Pricing and Hedging of European Plain Vanilla Options under Jump Uncertainty

by

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A Quotation

Two men are preparing to go hiking. While one is lacing up hiking boots, he sees that the other man is forgoing his usual boots in favor of sporty running shoes. "Why the running shoes?" he asks. The second man responds, "I heard there are bears in this area and I want to be prepared." Puzzled, the first man points out, "But even with those shoes, you can’t outrun a bear." The second man says, "I don’t have to outrun the bear, I just have to outrun you."

(See Hubbard (2009), p. 157/158)
Outline

1. Literature Review

2. Worst Case Option Pricing

3. Superhedging Strategy

4. Model Calibration

5. Conclusion
1. Literature Review

Option Pricing – Some References:

- Bachelier (1900) [Derivative Pricing using Brownian Motion],
- Black and Scholes (1973) [Reference Model for Option Pricing, No Jumps],
- Cox (1975) [CEV Model],
- Heston (1993) [Stochastic Volatility Model],
- Madan, Carr, and Chang (1998) [Variance Gamma Model],

See Cont and Tankov (2004) or Rebonato (2004) for more details and for other models as well.
Worst Case Scenario Optimization – Some References:

- Hua and Wilmott (1997) [→ Binomial Model Derivative Pricing],
- Korn and Wilmott (2002), [→ Portfolio Optimisation],
- Mönnig (2012), [→ Stochastic Target Approach],
- Belak, M. (2016) [→ BSDE Approach].

Remark: The worst case scenario optimisation problem is also known as Wald’s Maximin approach (Wald 1945, 1950), which is a well-known concept in decision theory. There, this approach is known as robust optimisation (e.g. Bertsimas et al. (2011)) [→ usually involves optimisation procedure done by a computer].


[→ parameter uncertainty, perturbation analysis].
Interpretation of Worst Case Scenarios

\[ \mathbb{E} \left[ \ln \left( X^{t,x,\pi,\tau,k}(T) \right) \right] \]
2. Worst Case Scenario Option Pricing

Consider the initial model with one bond and one risky asset. The aim is to price a contingent claim $\xi$.

**Definition 2.1 (Worst-case price; superhedging strategy)**
(see Belak and M. (2016))

The worst-case price $V_1(t; \xi)$ of $\xi$ at time $t \in [0, T]$ is defined as

$$V_1(t; \xi) \triangleq \essinf \left\{ x \in L_t^+ : \exists (\zeta_1, \zeta_0) \in A_1(t, x) \times A_0(\zeta_1) \text{ s.t.} \quad X_{t,x}^{\zeta_1,\zeta_0,\vartheta}(T) \geq \xi(P^0(T), P^{\vartheta}(T)) \text{ for all } \vartheta \in B(t) \right\}.$$ 

Furthermore, a strategy $(\zeta_1, \zeta_0) \in A_1(t, x) \times A_0(\zeta_1)$ is referred to as a superhedging strategy against $\xi$ if $X_{t,x}^{\zeta_1,\zeta_0,\vartheta}(T) \geq \xi(P^0(T), P^{\vartheta}(T))$ for all $\vartheta \in B(t)$.

We let $\xi$ be a European call option with strike price $K > 0$, i.e.

$$\xi(p) = [p - K]^+.$$
It is well-known (see Black and Scholes (1973)) that the fair price $V_0$ is given by

$$V_0(t, p) = p\Phi (d_1 (K, t, p)) - Ke^{-r[T-t]} (d_2 (K, t, p))$$

with

$$d_1 (K, t, p) = \frac{\log \left( \frac{p}{K} \right) + \left[ r + \frac{\sigma^2}{2} \right] [T - t]}{\sigma \sqrt{T - t}},$$

$$d_2 (K, t, p) = (d_1 (K, t, p)) - \sigma \sqrt{T - t},$$

and where $\Phi$ denotes the standard normal cumulative distribution function. Equivalently, the fair price is given as the unique classical solution of the Black-Scholes PDE

$$-\frac{\partial}{\partial t} V_0(t, p) - rp\frac{\partial}{\partial p} V_0(t, p) - \frac{\sigma^2}{2} p^2 \frac{\partial^2}{\partial p^2} V_0(t, p) + rV_0(t, p) = 0,$$

$$V_0(T, p) = [p - K]^+. $$

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In the jump-threatened market, we assume the minimum and maximum jump sizes to be given by constants $\beta_D \in (-1, 0]$ and $\beta_U \in [0, \infty)$. With this, the pricing PDE for the worst-case price $\mathcal{V}_1(t, p)$ can be written as

$$
\min \left\{ -\frac{\partial}{\partial t} \mathcal{V}_1(t, p) - rp \frac{\partial}{\partial p} \mathcal{V}_1(t, p) - \frac{\sigma^2}{2} p^2 \frac{\partial^2}{\partial p^2} \mathcal{V}_1(t, p) + r \mathcal{V}_1(t, p),
\right.
$$

$$
\mathcal{V}_1(t, p) - \max_{\beta \in \{\beta_D, \beta_U\}} \left[ \mathcal{V}_0(t, [1 + \beta] p) - \beta p \frac{\partial}{\partial p} \mathcal{V}_1(t, p) \right] \right\} = 0,
$$

which is the pricing PDE obtained in both Mönnig (2012) and Belak and M. (2016). Notice that we have used the strict convexity of $\mathcal{V}_0$ to replace the supremum over all $\beta \in [\beta_D, \beta_U]$ with the maximum over $\beta_D$ and $\beta_U$. In a similar fashion, the terminal condition can be written as

$$
\min \left\{ \mathcal{V}_1(T-, p) - \max_{\beta \in \{\beta_D, \beta_U\}} \left[ [(1 + \beta)p - K]^+ - \beta p \frac{\partial}{\partial p} \mathcal{V}_1(T-, p) \right],
\right.
$$

$$
\mathcal{V}_1(T-, p) - [p - K]^+ \right\} = 0.
$$
We define the constants

\[
L := \frac{K}{[1 + \beta_D][1 + \beta_U]},
\]

\[
\alpha_{D/U} := \frac{\beta_{D/U}^2}{\beta_U - \beta_D} \frac{K}{1 + \beta_{D/U}} \frac{1}{L^{\beta_D}} \quad \text{and}
\]

\[
\eta_{D/U}(t) := \exp \left( - \left[ r - \frac{\sigma^2}{2\beta_{D/U}} \right] \left[ 1 + \frac{1}{\beta_{D/U}} \right] [T - t] \right).
\]

The terminal condition can be computed explicitly.

**Lemma 2.2 (Explicit Formula for the Terminal Condition)**

Let \( \beta_D \leq 0 \leq \beta_U \). Then the unique solution of (1) is given by

\[
\mathcal{V}_1(T-, p) = \alpha_{Dp} L^{-\frac{1}{\beta_D}} \mathbb{1}_{\{p < L\}} + \left[ \alpha_{Up} - \frac{1}{\beta_U} + p - K \right] \mathbb{1}_{\{p \geq L\}}.
\]
The Terminal Boundary

Downward jump of max. size is worst case.

Upward jump of max. size is worst case.

This Figure is plotted assuming $\beta_U = -\beta_D = 0.5$ and $K = 100$. 

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The Terminal Boundary

This Figure is plotted assuming $\beta_D = 0.5$, $\beta_U = 0$, and $K = 100$.

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The Terminal Boundary

This Figure is plotted assuming $\beta_D = 0$, $\beta_U = 0.5$, and $K = 100$. 

No jump is worst case.
Upward jump of max. size is worst case.
Theorem 2.3 (Explicit Solution for the Worst Case Option Price)

For \((t, p) \in [0, T) \times \mathbb{R}_+\), the worst-case price \(\mathcal{V}_1\) in a Black-Scholes market with constant minimum and maximum jump sizes is given explicitly as

\[
\mathcal{V}_1(t, p) = p\Phi(d_1(L, t, p)) - Ke^{-r(T-t)}\Phi(d_2(L, t, p)) \\
+ \alpha_D\eta_D(t)p^{-\frac{1}{\beta_D}}\Phi\left(-d_2(L, t, p) + \frac{\sigma}{\beta_D}\sqrt{T-t}\right) \\
+ \alpha_U\eta_U(t)p^{-\frac{1}{\beta_U}}\Phi\left(d_2(L, t, p) - \frac{\sigma}{\beta_U}\sqrt{T-t}\right).
\]
This price can be decomposed to:

- one gap option (with strike $K$ and trigger $L$)
- plus $\alpha_D$ number of short standard power gap put options (where the standard power option is defined in Haug (2007)) with strike 0 and trigger $L$, and
- plus $\alpha_U$ number of standard power gap call options with strike 0 and trigger $L$,

where these latter three options live in the underlying Black–Scholes market (that is without jump risk).
Options Prices

This Figure is plotted assuming $K = 100$, $\sigma = 0.4$, $r = 0.03$, $\beta_D = -0.5$, $\beta_U = 0$, and $T = 1$. The used values are: $K = 100$, $\sigma = 0.4$, $r = 0.03$, $\beta_D = -0.5$, $\beta_U = 0$, and $T = 1$. Thus, $K_D = 200$. 

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The used values are: \( K = 100, r = 0.03, \sigma = 0.4, \beta_D = 0, \beta_U = 0.5, \) and \( T = 1. \)

This Figure is plotted assuming \( K = 100, \sigma = 0.4, r = 0.03, \beta_D = 0, \beta_U = 0.5, \) and \( T = 1. \)
Theorem 2.4 (Greeks)

\[ \partial_p V_1(t,p) = \Phi(d_1(L)) - \frac{\alpha_D}{\beta_D} p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \]
\[ - \frac{\alpha_U}{\beta_U} p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right), \]

\[ \partial_{p^2} V_1(t,p) = \frac{1 + \beta_D}{\beta_D^2} \frac{\alpha_D}{\beta_D} p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \]
\[ + \frac{1 + \beta_U}{\beta_U^2} \frac{\alpha_U}{\beta_U} p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right), \]

\[ \partial_t V_1(t,p) = \left[ r - \frac{\sigma^2}{2\beta_D} \right] \left[ 1 + \frac{1}{\beta_D} \right] \frac{\alpha_D}{\beta_D} p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \]
\[ + \left[ r - \frac{\sigma^2}{2\beta_U} \right] \left[ 1 + \frac{1}{\beta_U} \right] \frac{\alpha_U}{\beta_U} p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right) + \]
\[ - r Ke^{-r(T-t)} \Phi \left( d_2(L) \right), \]
\[
\begin{align*}
\partial_{\sigma} V_1(t, p) &= \frac{\sigma K [T-t]}{\beta_{U} - \beta_{D}} \eta_{D} \left[ \frac{L}{p} \right]^{\frac{1}{\beta_{D}}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_{D}} \sqrt{T-t} \right) + \\
&\quad + \frac{\sigma K [T-t]}{\beta_{U} - \beta_{D}} \eta_{U} \left[ \frac{L}{p} \right]^{\frac{1}{\beta_{U}}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_{U}} \sqrt{T-t} \right), \\
\partial_{\beta_{D}} V_1(t, p) &= \frac{\alpha_{D}}{\beta_{D}^2} \eta_{DP} \frac{1}{\beta_{D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_{D}} \sqrt{T-t} \right) \times \\
&\quad \times \left\{ \frac{\beta_{D} \beta_{U}}{\beta_{U} - \beta_{D}} + \sigma \sqrt{T-t} \left[ d_2(L) - \frac{\sigma}{\beta_{D}} \sqrt{T-t} \right] \right\} + \\
&\quad + \frac{\beta_{D} \left[ 1 + \beta_{U} \right]}{\beta_{U} \left[ \beta_{U} - \beta_{D} \right] \left[ 1 + \beta_{D} \right]} \alpha_{U} \eta_{UP} \frac{1}{\beta_{U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_{U}} \sqrt{T-t} \right) + \\
&\quad - \frac{1 + \beta_{U}}{\beta_{U} - \beta_{D}} \sigma \sqrt{T-t} Le^{-r[T-t]} \Phi \left( d_2(L) \right).
\end{align*}
\]
This Figure is plotted assuming $K = 100$, $\sigma_0 = 0.4$, $r_0 = 0.03$, $\beta_U = -\beta_D = 0.5$, and $T = 1$. 

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This Figure is plotted assuming $K = 100, \sigma_0 = 0.3, r_0 = 0.03, \beta_U = -\beta_D = 0.25$, and $T = 1$. 

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This Figure is plotted assuming $K = 100$, $\sigma_0 = 0.3$, $r_0 = 0.03$, $\beta_U = -\beta_D = 0.25$, and $T = 1$. 

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Vega

The used values are: $K = 100$, $r^1 = 0.03$, $r^0 = 0.03$, $\sigma^1 = 0.3$, $\sigma^0 = 0.3$, $\beta^D = -0.25$, and $\beta^U = 0.25$, $T = 1$.

This Figure is plotted assuming $K = 100$, $\sigma^0 = 0.3$, $r^0 = 0.03$, $\beta^U = -\beta^D = 0.25$, and $T = 1$. 

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This Figure is plotted assuming $K = 100$, $\sigma_0 = 0.3$, $r_0 = 0.03$, $\beta_U = -\beta_D = 0.25$, and $T = 1$.
This Figure is plotted assuming $\sigma = 0.3$, $r = 0.03$, $\beta_U = -\beta_D = 0.25$, $T = 1$. 

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Implied Volatility Surface for $\beta_U = 0$

This Figure is plotted assuming $\sigma = 0.3$, $r = 0.03$, $\beta_D = -0.25$, $\beta_U = 0$, $T = 1$. 

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Implied Volatility Surface for $\beta_D = 0$

This Figure is plotted assuming $\sigma = 0.3$, $r = 0.03$, $\beta_U = 0.25$, $\beta_D = 0$, $T = 1$. © Olaf Menkens

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3. Superhedging Strategy

Define

\[ \mathcal{H}(t, p; \beta) \triangleq \mathcal{V}_1(t, p) + \beta p \frac{\partial}{\partial p} \mathcal{V}_1(t, p) - \mathcal{V}_0(t, [1 + \beta] p). \]  (2)

Observe that \( \mathcal{H} \) is the value of a portfolio. This portfolio consists of one call option and delta shares of the underlying risky asset—hence this is the classical delta–hedge of Black–Scholes for a plain vanilla call option. \( \mathcal{H} \) is the value of this portfolio if at time \( t \) a jump with jump size \( \beta \) happens and the price of the risky asset is \( p \) (just prior to the jump).
Theorem 3.5 (Superhedging Strategy)

One has that

\[ \mathcal{H}(t, p; \beta) = \left[ 1 - \frac{\beta}{\beta_D} \right] \alpha_D \eta_D \ p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \]

\[ + \left[ 1 - \frac{\beta}{\beta_U} \right] \alpha_U \eta_U \ p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right) + \]

\[ + [1 + \beta] \ p \Phi \left( d_1(L) \right) - Ke^{-r[T-t]} \Phi \left( d_2(L) \right) + \]

\[ - [1 + \beta] \ p \Phi \left( d_1 \left( \frac{K}{1 + \beta} \right) \right) + Ke^{-r[T-t]} \Phi \left( d_2 \left( \frac{K}{1 + \beta} \right) \right) , \]

where

\[ \mathcal{H}(t, p; \beta) \geq 0 \quad \text{for all} \ t \in [0, T], p \in (0, \infty) \ \text{and} \ \beta \in [\beta_D, \beta_U] ; \]

and equality holds at least for one \( (t, p, \beta) \).
The best jump size for a European call option with a possible jump in both directions.
The worst jump size (bottom) for a European call option with a possible jump in both directions.
\( \mathcal{H}(t, p, \beta) \) for a European call with \( \beta = \beta_* \) with a possible jump in both directions.
$H(t, p, \beta^*)$ for a European call with $\beta = \beta^*$ with a possible jump in both directions.
\( \mathcal{H}(t, p, \beta) \) for a European call with \( \beta = 2\beta^* \) (bottom) with a possible jump in both directions.
4. European Plain Vanilla Put

Now, let $\xi$ be a European put option with strike price $K > 0$, i.e.

$$\xi(p) = [K - p]^+.$$ 

The boundary condition (1) for a European plain vanilla put writes to

$$\min \left\{ P_1(T-, p) - (K - p)^+, 
\mathcal{P}_1(T-, p) - \sup_{\beta \in [\beta_D, \beta_U]} \left[ (K - (1 + \beta)p)^+ - \beta p \frac{\partial}{\partial p} \mathcal{P}_1(T-, p) \right] \right\} = 0. \quad (5)$$

It is straightforward to verify (either by direct computation or by using the Put–Call–Parity (see e.g. Seydel (2006, Exercise 1.1, p. 52) or Cont and Tankov (2004, p. 356))) that the solution is given by
Corollary 4.6

\[ P_1(T-, p) = \left[ \alpha_D p^{-\frac{1}{\beta_D}} + K - p \right] \mathbb{1}_{\{p \leq L\}} - \alpha_U p^{-\frac{1}{\beta_U}} \mathbb{1}_{\{p > L\}} , \]

where \( p \in (0, \infty) \),

Moreover, one has the following

Corollary 4.7

The worst case price of a European plain vanilla call is given by

\[ P_1(t, p) = K e^{-r(T-t)} \Phi(-d_2(L)) - p \Phi(-d_1(L)) + \] 

\[ + \alpha_D \eta_D p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \]

\[ + \alpha_U \eta_U p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right) . \]
Furthermore, \( \mathcal{H}_P(t, p; \beta) \) is given by:

\[
\mathcal{H}_P(t, p; \beta) = \left[ 1 - \frac{\beta}{\beta_D} \right] \alpha_D \eta_D p^{-\frac{1}{\beta_D}} \Phi \left( -d_2(L) + \frac{\sigma}{\beta_D} \sqrt{T-t} \right) + \left[ 1 - \frac{\beta}{\beta_U} \right] \alpha_U \eta_U p^{-\frac{1}{\beta_U}} \Phi \left( d_2(L) - \frac{\sigma}{\beta_U} \sqrt{T-t} \right) +
\]

\[
+ Ke^{-r[T-t]} \Phi \left( -d_2(L) \right) - [1 + \beta] p \Phi \left( -d_1(L) \right) +
\]

\[
- Ke^{-r[T-t]} \Phi \left( -d_2 \left( \frac{K}{1+\beta} \right) \right) +
\]

\[
+ [1 + \beta] p \Phi \left( -d_1 \left( \frac{K}{1+\beta} \right) \right).
\]
5. Model Calibration
Calibrated implied volatilities for maturities $T = 0.0712$ (next slide) and $T = 1.1452$ (second next slide). The blue circles are the implied volatilities observed in the market while the blue dash–dotted lines are the interpolation of the blue circles. The black solid lines give the implied volatility of the worst case option price formula where the parameters have been calibrated using the market data with penalty $a = 10^{30}$. In particular, note that the calibration is done in such a way that the calibrated curve (black solid lines) should be greater or equal the curve plotted from market data (blue dash–dotted lines). For comparison reasons, the usual calibration (that is without penalty, meaning $a = 0$) is given as well (green dashed lines).

<table>
<thead>
<tr>
<th></th>
<th>$T = 0.0712$</th>
<th>$T = 0.1479$</th>
<th>$T = 0.3973$</th>
<th>$T = 0.6466$</th>
<th>$T = 1.1452$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>0.00024</td>
<td>0.09452</td>
<td>0.12341</td>
<td>0.00043</td>
<td>0.20529</td>
</tr>
<tr>
<td>$\beta_D$</td>
<td>$-0.09958$</td>
<td>$-0.12509$</td>
<td>$-0.17554$</td>
<td>$-0.21514$</td>
<td>$-0.21988$</td>
</tr>
<tr>
<td>$\beta_U$</td>
<td>0.21213</td>
<td>0.32089</td>
<td>0.44456</td>
<td>0.49974</td>
<td>0.53858</td>
</tr>
<tr>
<td>$O(\cdot)$</td>
<td>0.0381</td>
<td>0.12859</td>
<td>0.09537</td>
<td>0.14105</td>
<td>0.04946</td>
</tr>
</tbody>
</table>
\( T = 0.0712 \)
\[ T = 0.1479 \]

A graph showing the relationship between the strike price of an option and implied volatility for different time periods. The graph indicates that implied volatility decreases as the strike price increases, which is typical in financial modeling. The specific time period represented on the x-axis is 0.1479.
$T = 0.3973$
$T = 0.6466$
\[ T = 1.1452 \]
6. Conclusion

To summarize, one has the following properties:

• jumps are not averaged out but are fully taken into account (compare with liability insurance),

• first explicit non–trivial superhedging price and superhedging strategy,

• it explains the volatility smile (as well as the smirk),

• and the closed form solution is numerically of the same level as the solution of Black and Scholes.
Thank you very much for your attention!

The corresponding paper can be downloaded from SSRN:
http://ssrn.com/abstract=2773246