Efficient Valuation of GMWB Annuities: A Variance Reduction Approach

Jennifer L. Wang
Ming-hua Hsieh
Dept. of Risk Management and Insurance
National Chengchi University, TAIWAN

Yu-Fen Chiu
Dept. of Financial Engineering and Actuarial Mathematics,
Soochow University, TAIWAN

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1 Valuation of a VA contract with GMWB
   - Discrete Withdrawal GMWB
   - GMWB Option
   - Valuation of GMWB option

2 Variance Reduction Techniques
   - Control Variates
   - Numerical examples
We consider the following contract:

- Single premium $f_0$
- The initial value of the sub-account $F_0$ equals $f_0$.
- Annually a certain percentage ($g$) of the guaranteed amount $f_0$ will be withdrawn from the sub-account for $T$ years ($gT$ is usually equals 1).
- At the beginning of year $t$ ($t = 0, 1, \ldots, T - 1$), guarantee fee ($\alpha$ times $F_t$) and a fixed management fee $K$ are withdrawn from the sub-account by the insurer.
Contract Specification

Above GMWB contract provides the following cash-flows to the policy holder

\[
\begin{align*}
  c_t &= gf_0, \quad t = 1, 2, \ldots, T - 1; \\
  c_T &= \max(gf_0, F_T)
\end{align*}
\]
The cash-flow received at time $T$ can be decomposed into

$$c_T = \max(gf_0, F_T) = gf_0 + \max(F_T - gf_0, 0)$$

These cash-flows can be decomposed into a term annuity with annual payment $gf_0$ and an option-like payment $\max(F_T - gf_0, 0)$. We call the option-like payment the GMWB option.
The fair value of a VA contract with GMWB is the sum of fair values of the term annuity and the GMWB option.

The fair value of the term annuity is easy to compute, since its value only depends on the current term structure of interest rates.

The problem of fair valuation of a VA contract with GMWB reduced to the valuation problem of GMWB option.
Based on risk-neutral valuation principle, the fair value of GMWB option can be expressed as

$$\mathbb{E}_Q \left[ \max(F_T - g_f_0, 0) \right] \over B(T)$$

where $\mathbb{E}_Q$ denote the expectation under risk neutral measure and $B(T)$ denotes the account value of a money market account with initial account value 1.
Dynamics of the sub-account

The value of the sub-account depends on the annual returns of the invested mutual fund. Let $S(t)$ be NAV of the invested mutual fund at year $t$. Then the annual return of the invested mutual fund over the $t$-th year would be:

$$R_t = \frac{S(t)}{S(t-1)}$$

Let us denote $F_t^-$ the account value at year $t$ before withdraws and $F_t^+$ the account value at year $t$ after withdraws.
The process of the account value can then be described

\[
\begin{align*}
F_0^- &= f_0, \\
F_0^+ &= \max((1 - \alpha)F_0^- - K, 0), \\
F_t^- &= R_tF_{t-1}^+, \quad t = 1, 2, \cdots, T \\
F_t^+ &= \max((1 - \alpha)F_t^- - K - gf_0, 0), \quad t = 1, 2, \cdots, T - 1
\end{align*}
\]
The value of GMWB option only depends on $F_T$ and $F_T$ in turns only depends on the joint distribution of $(R_1, \cdots, R_T)$. Therefore, the dynamic of $S(t)$ can be very flexible.

For simulation based method, the only restriction is that the sample of $(R_1, \cdots, R_T)$ is easy to generate.

The dynamic of $B(t)$ can be derived from the selected interest rate model.
Suppose that we wish to estimate $\beta = E_X$, where $X$ is the output of a complex stochastic process. In our case,

$$X = \frac{\max(F_T - gf_0, 0)}{B(T)}.$$

A naive Monte Carlo procedure would generate $n$ independent copies of $X$, and produce the standard estimate

$$\beta_{\text{naive}} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

where $X_1, \ldots, X_n$ are independent copies of $X$. 

**Monte Carlo method**

Jennifer L. Wang Ming-hua Hsieh Dept. of Risk Management and Insurance National Chengchi University, TAIWAN Yu-Fen Chiu Dept. of Financial Engineering and Actuarial Mathematics, Soochow University, TAIWAN
Control Variates

Let $Y$ be a $d$ by 1 random vector and each component of $Y$ is correlated with $X$. Let $(\mu, \Sigma)$ denote the mean vector and covariance matrix of $Y$. The mean vector is known. Suppose that the covariance between $X$ and $Y_i$ is $c_i$ and $c = (c_1, \ldots, c_d)^T$. Define control variates 

$$C = Y - \mu$$

It is clear that the mean vector of $C = 0$, covariance matrix of $C = \Sigma$, and the covariance between $X$ and $C_i$ is $c_i$. 
Control Variates

Let $\lambda \in \mathbb{R}^d$ and define

$$X_C(\lambda) = X - \lambda^T C$$

It is obvious that $E[X_C(\lambda)] = \beta$ and

$$\text{Var}[X_C(\lambda)] = \sigma_X^2 - 2\lambda^T c + \lambda^T \Sigma \lambda$$

The minimizer of above formula

$$\lambda^* = \Sigma^{-1} c$$

and

$$\text{Var}[X_C(\lambda^*)] = \sigma_X^2 - 2(\Sigma^{-1} c)^T c + (\Sigma^{-1} c)^T \Sigma (\Sigma^{-1} c)$$
Control Variates

Hence

\[
\text{Var}[X_C(\lambda^*)] = \sigma_X^2 - c^T\Sigma^{-1}c < \sigma_X^2
\]

Let \(X^{(i)}_C(\lambda^*), i = 1, \ldots, n\), be independent copies of \(X_C(\lambda^*)\). Then it is obvious that

\[
\beta_{\text{control}} = \frac{1}{n} \sum_{i=1}^{n} X^{(i)}_C(\lambda^*)
\]

is a more efficient estimate for \(\beta\).
Assume $S(t)$ is a Levy process and then $R_1, \ldots, R_T$ are independent. We propose two estimators with control variates. Define

$$H_1 = ((1 - \alpha)f_0 - K)R_1,$$
$$H_t = ((1 - \alpha)H_{t-1} - K - gf_0)R_t, \quad t = 2, \ldots, T$$

and set

$$C_1 = H_T - E[H_T], \quad C_2 = \prod_{t=1}^{T} R_t - E \left[ \prod_{t=1}^{T} R_t \right]$$
Table: Point estimates $\times 10^5$ ($f_0 = 1000000$, $r = 0.04$, $T = 20$, $g = 0.05$, $\sigma = 0.16$, $n = 1000000$, $\alpha = 0.008$)

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\beta_{\text{naive}}$</th>
<th>$\beta_{C_1}$</th>
<th>$\beta_{C_2}$</th>
<th>$\beta_{C_1,C_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>2.5517</td>
<td>2.5585</td>
<td>2.5566</td>
<td>2.5584</td>
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<td>2000</td>
<td>2.4681</td>
<td>2.4666</td>
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<td>2.4667</td>
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<tr>
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<td>2.3783</td>
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<td>2.2856</td>
<td>2.2881</td>
<td>2.2886</td>
<td>2.2882</td>
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</table>
**Table:** Standard errors ($f_0 = 1000000$, $r = 0.04$, $T = 20$, $g = 0.05$, $\sigma = 0.16$, $n = 1000000$, $\alpha = 0.008$)

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<th>$K$</th>
<th>$\beta_{\text{naive}}$</th>
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</thead>
<tbody>
<tr>
<td>1000</td>
<td>440.7616</td>
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<td>2000</td>
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<td>151.6640</td>
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Numerical example I

Table: Variance Ratios \((f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.008)\)

<table>
<thead>
<tr>
<th>(K)</th>
<th>(\beta_{C_1})</th>
<th>(\beta_{C_2})</th>
<th>(\beta_{C_1,C_2})</th>
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<tbody>
<tr>
<td>1000</td>
<td>46.7305</td>
<td>9.0460</td>
<td>48.0552</td>
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<tr>
<td>2000</td>
<td>41.5615</td>
<td>8.5656</td>
<td>42.8352</td>
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<tr>
<td>3000</td>
<td>36.6345</td>
<td>8.0746</td>
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<td>4000</td>
<td>32.5992</td>
<td>7.6343</td>
<td>33.8448</td>
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</table>
**Table:** Point estimates $\times 10^5 \ (f_0 = 1000000, \ r = 0.04, \ T = 20, \ g = 0.05, \ \sigma = 0.16, \ n = 1000000, \ \alpha = 0.005)$

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<thead>
<tr>
<th>$K$</th>
<th>$\beta_{\text{naive}}$</th>
<th>$\beta_{C_1}$</th>
<th>$\beta_{C_2}$</th>
<th>$\beta_{C_1,C_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>2.8568</td>
<td>2.8607</td>
<td>2.8604</td>
<td>2.8608</td>
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<tr>
<td>2000</td>
<td>2.7634</td>
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<tr>
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<td>2.6651</td>
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<tr>
<td>4000</td>
<td>2.5683</td>
<td>2.5713</td>
<td>2.5708</td>
<td>2.5713</td>
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</tbody>
</table>
Table: Standard errors ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.005$)

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<tr>
<th>$K$</th>
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<th>$\beta_{C_1,C_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
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<td>62.7985</td>
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<td>62.0168</td>
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<td>2000</td>
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<tr>
<td>3000</td>
<td>462.5048</td>
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<td>4000</td>
<td>454.2942</td>
<td>71.8611</td>
<td>157.7726</td>
<td>70.7243</td>
</tr>
</tbody>
</table>
**Table: Variance Ratios**

\( f_0 = 1000000, \ r = 0.04, \ T = 20, \ g = 0.05, \sigma = 0.16, \ n = 1000000, \alpha = 0.005 \)

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<tr>
<th>( K )</th>
<th>( \beta_{C_1} )</th>
<th>( \beta_{C_2} )</th>
<th>( \beta_{C_1,C_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>56.6435</td>
<td>9.7624</td>
<td>58.0804</td>
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<tr>
<td>2000</td>
<td>50.8874</td>
<td>9.2884</td>
<td>52.2561</td>
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<tr>
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<td>46.1033</td>
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<tr>
<td>4000</td>
<td>39.9656</td>
<td>8.2911</td>
<td>41.2607</td>
</tr>
</tbody>
</table>
The selected control variates are effective from the numerical examples.

The algorithm is easy to generalize to more complex $S(t)$ and $B(t)$ processes.

The algorithm is easy to generalize to life-long GMWB (model $T$ driven by a specific mortality model).

The algorithm can extend to value contracts with $g$ is time dependent.