tau^x}(\phi_s^x-1)\mu_s^x\mathrm{d}s\right), \tag{10}$$

where the $\mathbb{R}^d$-valued process η and the $\mathbb{R}_{++}$-valued processes $(\phi^x)_{x\in I}$ are $\mathbb{F}$-predictable and satisfy suitable integrability conditions. Let η and $(\phi^x)_{x\in I}$ be $\mathbb{G}$-predictable (rather than just $\mathbb{F}$-predictable) and let us assume that for each x

$$\phi_t^x = \exp\left(a^x(x+t) + b^x(x+t)\cdot\kappa_t\right), \tag{11}$$

for some functions $(a^x)_{x\in I}$ and $(b^x)_{x\in I}$. Then, under $\widetilde{\mathbb{P}}$ the process $\widetilde{W} = W + \int_0^{\cdot} \eta_s \mathrm{d}s$ is an $\mathbb{F}$-Brownian motion and each stopping time τ^x has intensity $\widetilde{\mu}^x = \phi^x \mu^x$, i.e. an intensity of the generalized LC type given by

$$\widetilde{\mu}_t^x = \exp(\widetilde{\alpha}^x(x+t) + \widetilde{\beta}^x(x+t)\cdot\kappa_t), \tag{12}$$

with $\widetilde{\alpha}^x = \alpha + a^x$, $\widetilde{\beta}^x = \beta + b^x$. Under $\widetilde{\mathbb{P}}$, the dynamics of the time-trend κ are described by the SDE

$$\mathrm{d}\kappa_t = (\delta(t,\kappa_t) - \sigma(t,\kappa_t)\eta_t)\mathrm{d}t + \sigma(t,\kappa_t)\mathrm{d}\widetilde{W}_t. \tag{13}$$

From (12)-(13), it is apparent that the change of measure considered has a twofold effect on the new intensities: one the one hand, it affects the drift of the time-index κ through the process η entering (9); on the other hand, it acts on each compensated jump process M^x through the strictly positive process ϕ^x in (8), leading to an actual *change in the intensity process itself*. Put another way, if the processes $(\phi^x)_{x\in I}$ are not identically 1, then the processes $(\mu^x)_{x\in I}$ are *not* the intensities of $(\tau^x)_{x\in I}$ under $\widetilde{\mathbb{P}}$. This will be exploited in Sec. 3.2 to specify a rich class of mortality risk-premiums.

3. A Stochastic Model for Insurance Securities

In Sec. 3.1, we recall some basic results concerning the fair valuation of insurance securities, in the spirit of the IASB proposals (e.g. IASB, 2004). Then, Sec. 3.2 describes the mortality risk-premiums that can be introduced by exploiting the changes of measure described in the previous section.

3.1. **Insurance Market Model.** In a financial market, the absence of arbitrage is essentially equivalent to the existence of an equivalent probability measure under which the gain from holding a security is a martingale after deflation at the risk-free rate (see Duffie, 2001, and references therein). The setup described in Sec. 2.1 allows us to easily extend such result to markets involving mortality-contingent securities (e.g. Artzner and Delbaen, 1995). This is remarkably convenient for the valuation of insurance liabilities according to the principles of IASB (2004).

Let us take as given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the information structure is the same as the one described in Sec. 2.1, but with the Brownian subfiltration $\mathbb{G} \subset \mathbb{F}$ now carrying also information about financial security prices and not just the evolution of mortality risk factors. We refer to $\mathbb{P}$ as to the physical (real-world) probability measure and take T^* to be a final trading date and r to be a bounded short rate process adapted to $\mathbb{G}$, from which a money-market account $B_. = \exp(\int_0^\cdot r_s \mathrm{d}s)$ can be defined.

Let V be an $\mathbb{F}$-adapted process representing the price of an insurance security issued to an individual aged x at time 0 and paying a continuous dividend-stream contingent on survival and a lump sum in case of death (see Biffis, 2004, for other examples). For ease of notation, we drop reference to age in the sequel. We let S and D represent the $\mathbb{G}$-predictable survival dividend and death benefit processes. In the absence of arbitrage, there exists an equivalent probability measure $\widetilde{\mathbb{P}}$ such that the discounted gain process $B_t^{-1} V_t + \int_0^t B_s^{-1} D_s \mathrm{d}N_s + \int_0^t B_s^{-1} S_s 1_{\tau > s} \mathrm{d}s$ is an $\mathbb{F}$-martingale. We take[2] $\widetilde{\mathbb{P}}$ to belong to the class of measure changes described in Sec. 2.4, so that it is actually parametrized by η and $(\phi^x)_{x \in I}$ (i.e. $\widetilde{\mathbb{P}} \doteq \widetilde{\mathbb{P}}^{(\eta, (\phi^x))}$). In view of the assumed information setup, assuming that a security has zero price if it does not pay any dividends, we can employ (2)-(3) to obtain (e.g. Lando, 1998)

$$\begin{aligned} V_t &= B_t \widetilde{E} \left[\int_t^T B_s^{-1} D_s \mathrm{d}N_s + \int_t^T B_s^{-1} S_s 1_{\tau > s} \mathrm{d}s + B_T^{-1} V_T \Big| \mathcal{F}_t \right] \\ &= 1_{\tau > t} B_t \widetilde{E} \left[\int_t^T \widehat{B}_s^{-1} D_s \widetilde{\mu}_s \mathrm{d}s + \int_t^T \widehat{B}_s^{-1} S_s \mathrm{d}s + \widehat{B}_T^{-1} \overline{V}_T \Big| \mathcal{G}_t \right], \end{aligned} \tag{14}$$

where $\widehat{B}_. = \exp(\int^\cdot (r_s + \widetilde{\mu}_s) \mathrm{d}s)$ represents a 'mortality risk-adjusted' money market account, $\widetilde{\mu}$ is the $\mathbb{G}$-predictable intensity of τ under $\widetilde{\mathbb{P}}$ and $\overline{V}_t$ is the pre-death price of the security, in the sense that $V_t = 1_{\tau > t} \overline{V}_t$. As a result, we see that the standard risk-neutral machinery can be employed for the fair valuation

[2] The meaning of such assumption is that the dynamics of financial securities are the same in the financial market and in the insurance market (see Blanchet-Scalliet and Jeanblanc, 2004, for details).

of mortality-contingent claims, provided we consider fictitious securities paying a fictitious dividend $D\widetilde{\mu} + S$ under a fictitious short rate $r + \widetilde{\mu}$.

If the insurance market is incomplete, i.e. the insurance assets available do not span the whole space of mortality-contingent claims, there are infinitely many equivalent martingale measures making (14) hold. Indeed, expression (14) determines a whole (open) interval of prices consistent with the absence of arbitrage. We narrow down the price range by mimicking the market practice of specifying $\widetilde{\mu}$ in terms of adjustments to a reference physical intensity of mortality, eventually identifying $\widetilde{\mathbb{P}}$ through calibration to observed prices. To do this properly, we need to understand the effects of the parametrization of $\widetilde{\mathbb{P}}$ on the dynamics of security prices (Sec. 3.2) and intensities (Sec. 4.1).

3.2. **Mortality Risk Premiums.** According to the IASB's principles, all risk sources should be taken into account when computing the fair value of life insurance liabilities. In particular, suitable margins for non-diversifiable risks (e.g. the longevity risk) should be allowed for. Our framework allows us to identify pretty well three sources of mortality risk affecting insurance securities.

The first one is the so called *unsystematic* risk associated with the fluctuations of death occurrences in a portfolio of insureds. In our context, such risk is associated with the randomness in the jumps of the processes $(N^x)_{x\in I}$, conditional on the information carried by $\mathbb{G}$, i.e. conditional on knowledge of the whole path of the intensities $(\mu^x)_{x\in I}$. Bearing in mind the construction given in (1), this is the risk of fluctuations associated with the draws from each unit exponential random variable Θ^x. Such risk can be pooled, in the sense that a large enough portfolio reduces its impact. As a consequence, it may not be priced under $\widetilde{\mathbb{P}}$.

The second source of risk is the *systematic* risk affecting simultaneously the intensities of the stopping times $(\tau^x)_{x\in I}$. When it manifests itself through persistent downward trends in the intensities, such risk is called longevity risk. In our context, it is associated with the information carried by $\mathbb{G}$ and is represented by the Brownian risk sources common to the processes $(\mu^x)_{x\in I}$ and impacting an insurance portfolio at the same time and in the same direction. Such risk cannot be diversified away, and a suitable margin should be allowed for.

There is another source of risk which can reside in the timing of the jumps of $(N^x)_{x\in I}$. It is the uncertainty concerning the likelihood of death as described by the information carried by $\mathbb{G}$. As often happens, the dynamics of the intensities $(\mu^x)_{x\in I}$ are specified on the basis of some reference population with demographic characteristics *only broadly* matching those of the insureds of concern. The remaining risk may be relevant and should be accounted for by specifying suitable margins. In our framework, this can be achieved by employing non trivial processes $(\phi^x)_{x\in I}$ making the compensated processes $(M^x)_{x\in I}$ come into play in the Radon-Nykodym density (8).

Let us now identify the specific adjustments allowed for by the model for each of the risks described. Let us focus on the security introduced in the previous section and suppose, for ease of interpretation, that D is proportional to V_-, i.e. $D_t = d_t V_{t-}$ for all t and suitable $\mathbb{G}$-predictable process d. Let us assume that the physical (real-world) dynamics of our security are given by

$$\begin{aligned} \mathrm{d}V_t &= V_{t-}\Big((\nu_t \mathrm{d}t + \sigma_t^V \cdot \mathrm{d}W_t + (d_t-1)\mathrm{d}N_t\Big) \\ &= V_{t-}\Big(\delta_t^V \mathrm{d}t + \sigma_t^V \cdot \mathrm{d}W_t + (d_t-1)\mathrm{d}M_t\Big), \end{aligned}$$

where the physical drift δ^V is given by $\delta_t^V \doteq \nu_t + (d_t-1)\mu_t 1_{\tau>t}$ and where we must consider the process V stopped at τ, since the value of the contract is zero after τ. As an example, we can think of a unit-linked contract backed by a unit reserve with value process V: at death, the beneficiary is paid the amount $D_\tau = d_\tau V_{\tau-}$ financed through liquidation of the reserve $V_{\tau-}$ and through provision of the top-up $(d_\tau - 1)V_{\tau-}$. Now, the measure change $\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t} = \xi_t$ allows us to write the risk-neutral dynamics as follows

$$\mathrm{d}V_t = V_{t-}\Big((\delta_t^V - \eta_t \cdot \sigma_t^V + (d_t-1)\widetilde{\mu}_t 1_{\tau>t})\mathrm{d}t + \sigma_t^V \cdot \mathrm{d}\widetilde{W}_t + (d_t-1)\mathrm{d}\widetilde{M}_t\Big),$$

with the no-arbitrage restriction $\delta_t^V - \eta_t \cdot \sigma_t^V + (d_t-1)\widetilde{\mu}_t 1_{\tau>t} = r_t - s_t 1_{\tau>t}$, where we express the the survival payouts as $S_t = s_t V_{t-}$ for suitable $\mathbb{G}$-adapted process s.

The latter two expressions make it clear the effect of the measure change on the dynamics of the insurance security. Indeed, the spread between the risk-neutral and the physical drift is equal to

$$-\eta_t \cdot \sigma_t^V - (d_t-1)(1-\phi_t)\mu_t,$$

showing a different role played by the processes η and ϕ entering (9)-(10). If the Brownian sources of risk carries a risk-premium (i.e., $\eta \neq 0$), this is reflected by a drift adjustment $-\eta_t \cdot \sigma_t^V$ proportional to the volatility process σ^V. If the compensated process carries a risk-premium (i.e., $\phi \neq 1$), this is reflected by a drift adjustment $(d_t - 1)(\phi_t - 1)\mu_t$ proportional to the physical intensity and to the relative size of the death benefit, which can in turn affect the sign of the adjustment.

4. Lee-Carter in the Risk-Neutral world

In Sec. 4.1, we show how an LC intensity behaves in the risk-neutral world and how it can incorporate margins for the sources of risk described in the previous section. Sec. 4.2 examines some calibration issues.

4.1. **The Risk-neutral Lee-Carter Model.** Let us focus again on a single stopping time τ with LC intensity μ under the physical measure $\mathbb{P}$. The $\mathbb{P}$-dynamics of the process μ can be written as

$$\mathrm{d}\mu_t = \mu_t(\delta_t^\mu \mathrm{d}t + \sigma_t^\mu \cdot \mathrm{d}W_t),$$

where the $\mathbb{R}$-valued drift process δ^μ and the $\mathbb{R}^d$-valued volatility process σ^μ can be recovered by applying Itô's formula to (6). The dynamics of μ under $\widetilde{\mathbb{P}}$ are instead given by

$$\mathrm{d}\mu_t = \mu_t((\delta_t^\mu - \eta_t \cdot \sigma_t^\mu)\mathrm{d}t + \sigma_t^\mu \cdot \mathrm{d}\widetilde{W}_t),$$

so that the drift process under $\widetilde{\mathbb{P}}$ is $\widetilde{\delta}^\mu = \delta^\mu - \eta \cdot \sigma^\mu$. However, we know that μ needs not be the intensity process of τ under $\widetilde{\mathbb{P}}$, unless $\phi = 1$. If that is the case, it is clear that

$$\widetilde{\delta}_t^\mu - \delta_t^\mu = -\eta_t \cdot \sigma_t^\mu \tag{15}$$

is the only margin allowed by the model for the systematic risk affecting the evolution of the intensity over time. More generally, if ϕ is not identically 1, the intensity of τ is represented by a different process, $\widetilde{\mu} = \phi\mu$. From (11), the dynamics of ϕ under $\widetilde{\mathbb{P}}$ can be written as

$$\mathrm{d}\phi_t = \phi_t(\widetilde{\delta}_t^\phi \mathrm{d}t + \sigma_t^\phi \cdot \mathrm{d}\widetilde{W}_t),$$

with obvious meaning of the notation. Now, integration by parts yields:

$$\begin{aligned} \mathrm{d}\widetilde{\mu}_t &= \phi_{t-}\mathrm{d}\mu_t + \mu_{t-}\mathrm{d}\phi_t + \mathrm{d}[\phi,\mu]_t \\ &= \phi_t\mu_t(\widetilde{\delta}^\mu_t \mathrm{d}t + \sigma^\mu_t \cdot \mathrm{d}\widetilde{W}_t) + \mu_t\phi_t(\widetilde{\delta}^\phi_t \mathrm{d}t + \sigma^\phi_t \cdot \mathrm{d}\widetilde{W}_t) + (\phi_t\sigma^\phi_t)\cdot(\mu_t\sigma^\mu_t)\mathrm{d}t \\ &= \widetilde{\mu}_t\left((\widetilde{\delta}^\mu_t + \widetilde{\delta}^\phi_t + \sigma^\mu_t\cdot\sigma^\phi_t)\mathrm{d}t + (\sigma^\mu_t + \sigma^\phi_t)\cdot \mathrm{d}\widetilde{W}_t\right). \end{aligned} \tag{16}$$

The LC dynamics (16) provide us with some useful insights into the behavior of LC intensities in the risk-neutral world. First, the drift adjustment (15) is replaced by the more complex spread

$$\widetilde{\delta^\mu_t} - \delta^\mu_t = -\eta_t\cdot\sigma^\mu_t + \widetilde{\delta}^\phi_t + \sigma^\mu_t\cdot\sigma^\phi_t,$$

allowing the timing of the jump process N in (10) to affect the drift through ϕ under the equivalent martingale measure. Second, the volatility of the $\widetilde{\mathbb{P}}$-intensity is in general different from that of the $\widetilde{\mathbb{P}}$-intensity process. Third, there are non-trivial specifications of $\widetilde{\delta}^\phi$, σ^ϕ that allow some Brownian sources of risk to disappear from the dynamics of $\widetilde{\mu}$ when switching from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$. This is particularly relevant when dealing with issues regarding the orthogonality of financial and mortality risk factors, as we explain below.

4.2. **Model Calibration.** In Sec. 3.1, we have taken $\mathbb{G}$ to be the filtration generated by financial security prices and mortality risk factors, *without* making any assumptions on their correlation under the physical measure $\mathbb{P}$. It is clear, however, that having independence between financial and mortality risk factors under $\widetilde{\mathbb{P}}$ greatly facilitates the calibration of the model, as we show below. Expression (16) tells us that LC intensities enable to consider situations in which such assumption needs not hold under the physical measure.

As a simple example, consider the case of an annuity paying unitary amounts at dates in $\mathcal{T} = \{t_1, \dots, t_m\}$. From the results in Sec. 3.1, we can exploit the independence assumption (under $\widetilde{\mathbb{P}}$) to write the time-0 price of such security as

$$\widetilde{E}\left[\sum_{t\in\mathcal{T}} B_t^{-1}1_{\tau>t}\right] = \sum_{t\in\mathcal{T}}\widetilde{E}\left[\widehat{B}_t^{-1}\right] = \sum_{t\in\mathcal{T}}\widetilde{E}\left[B_t^{-1}\right]\widetilde{\mathbb{P}}(\tau>t).$$

The last expression shows that we can separately calibrate, the financial component to zero-coupon bond prices, the mortality component to risk-neutral survival probabilities. We provide an example of the latter in the next section.

We note that the same holds true for more general annuities (e.g. unit-linked) provided the randomness in the payouts only depends on financial risk factors.

5. Numerical Illustrations

In this section, we provide a simple example of calibration of risk-neutral LC intensities to the survival probabilities implied by a mortality table employed in the Italian annuity market. In particular, we estimate the margins to be added to a classical LC model fitted to Italian population data.

5.1. **Lee-Carter modeling for the population mortality rates.** We use data concerning Italian males, general population. They have been downloaded from the Human Mortality Database (www.mortality.org). The period considered is 1960-2001 and the age range is 50-100. Fig. 1 depicts the shape of the mortality surface. More specifically, the $\ln \widehat{m_x}(t)$ are displayed in function of age x and time t. The fit of the Lee-Carter model on the Italian population data gives the results displayed in Fig. 2. The parameters of the random walk with drift are given next: $\widehat{\delta} = -0.6630901$ and $\widehat{\sigma}^2 = 1.974014$.

< Fig. 1 about here>

< Fig. 2 about here>

5.2. **Lee-Carter modelling for the IPS55 life table.**

5.2.1. *Presentation of the IPS55.* The ANIA (for *Associazione Nazionale fra le Imprese Assicuratrici*, Italian Association of Insurers) recently defined a set of life tables for pricing and reserving in the life anuity business: the IPS55 life table. It replaces the former RG48 annuitants life table released by ANIA.

IPS55 life tables are based on mortality projections performed by the Italian National Institute for Statistics (*Istituto Nazionale di Statistica*). These projections have been obtained from the Lee-Carter model, fitted to population data. ANIA then applied self-selection factors to the death probabilities of the projected 1955 cohort life table. Age shifts are then used to take mortality improvements into account. Specifically, the technical ages are obtained by adding 3 years to the real age for generations between 1908 and 1925, 2 years for generations between 1926 and 1938, 1 year for generations between 1939 and 1947, and by substracting 1 year for generations between 1961 and 1970, and 2 years for the generations from 1971. No correction is applied for the generations between 1948 and 1960 (the cohort life table for the generation 1955 is thus directly applied).

Fig. 3 depicts the reference life table for the generation 1955 as well as the shape of the mortality surface given by the IPS55 life table. More specifically, the $\ln {m_x}^{\text{IPS55}}(t)$ are displayed in function of age x and time t, where ${m_x}^{\text{IPS55}}(t)$ is the force of mortality prevailing at age x in year t according to the IPS55. Note that the ${m_x}^{\text{IPS55}}(t)$'s do not fill a rectangular array of data.

< Fig. 3 about here>

5.2.2. *Estimation of the κ_t's keeping the population α_x's and β_x's.* We first carry out the classical Lee-Carter estimation on the basis of the IPS55 life table by keeping the α_x's and β_x's fixed at their population values. The estimated κ_t's are then obtained by the linear regressing of $\ln \widehat{m_x}(t) - \widehat{\alpha}_s$ on the $\widehat{\beta}_x$, without intercept and separately for each value of t. The resulting κ_t's are displayed in Fig. 4.

The third plot in Fig. 4 shows that the resulting time index is driven by a totally different process, with an estimated volatility of 0.01106657. It is thus impossible to get a Lee-Carter model consistent with the IPS55 life table if the change of measure only involves the Brownian motion driving the time index.

< Fig. 4 about here>

5.2.3. *Estimation of the α_x's and κ_t's keeping the population β_x's.* We consider the change of measure (9)-(10) with $\eta \in \mathbb{R}$ and $\phi_t^x = \phi_t = a(x+t) + b(x+t)\kappa_t$ for all x, where the dynamics of κ under $\widetilde{\mathbb{P}}$ are assumed to be described by

$$\mathrm{d}\kappa_t = (\delta - \eta\sigma)\mathrm{d}t + \sigma \mathrm{d}\widetilde{W}_t, \tag{17}$$

with $\widetilde{W}$ a one-dimensional Brownian motion and where the coefficients δ, σ have been estimated in Sec. 5.1. Our aim is to estimate the functions a and b, and the parameter η entering the drift of κ under $\widetilde{\mathbb{P}}$.

We start by keeping the population β_x's and by estimating the α_x's and the time-index *implied* by the IPS55 table. We denote the implied time-index by $\widetilde{\kappa}$. The new estimates for the drift and volatility of $\widetilde{\kappa}$ will enable us to recover the adjustment function b underlying the change of measure, as we now show.

The fit of the Lee-Carter model to the IPS55 data gives the results displayed in Fig. 5. Note that the α_x's have been modified in order to satisfy the constraints (5). The parameters of the random walk with drift are given next: $\widetilde{\delta}^{\widetilde{\kappa}} = -0.18834872$ and $(\sigma^{\widetilde{\kappa}})^2 = 0.01273341$. Now, using a superscript 'IM' to indicate that the quantity relates to Italian males, and 'IPS55' to the Italian IPS55 life table, the force of mortality can be written as

$$\widetilde{\mu}_t = \exp\left(\alpha^{\mathrm{IPS55}}(x+t) + \beta^{\mathrm{IM}}(x+t)\widetilde{\kappa}_t\right)$$

where $\alpha^{\mathrm{IPS55}} = \alpha^{\mathrm{IM}} + a$ and $\widetilde{\kappa}$ has $\widetilde{\mathbb{P}}$-dynamics

$$d\widetilde{\kappa}_t = \widetilde{\delta}^{\widetilde{\kappa}} dt + \sigma^{\widetilde{\kappa}} d\widetilde{W}_t,$$

with $\sigma^{\widetilde{\kappa}}$ different from the volatility coefficient σ of κ under $\mathbb{P}$ (as seen from the estimates), so that $\widetilde{\kappa}$ and κ are two different processes under $\widetilde{\mathbb{P}}$. Now, we can also write

$$\widetilde{\mu}_t = \exp\left(\alpha^{\mathrm{IPS55}}(x+t) + \left(\beta^{\mathrm{IM}}(x+t) + b(x+t)\right)\kappa_t\right)$$

with κ having $\widetilde{\mathbb{P}}$-dynamics (17) and with

$$b(x+t) = \beta^{\mathrm{IM}}(x+t)\left(\frac{\sigma^{\widetilde{\kappa}}}{\sigma} - 1\right).$$

The interpolated point estimates of the functions a and b are displayed in Fig. 6. Finally, we can compute η as

$$\eta = \frac{\delta - \widetilde{\delta}^{\widetilde{\kappa}}\sigma/\sigma^{\widetilde{\kappa}}}{\sigma} = 1.197179.$$

The resulting estimates for the adjustment functions a,b and for the coefficient η can be employed for the fair valuation of annuity business in the framework of Sec. 3. In the same context, they can be employed to quantify the adjustments implied by table IPS55 for the different types of risk described in Sec. 3.2.

< Fig. 5 about here>

< Fig. 6 about here>

6. Conclusion

In this work, we have introduced a class of stochastic intensities of mortality generalizing the model proposed by Lee and Carter (1992). We have described their stability under a suitable class of measure changes, which can be employed in the context of risk-neutral valuations. We have examined the way LC intensities behave in the risk-neutral world and the way they affect the dynamics of insurance security prices. To conclude, we have provided an example of parameter calibration with reference to the Italian annuity market.

References

Artzner, P. and F. Delbaen (1995). Default risk insurance and incomplete markets. *Mathematical Finance*, **5**(3):187–195.

Biffis, E. (2004). Affine processes for dynamic mortality and actuarial valuations. Forthcoming in *Insurance: Mathematics & Economics.*

Biffis, E., M. Denuit and P. Devolder (2005). Stochastic mortality under measure changes. Working Paper.

Blanchet-Scalliet, C. and M. Jeanblanc (2004). Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics*, **8**:145–159.

Dahl, M. (2004). Stochastic mortality in life insurance: Market reserves and mortality-linked insurance contracts. *Insurance: Mathematics & Economics*, **35**(1):113–136.

Duffie, D. (2001). *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton, third edn.

IASB (2004). *International Financial Reporting Standard N. 4*. International Accounting Standards Board, London.

Jeanblanc, M. and M. Rutkowski (2000). Modeling of default risk: Mathematical tools. Working paper, Départment of Mathématiques, Université d'Évry.

Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research*, **2**(2/3):99–120.

Lee, R. (2000). The Lee-Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal*, **4**(1):80–93.

Lee, R. and L. Carter (1992). Modeling and forecasting the time series of us mortality. *Journal of The American Statistical Association*, **87**(419):659–671.

Milevsky, M. and S. Promislow (2001). Mortality derivatives and the option to annuitise. *Insurance: Mathematics & Economics*, **29**(3):299–318.

Pitacco, E. (2004). Survival models in a dynamic context: a survey. *Insurance: Mathematics & Economics*, **35**(2):279–298.

Protter, P. (2004). *Stochastic Integration and Differential Equations*. Springer-Verlag, Heidelberg, second edn.

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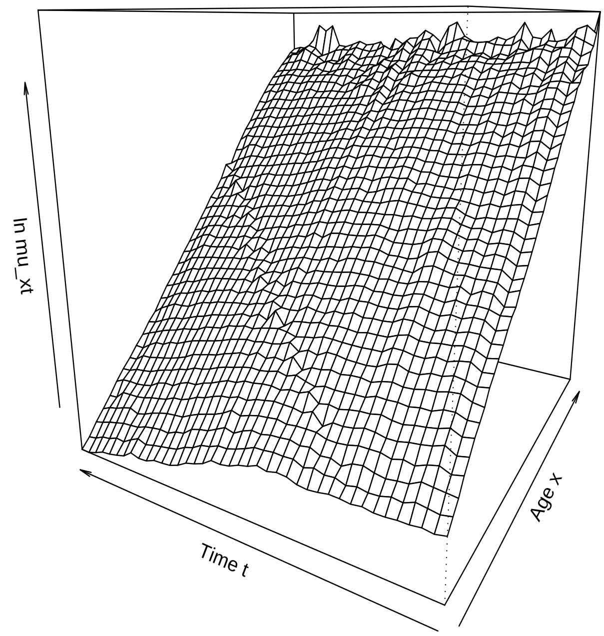

Figure 1: Mortality surface for the Italian males.

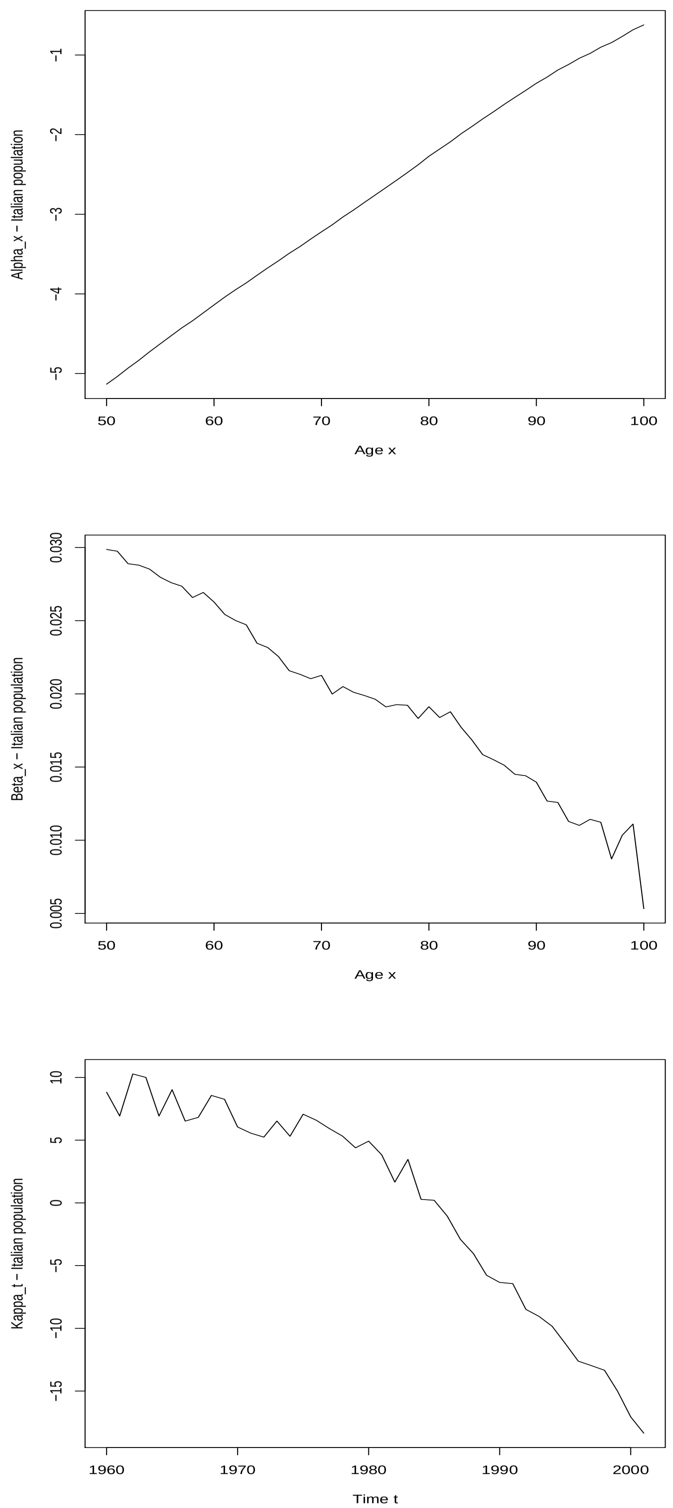

Figure 2: Estimated Lee-Carter parameters for Italian males.

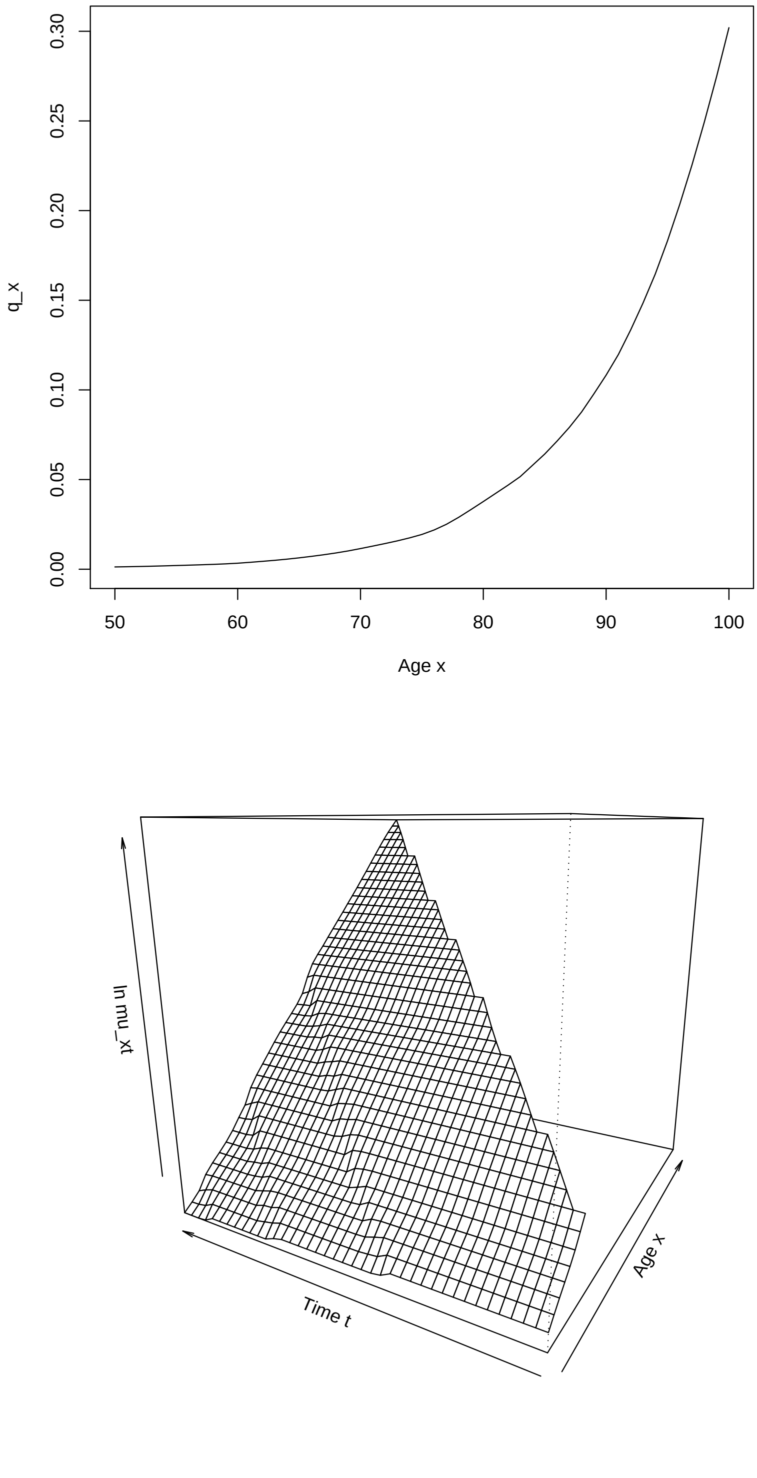

Figure 3: One-year death probabilities for the reference generation 1955 (top panel) and mortality surface for the Italian IPS55 life table (bottom panel), male annuitants.

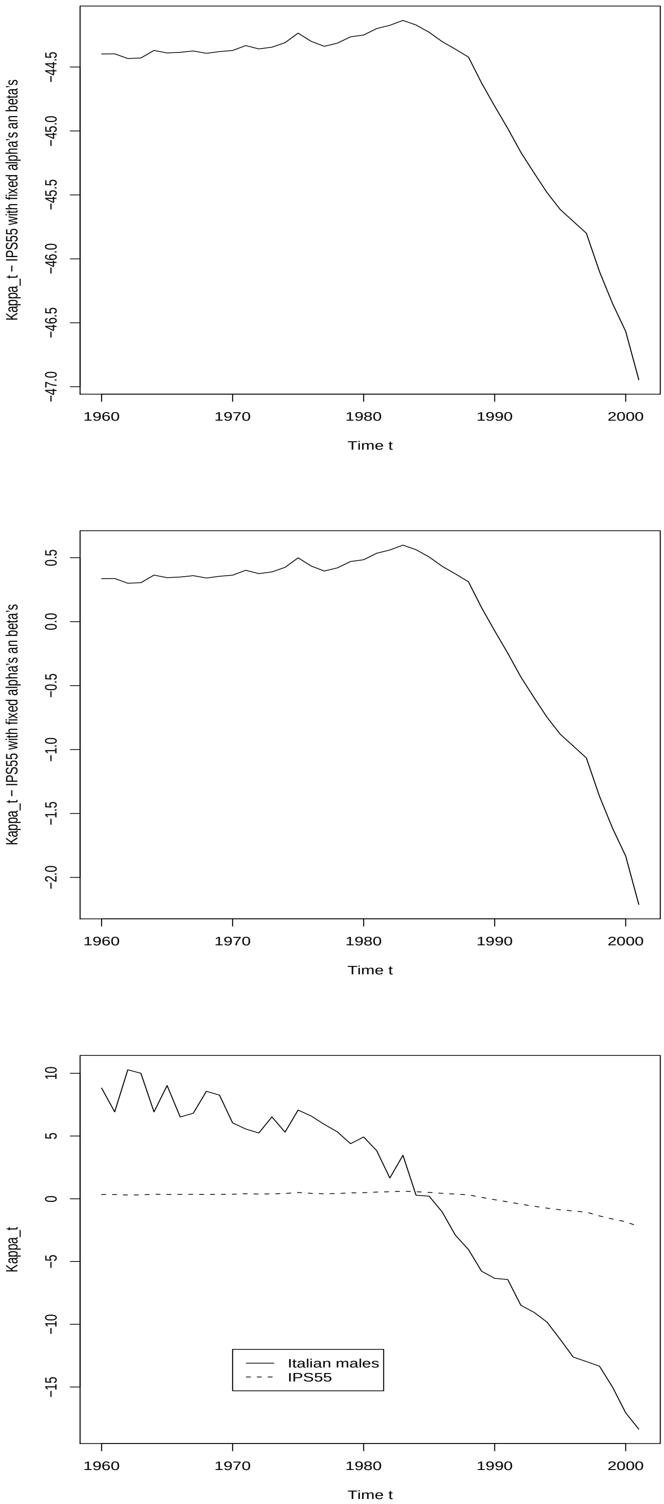

Figure 4: Estimated κ_t's for the IPS55 life table: the resulting $\widehat{\kappa_t}$'s before the standardization (5) are diplayed in the top panel, the transformed values meeting (5) are displayed in the middle panel and a comparison with population κ_t's is given in the bottome panel.

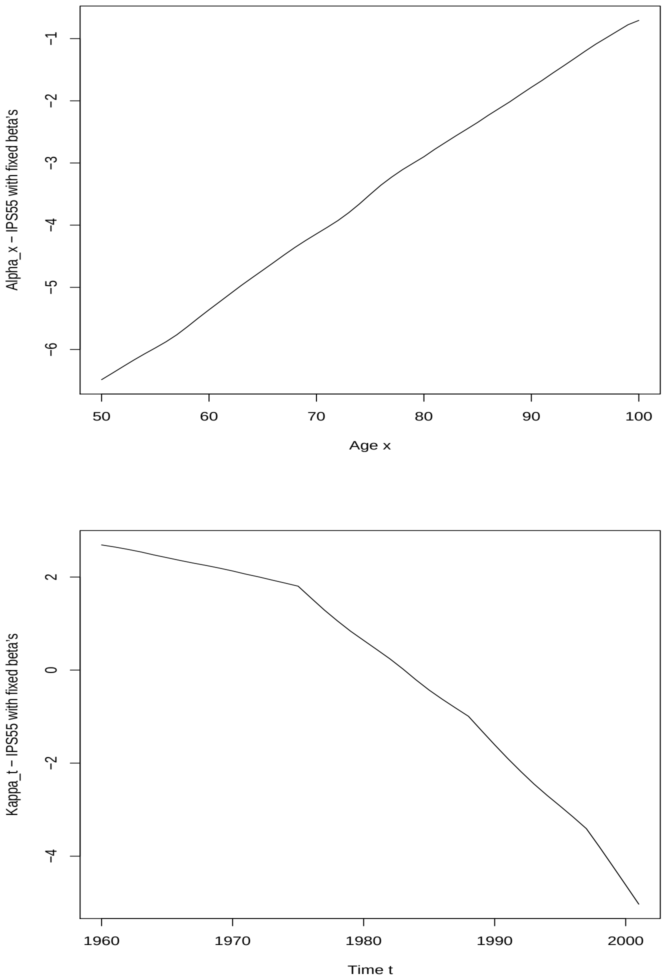

Figure 5: Estimated Lee-Carter parameters for the IPS55 life table, keeping the population β_x's.

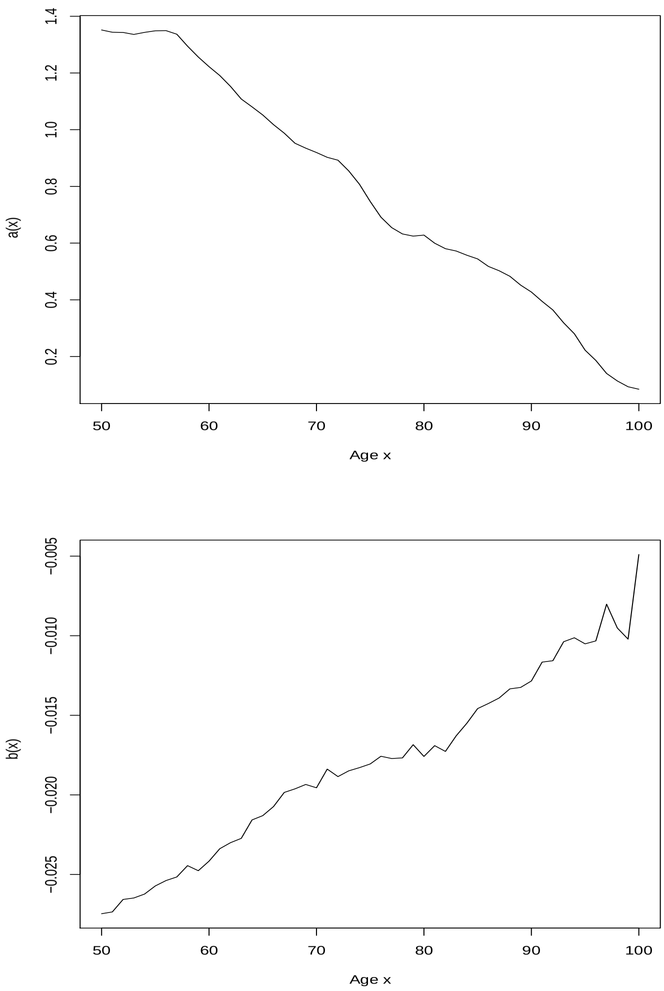

Figure 6: The functions a and b.

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